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A DETERMINISTIC INVENTORY MODEL FOR TIME-VARYING DEMAND AND HOLDING COST WITH PARTIAL BACKLOGGING

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Abstract. In this article, we propose inventory model for deteriorating items with time-varying demand and holding cost. Deterioration rate is assumed to be constant. The shortage is allowed and assumed to be partially backlogged. The demand rate varies with time until the shortage occurs; during shortages, demand rate becomes constant. The backlogging rate is variable and inversely proportional to the length of the waiting time for next replenishment. Taylor series to used for exponential terms approximating up to second-degree terms. We solve the proposed inventory model to obtain the optimal value of order quantity and total cost. The purpose of this paper is to minimize the total cost and maximize total profit of inventory with optimal order quantity. The convexity of the cost function shown graphically. A numerical example is given to show the applicability of the proposed model and sensitivity analysis is also carried out to identify the more sensitive parameters in the system.

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1. Introduction

In the literature regarding inventory models, there are different types of demand: such as constant demand, time-dependent demand, probabilistic demand and stock-dependent demand. We noticed that request of products does not stay steady. Maybe, it changes with time or cost or even with the immediate level of stock. Regularly, we experience items, for example, organic products, drain, medication, vegetables, and photographic moves and so on which have a characterized time of life time. Decay implies harm,

deterioration, dryness, vaporization, and so on. Because of weakening, stock framework confronts the issue of deficiencies and loss of cooperative attitude or loss of benefit. Along these lines, the weakening must likewise consider in stock control models. In writing, there are such a variety of papers on the assurance of the stock level of breaking down things with deficiencies and without lack.

In real-life inventory model, the holding cost is also time dependent and constant deterioration rate. Dutta and Kumar (2015) have analysed a partial backlogging inventory model with time-varying demand and holding cost whereas deterioration rate is constant. Mandal (2010) has proposed an EOQ inventory model for Weibull distributed deteriorating items under ramp type demand and shortages. Begum et al. (2010) have presented an EOQ model for deteriorating items with quadratic demand and shortages whereas Weibull distribution deterioration rate considered. Skouri et al. (2009) have proposed some inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate. Nahmias (2009) has developed several inventory models with production and operations analysis.

Authors have focussed the effect of investing in reducing the rate of product decoration, but still it needs more attention. In practice the deterioration rate of products can be controlled and reduced through various effects such as procedural changes and specialized equipment acquisition etc.. The consideration of preservation technology is important due to rapid social changes. Many researchers have studied the effect of preservation technology investment. Among these, Bardhan et al. (2017) have analysed an optimal replenishment policy for a non-instantaneous deteriorating item with stock-dependent demand using preservation technology investment. Nita et at. (2015) have studied the impact of deploying suitable preservation technology for an inventory system with demand as a function of time and retail price. Soni and Patel (2014) have dealt with optimal policies for the vendor-buyer inventory system with variable lead time and service level constraint involving defective items.

Mahapatra et al. (2017) have considered an inventory model for uniform deteriorating items with time and reliability dependent demand under partial backorder. Pal and Adhikari (2017) have classified price-sensitive imperfect production inventory model with exponential partial backlogging. Sarkar et al. (2015) have focused an inventory model with variable deterioration for fixed lifetime products trade-credit policy. Chung et al. (2014) have considered an inventory model with non-instantaneous exponentially deteriorating items for an integrated three layer supply chain system under the assumption of two levels of trade credit. Sarkar and Sarkar (2013) have presented an improved inventory model with a stock dependent demand under partial backlogging whereas varying time deterioration. Hung (2011) proposed an inventory model with generalized type demand, decline, and backorder rates.

In this paper, we develop a deteriorating inventory model with time dependent demand rate. The deterioration rate assumed to be a constant and the unsatisfied demand is partially backlogged. The backlogging rate is variable and inversely proportional to the length of the waiting time for next replenishment. We aim to minimize the total integrated cost of inventory with optimal order quantity and optimal time of inventory exhausting. This paper describes as follows: Section 2 describes the notations and assumptions, Section 3 presents the mathematical formulation of the given problem, Section 4 provides a numerical example to illustrate the proposed inventory model and graphical analysis. Section 5 discusses the sensitivity analysis and finally, Section 6 presented the conclusions.

