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GENERALIZED $F_\delta-$ CONTRACTIONS AND MULTIVALUED COMMON FIXED POINT THEOREM

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Abstract. In this paper, using Wardowski technique, we mainly study common fixed point theorem for pair of multivalued mappings with δ-distance satisfying generalized $F_\delta-$ multivalued contraction in complete metric spaces. Our main result extend the result of Wardowski (Fixed Point Theory and Applications 2012 (2012), Article Id:94), and generalize the result of Acar (Abstract and Applied Analysis, 2014 (2014), Article ID:497092, 5 pages).

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1. Introduction and preliminaries

Banach fixed point theorem is one of the pioneer findings of nonlinear analysis. Since its appearance, Banach [1] result extended and generalized in several directions for mappings in different metric space for various types of contractions, see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Nadler [15], on using Pompeiu-Hausdorff metric, employed the notions of multivalued contraction mapping and derive that such types of map preserve fixed-point in metric space. On other side, Fisher [16] defined the notion of δ -distance between the pair of subsets of Y which are bounded and, derived various type of invariant results for multivalued mappings. Some results on δ -distance and multivalued mappings, can be found in [17, 18, 19, 20].

Let $\mathcal{F}$ be the set of all functions $F:(0,\infty)\to\Re$ satisfying the conditions:

(F_1) $\forall\ \gamma,\lambda\in(0,\infty)$, F is such that $\gamma<\lambda$, implies $F(\gamma)<F(\lambda)$.

(F_2) for every sequence of positive numbers $\{b_n\}$, $\lim_{n\to\infty} b_n=0$ if and only if $\lim_{n\to\infty}F(b_n)=-\infty$.

(F_3) there exists $0<k<1$ s.t. $\lim_{\gamma\to0^+}\gamma^kF(\gamma)=0$.

Definition 1.1. [17] *Let (Y,d) be a metric space and let $P:Y\to Y$ be a mapping. Given $F\in\mathcal{F}$, we say that P is F-contraction, if there exists $\kappa>0$ such that for all $u,v\in Y$*

$$d(Pu,Pv)>0\implies \kappa+F(d(Pu,Pv))\le F(d(u,v)).$$

Some beautiful examples of F-contraction are given in [17, 19]. Wardowski[17] also mentioned that every F-contraction P is a contractive mapping, and so it is continuous. Wardowski[17] also gave a real generalization of Banach results.

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Theorem 1.1. [17] *Let (Y, d) be a complete metric space and let $P : Y \to Y$ be a F-contraction. Then P has a unique fixed point in Y.*

On the line of Wardowski, Minak et al.[19], Acar and Altun [20] have also discussed some fixed point results satisfying F-contraction for different type of mappings.

Acar and Altun [20] defined and proved following result.

Definition 1.2. [20] *Let (Y, d) be a metric space and let $P : Y \to B(Y)$ be a mapping. Given $F \in \mathcal{F}$, we say that P is a generalized multivalued F-contraction, if there exists $\kappa > 0$ such that for all $u, v \in Y$*

$$\kappa + F(\delta(Pu, Pv)) \leq F(M(u, v)).$$

with $\min\{\delta(Pu, Pv), d(u, v)\} > 0$, *where*

$$M(u, v) = \max\left\{d(u, v), \mu(u, Pu), \mu(v, Pv), \frac{1}{2}[\mu(u, Pv) + \mu(v, Pu)]\right\}.$$

Theorem 1.2. *Let (Y, d) be a complete metric space, and let $P : Y \to B(Y)$ be a multivalued F -contraction. If F is continuous and Pu is closed for all $u \in Y$. Then P has a fixed point in Y.*

Recently, in 2017, Acar [21] extends the result of Acar and Altun [20] (Theorem 1.2) for multivalued almost $F_\delta-$ contraction in complete metric spaces.

First of all, we give here some basic definitions and notions that are important to derive our main result.

Let (Y, d) be a metric space and suppose there exists a family of bounded and non-empty subsets $B(Y)$ of Y.

Define, for all $H, K \in B(Y)$

$$\delta(H, K) = \sup\{d(u, v) : u \in H, v \in K\},$$
$$\mu(u, K) = \inf\{d(u, v) : v \in K\}.$$

If $H = \{u\}$ we write $\delta(H, K) = \delta(u, K)$ and also if $K = \{v\}$, then $\delta(u, K) = d(u, v)$. For all $H, K, M \in B(Y)$, we can easily prove that

$$\delta(H, K) = \delta(K, H) \geq 0,$$
$$\delta(H, K) \leq \delta(H, M) + \delta(M, K).$$
$$\delta(H, H) = \sup\{d(u, v) : u, v \in H\} = \text{diam H},$$
$$\delta(H, K) = 0, \implies H = K = \{u\}$$