2. Notations & Assumptions

Throughout this paper, the following notations and assumptions used:

2.1. **Notation.**

(i) O_c Ordering cost per unit.

(ii) θ Deterioration cost.

(iii) M Maximum inventory level during a cycle length.

(iv) c_d Deterioration cost per cycle.

(v) s_c Shortage cost per cycle.

(vi) l_c Lot sale per cycle.

(vii) T Total cost.

(viii) $Q[t_1, T]$Order quantity during a cycle of length.

(ix) t_1 A time where shortage start.

(x) T Cycle time.

(xi) $(T - t_1)$ Length of waiting time.

(xii) $I(t)$ Inventory level at a time, $0 \leq t \leq T$.

(xiii) $I_1(t)$ Inventory level at time, $[0, t_1]$.

(xiv) $I_2(t)$ Inventory level at time, $[t_1, T]$.

2.2. **Assumption.**

(i) Demand rate $D(t) = \begin{cases} D_1 + \beta t, for\ 0 \leq t < t_1 \\ D_2, \quad for\ t_1 \leq t \leq T \end{cases}$ where $D_1, D_2 \& \beta$ are arbitrary constants and non negative.

(ii) Inventory system includes one item.

(iii) The planning horizon is infinite.

(iv) Replenishment rate is assumed to be infinite, which means the lead time to be zero.

(v) The rate of deterioration is constant, $0 < \theta < 1$

(vi) There is no replacement and repair of deteriorated units.

(vii) The shortage is allowed which partially backlogged with Back ordered rate; $B(t) = \frac{1}{1+\delta(T-t)}$.

(viii) The backlogging parameter δ is a positive constant. we assume $\delta < 1$ for the approximation by Taylor's series.

(ix) Holding cost is a linear function of time $h_c(t) = \alpha_1 + \alpha_2 t$, where $\alpha_1 \& \alpha_2$ are the holding cost scale parameters both are positive.

3. Formulation of the inventory model

The objective of the model determines the optimal order quantity to keep the total relevant cost as low as possible. The inventory replenished at the initial time $t = 0$, when the inventory level is at its maximum point M. Now, both the demand and deterioration of the item, the inventory level decrease during the period $(0, t_1)$, and finally becomes zero at time $t = t_1$.

Further, during the period t_1, T the shortage is considered, and the demand is assumed to be partially backlogged.

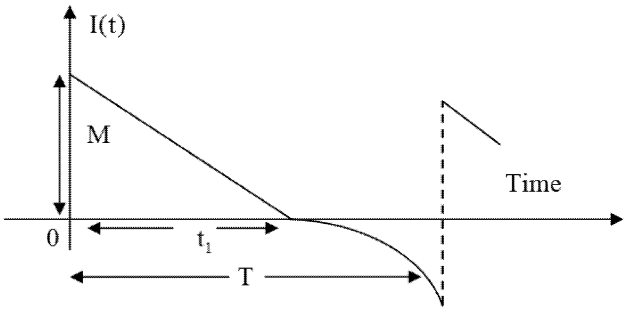

FIGURE 1. Graphical presentation of inventory system

The following differential equation during the periods $(0, t_1) and (t_1, T)$ are respectively given by:

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = (-D_1 + \alpha t); \;\; 0 < t < t_1 \tag{1}$$

$$\frac{dI_2(t)}{dt} = \frac{-D_2}{1 + \delta(T - t)}; \;\; t_1 \leq t \leq T \tag{2}$$

With boundary conditions:

$$I_1(t) = I_2(t) = 0, \; at \; t = T_1 \; and \;\; I_1(t) = M \; at \; t = 0 \tag{3}$$

The objective of inventory system to develop the order quantity and length of ordering cycle to keep the total costs as low as possible. That is, to determine the optimal value and minimized the total cost. Now we discussed two cases in this paper:

Case I: Without shortage in the time interval $(0, t_1)$

In this case, the inventory level decreases due to the demand as well as deterioration, and the inventory level is described by (1) using the given boundary conditions (3) then the solution of (1) is given by

$$I_1(t) = \left[-\frac{D_1}{\theta} - \frac{\beta t}{\theta} + \frac{\beta}{\theta^2} + \left(\frac{D_1}{\theta} + \frac{\beta t_1}{\theta} - \frac{\beta}{\theta^2} \right) e^{\theta(t_1 - t)} \right] \;\; ; 0 \leqslant t \leqslant t_1 \tag{4}$$

So, the maximum inventory level for each cycle will be given as

$$M = I_1(0) = \left[-\frac{D_1}{\theta} + \frac{\beta}{\theta^2} + \left(\frac{D_1}{\theta} + \frac{\beta t_1}{\theta} - \frac{\beta}{\theta^2} \right) e^{\theta t_1} \right] \;\; ; 0 \leqslant t \leqslant t_1 \tag{5}$$

Case II: When shortage is allowed in the time interval $[t_1, T]$.

In this case, the inventory level depends due on demand. But a fraction of demand is partial backlogged. The inventory level is described by (2) using the boundary conditions (3), then the solution of (2) is given as follow

$$I_1(t) = \frac{D_2}{\delta}\left[Log\{1+\delta(T-t)\} - Log\{1+\delta(T-t)\}\right] \;\; ; t_1 \leqslant t \leqslant T \tag{6}$$

Put in equation (6) $t = T$ we obtain the maximum amount of demand backlogged per cycle as follows

$$D_B = -I_2(T) = \frac{D_2}{\delta}\left[Log\{1+\delta(T-t_1)\}\right] \;\; ; t_1 \leqslant t \leqslant T \tag{7}$$

So, the order quantity per cycle is given by $Q = M + D_B$

$$\left[-\frac{D_1}{\theta} + \frac{\beta}{\theta^2} + \left(\frac{D_1}{\theta} + \frac{\beta t_1}{\theta} - \frac{\beta}{\theta^2}\right)e^{\theta t_1} + \frac{D_2}{\delta}\left[Log\{1+\delta(T-t_1)\}\right]\right] \tag{8}$$

For the assumption of $\delta < 1$ and $\theta < 1$, using Taylor's series expansion the following 2nd-degree approximations will be as follow

$$Q = \left[D_1 t_1 + \left(\frac{D_1\theta + \beta}{2}\right)t_1^2 + \frac{\beta\theta t_1^3}{2} + D_2\left((T-t_1) - \frac{\delta(T-t_1)^2}{2}\right)\right] \tag{9}$$

Cost Calculation of Inventory model

(1) Inventory holding cost per cycle TH_c

$$TH_c = \int_0^{t_1}(\alpha_1 + \alpha_2 t)I_1(t)dt = \left[\frac{\alpha_1 D_1 t_1^2}{2} + \frac{\alpha_1\beta t_1^3}{2} + \frac{\alpha_2\beta t_1^3}{6\theta}\right] \tag{10}$$

(2) Deterioration cost per cycle TD_c

$$TD_C = c_d\left[M - \int_0^{t_1} D(t)dt\right] = \frac{c_d\theta}{2}\left[D_1 t_1^2 + \beta t_1^3\right] \tag{11}$$

(3) Preservation cost PT_c

$$PT_C = \xi t_1 \tag{12}$$

(4) Shortage cost per cycle TS_c

$$TS_C = s_c\left[-\int_{t_1}^{T} I_2(t)dt\right] = s_c D_2\left[\frac{(T-t_1)}{\delta} - \frac{1}{\delta^2}Log\{1+\delta(T-t)\}\right] \tag{13}$$

(5) Lost sale cost per cycle $TL_s = l_c D_2\left[1 - \frac{1}{1+\delta(T-t)}\right]D_2 dt$

$$TL_s = l_c D_2\left[(T-t_1) - \frac{1}{\delta}Log\{1+\delta(T-t)\}\right] \tag{14}$$

So, the average standard total cost per unit time per cycle

$TC = \frac{1}{T}\left[TO_c + TH_p + TD_c + PT_c + TS_c + TL_s\right]$

$$TC = \frac{1}{T}\left[\begin{array}{l} O_c + \dfrac{\alpha_1 D_1 t_1^2}{2} + \dfrac{\alpha_1 \beta t_1^3}{2} + \dfrac{c_d\theta\left(D_1 t_1^2 + \beta t_1^3\right)}{2} \\ + \dfrac{\alpha_2 \beta t_1^3}{6\theta} + \xi t_1 + \dfrac{(s_c + \delta l_c)}{2} D_2 (T - t_1)^2 \end{array}\right] \tag{15}$$

Equation (15) is a single-objective non-linear optimization problem. The objective function TC is a function of ξ, t_1 and T. To achieve the optimally we will partial derivatives concerning ξ, t_1 and T are equated to zero. The resulting equations can be solved simultaneously to obtain the optimal values of ξ, t_1 and T.