If $\{H_n\}$ is a sequence in $B(Y)$, then $H_n \to H$ for all $H \subseteq Y$ iff

(i) $u \in H$ implies that $u_n \to u$ for some sequence $\{u_n\}$ with $u_n \in H_n$ for $n \in N$,

(ii) for any $\epsilon > 0$, there exists $m \in N$ such that $H_n \subseteq H_\epsilon$ for $n > m$, where

$$H_\epsilon = \{z \in Y : d(z, u) < \epsilon \text{ for some } u \in H\}.$$

In this paper, by the use of $F_\delta-$ contraction and Wardowski [17] technique, we have prove some innovative unique common multivalued fixed point results for two selfmaps with δ distance in complete metric spaces.

2. Main Result:

Theorem 2.1. *Let (Y, d) be a complete metric space, and let $f, g : Y \to B(Y)$ are mappings. If there exists $F \in \mathcal{F}$ and $\kappa > 0$ such that for all $u, v \in Y$*

$$F(\delta(fu, gv)) \leq F(N(u, v)) - \kappa, \tag{1}$$

with $\min\{\delta(fu, gv), d(u, v)\} > 0$, *where*

$$N(u, v) = \max\{d(u, v), \mu(u, fu), \mu(v, gv)\} \tag{2}$$

Moreover, if for all $u \in Y$, gu is closed and F is continuous. Then f and g have unique common fixed point in Y.

Proof. Choose $u_0 \in Y$ such that $fu_0 = u_1$ and $gu_1 = u_2$. Continuing on this line, we construct sequences $\{u_j\}$ and $\{v_j\}$ in Y as

$$u_{2j+1} = fu_{2j} = v_{2j}, \quad u_{2j+2} = gu_{2j+1} = v_{2j+1}, \quad \forall\, j \in N. \tag{3}$$

If there exists $j_0 \in N \cup \{0\}$ for which $v_{j_0} = v_{j_0+1} = u(say)$, then clearly u is a common fixed point of f and g.
Thus assume that $v_{2j} \neq v_{2j+1}$ for all $j \in N \cup \{0\}$. Let $a_j = d(v_{2j}, v_{2j+1})$.
Clearly, $d(v_{2j}, v_{2j+1}) > 0$ and $\delta(v_{2j}, v_{2j+1}) > 0$ for all $j \in N$. Then we have from (2)

$$\begin{aligned} F(a_j) = F(d(v_{2j}, v_{2j+1})) &\leq F(\delta(v_{2j}, v_{2j+1})) = F(\delta(fu_{2j}, gu_{2j+1})) \\ &\leq F(N(u_{2j}, u_{2j+1})) - \kappa, \end{aligned} \tag{4}$$

where,

$$\begin{aligned} N(u_{2j}, u_{2j+1}) &= \max\{d(u_{2j}, u_{2j+1}), \mu(u_{2j}, fu_{2j}), \mu(u_{2j+1}, gu_{2j+1})\} \\ &= \max\{d(v_{2j-1}, v_{2j}), \mu(v_{2j-1}, v_{2j}), \mu(v_{2j}, v_{2j+1})\} \\ &\leq \max\{d(v_{2j-1}, v_{2j}), d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1})\} \\ &= \max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1})\} \end{aligned}$$

Hence, we have from (4)

$$\begin{aligned} F(a_j) &\leq F(\max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1})\}) - \kappa, \\ &= F(\max\{a_{j-1}, a_j)\}) - \kappa, \end{aligned} \tag{5}$$

If $a_{j-1} < a_j$ for all $j \in N$, then $F(a_j) \leq F(a_j) - \kappa$. Which is a contradiction, since $\kappa > 0$. Therefore

$$F(a_j) \leq F(a_{j-1}) - \kappa$$

and so,

$$F(a_j) \leq F(a_{j-1}) - \kappa \leq F(a_{j-2}) - 2\kappa \leq \cdots \leq F(a_0) - j\kappa \tag{6}$$

From (6), on applying $\lim_{j\to\infty}$, we get $\lim_{j\to\infty} a_j = -\infty$. Then from (F_2) we get

$$\lim_{j\to\infty} a_j = 0. \tag{7}$$

and, we have from (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{j\to\infty} a_j^k F(a_j) = 0. \tag{8}$$