The first objective ξ^* will we find from equation (16) using Mathematica software 9.0. Differentiating cost function TC with respect to ξ two times.

$$\frac{dTC}{d\xi} = \frac{1}{T}\left[t_1 + \frac{\alpha \mathrm{e}^{-\alpha\xi} t_1^3 \alpha_2 \beta}{6\theta_1} - \frac{c_d \theta_1 \alpha \mathrm{e}^{-\alpha\xi}\left(D_1 t_1^2 + \beta t_1^3\right)}{2}\right] \tag{16}$$

$$\frac{d^2TC}{d\xi^2} = \frac{1}{T}\left[\frac{\alpha^2 \mathrm{e}^{\alpha\xi} t_1^3 \alpha_2 \beta}{6\theta_1} + \frac{c_d \theta_1 \alpha^2 \mathrm{e}^{-\alpha\xi}\left(D_1 t_1^2 + \beta t_1^3\right)}{2}\right] > 0 \tag{17}$$

From equation (17) $\frac{d^2TC}{d\xi^2} > 0$.

Second objective t_1^* will be calculated Mathematica software 9.0 from equation (18). Now differentiating cost function TC with respect to t_1 two times.

(18)

$$\frac{dTC}{dt_1} = \left[\alpha_1 D_1 t_1 + \frac{3\alpha_1 \beta t_1^2}{2} + \frac{\alpha_2 \beta t_1^2}{2\theta} + \frac{c_d\theta\left(2D_1 t_1 + 3\beta t_1^2\right)}{2} + (s_c + \delta l_c) D_2 (T - t_1)\right]$$

(19)

$$\frac{d^2TC}{dt_1^2} = \left[\alpha_1 D_1 + 3\alpha_1 \beta t_1 + \frac{\alpha_2 \beta t_1}{\theta} + \frac{c_d\theta\left(2D_1 + 6\beta t_1\right)}{2} - (s_c + \delta l_c) D_2\right] > 0$$

Third optimal value T^* will be calculated Mathematica software 9.0 from equation (20). Now differentiating cost function TC with respect to T two times.

$$\frac{dTC}{dT} = \left[\begin{array}{l} \frac{(s_c + \delta l_c)\, D_2(T - t_1)}{T} + \frac{1}{T^2}\left\{ O_c + \frac{\alpha_1 D_1 t_1^2}{2} + \frac{\alpha_1 \beta t_1^3}{2} + \right. \\ \left. \frac{c_d \theta \left(D_1 t_1^2 + \beta t_1^3\right)}{2} + \frac{\alpha_2 \beta t_1^3}{6\theta} + \frac{(s_c + \delta l_c)\, D_2 (T - t_1)^2}{2} \right\} \end{array} \right] \tag{20}$$

$$\frac{d^2TC}{dT^2} = \left[\begin{array}{l} \frac{1}{3T^2\theta} \{-6\theta O_c + t_1 \left(3\theta D_2 \left(s_c + \delta l_c\right)\left(2T - t_1\right)\right. \\ \left. + t_1 \left(3\theta (D_2 + \beta t_1)(c_d\theta + \alpha_1) - \beta t_1 \alpha_2\right)\right)\} \end{array} \right] > 0 \tag{21}$$

Further, the nature of the cost function is highly non-linear. The convexity of the cost function TC is shown numerically and graphically in the next section. Hence, the optimal value of TC would be globally minimum.