By (6) for all $j \in N$, the following holds:

$$a_j^k F(a_j) - a_j^k F(a_0) \leq -a_j^k j\kappa \leq 0. \tag{9}$$

On taking limit as $j \to \infty$ in (9), we obtain that

$$\lim_{j\to\infty} j a_j^k = 0. \tag{10}$$

From (10), there exists $j_1 \in N$ such that $ja_j^k \leq 1$ for all $j \geq j_1$.
Thus, we have for all $j \geq j_1$

$$\lim_{j\to\infty} a_j \leq \frac{1}{j^{\frac{1}{k}}}. \tag{11}$$

For all $l, j \in N$ with $l > j > j_1$, consider

$$\begin{aligned} d(v_{2j}, v_{2l}) &\leq d(v_{2j}, v_{2j+1}) + d(v_{2j+1}, v_{2j+2}) + \cdots + d(v_{2l-1}, v_{2l}) \\ &= a_j + a_{j-1} + \cdots + a_{l-1} = \sum_{i=j}^{l-1} a_i \\ &\leq \sum_{i=j}^{\infty} a_i \leq \sum_{i=j}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

On using the convergence of series $\sum_{i=j}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ and taking limit $j \to \infty$, we get $\lim_{j\to\infty} d(v_{2j}, v_{2l}) = 0$. Thus the sequence $\{v_{2j}\}$ is Cauchy. Consequently, the sequence $\{u_{2j}\}$ is Cauchy. Therefore there exists a $z \in Y$ such that

$$\lim_{j\to\infty} u_{2j} = z. \tag{12}$$

From (3), we have

$$\lim_{j\to\infty} fu_{2j} = z, \qquad \lim_{j\to\infty} gu_{2j+1} = z, \quad \forall\, j \in N. \tag{13}$$

Suppose that $d(fz, z) \neq 0$. This implies $d(fz, z) > 0$ and $\delta(fz, z) > 0$ for all $z \in Y$. Then from (2), we get

$$\begin{aligned} F(d(fz, z)) \leq F(\delta(fz, z)) &= \lim_{j\to\infty} F(\delta(fz, gu_{2j+1})) \\ &\leq \lim_{j\to\infty} F(N(z, u_{2j+1})) - \kappa, \end{aligned} \tag{14}$$

where,

$$\begin{aligned} \lim_{j\to\infty} N(z, u_{2j+1}) &= \lim_{j\to\infty} \max\{d(z, u_{2j+1}), \mu(z, fz), \mu(u_{2j+1}, gu_{2j+1})\} \\ &\leq \lim_{j\to\infty} \max\{d(z, u_{2j+1}), d(z, fz), d(u_{2j+1}, gu_{2j+1})\} \\ &\leq d(z, fz) \end{aligned} \tag{15}$$

and so from (14) and by continuity of F, we get $F(d(fz, z)) \leq F(d(fz, z)) - \kappa$, which is a contradiction. Hence $d(fz, z) = 0$.
Next we claim that $z \in gz$. Suppose on contrary that $z \notin gz$. Therefore there exist a sub-sequence $\{u_{2j_k}\}$ of $\{u_{2j}\}$ such that $\mu(u_{2j_k+1}, gz) > 0$ for all $j_k > j_0 \in N$. Then from (2), we get

$$\begin{aligned} F(\mu(u_{2j_k+1}, gz)) &\leq F(\delta(fu_{2j_k}, gz)) \\ &\leq F(N(u_{2j_k}, z)) - \kappa, \end{aligned} \tag{16}$$

where,

$$N(u_{2j_k}, z) = \max\{d(u_{2j_k}, z), \mu(u_{2j_k}, fu_{2j_k}), \mu(z, gz)\}$$
$$\leq \max\{d(u_{2j_k}, z), d(u_{2j_k}, fu_{2j_k}), \mu(z, gz)\}. \tag{17}$$

Taking the limit $k \to \infty$ in (16), (17) and use the fact that F is continuous, we get $F(\mu(z, gz)) \leq F(\mu(z, gz)) - \kappa$, which is a contradiction to our assumption. Hence $z \in gz$. So z is a fixed point of g.
For uniqueness, assume that there exists $w \neq z$ such that $d(w, z) > 0$ and $\delta(w, z) > 0$. Then from (2), we get

$$F(d(w, z)) \leq F(\delta(w, z)) = F(\delta(fw, gz))$$
$$\leq F(N(w, z)) - \kappa, \tag{18}$$

where,

$$N(w, z) = \max\{d(w, z), \mu(w, fw), \mu(z, gz)\}$$
$$= \max\{d(w, z), d(w, w), d(z, z)\}$$
$$\leq d(w, z), \tag{19}$$

and so from (18), we get $F(d(w, z)) \leq F(d(w, z)) - \kappa$, which is a contradiction. Hence $d(w, z) = 0$.
This completes uniqueness and the proof of our main result. □

3. Conclusion

In this article, we have proved a unique common fixed point theorem for pair of multivalued maps in complete metric spaces with help of Wardowski [17] technique. Moreover, we have generalized the result of Acar [20] for two selfmaps without assuming the continuity of maps f and g and get a unique common fixed point.

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