4. Numerical Illustration

Consider the following data to illustrate the validate proposed inventory model. Let us assume demand rate of D_1=20, D_2=25 and β=40. Holding cost parameter $\alpha_1 = 0.5$ and $\alpha_2 = 0.011$. The ordering cost 60, shortage cost 12, lot sale cost is 15 and $\delta = 0.8$. Deterioration rate is 0.05 and deterioration cost is 4. Using Mathematica software to find the optimal value $\xi^*, t_1^*, T^*.Q^* \& TC^*$ shown in table-1.

ξ^*	t_1^*	T^*	Q^*	TC^*
5.823	1.026	1.143	45.03	82.45

Table 1: Optimal value

4.1. **Graphical Analysis (Convexity of Cost Function).** To show the convexity of the given cost function, we generate the graphs of total cost function TC based on the parameter values taken in above example. If we plot the total cost function with some values of $\xi, t1$ and T, then we get a strictly convex graph of the total cost function shown in Figure 2, 3, 4, respectively.

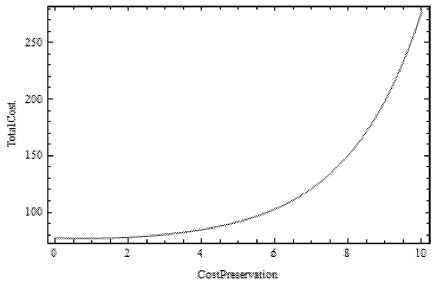

FIGURE 2. Convexity of TC and preservation cost.

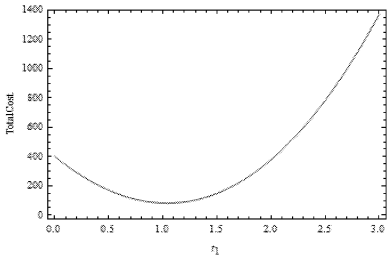

FIGURE 3

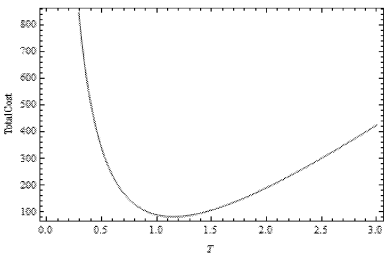

FIGURE 4

5. SENSITIVITY ANALYSIS

Sensitivity analysis is shifted by changing the value of each parameter by 20%, 10%, 10% and 20%, and keeping the remaining parameters unchanged. Then the current solutions become infeasible. Therefore, we analyse the effect of changes in various parameters: t_1^*, T^*, Q^*, C^*. The results displayed in the following Table-2.

Parameter	Percent	ξ^*	t_1^*	T^*	Q^*	TC^*
α_1	20	0.00	-1.17	0.40	0.00	-3.19
	10	0.00	-0.68	0.20	0.00	-1.62
	-10	0.00	0.49	-0.19	0.00	1.68
	-20	0.00	1.07	-0.39	0.00	3.41
α_2	20	2.87	-1.36	0.41	9.42	-3.31
	10	1.49	-0.78	0.21	4.71	-1.68
	-10	-1.77	0.58	-0.20	-4.71	1.73
	-20	-3.49	1.27	-0.41	-9.42	3.53
θ	20	-5.20	-0.49	0.18	9.22	-1.51
	10	-2.80	-0.29	0.10	4.62	-0.76
	-10	2.87	0.10	-0.09	-4.62	0.77
	-20	5.96	0.29	-0.18	-9.22	1.54
c_d	20	0.00	1.56	-1.87	1.35	-1.04
	10	0.00	0.78	-0.99	0.69	-0.52
	-10	0.00	-1.07	1.11	-0.67	0.52
	-20	0.00	-2.34	2.60	-1.35	1.05
D_1	20	-23.92	-0.20	0.00	0.00	-6.19
	10	-13.45	-0.10	0.00	0.00	-3.19
	-10	18.84	0.10	0.00	0.00	3.11
	-20	38.72	0.20	0.00	0.00	6.42
D_2	20	-14.05	-0.58	0.16	0.00	-1.30
	10	-7.64	-0.29	0.08	0.00	-0.65
	-10	9.39	0.19	-0.08	0.00	0.66
	-20	21.42	0.78	-0.16	0.00	1.34
s_c	20	0.00	0.39	-0.99	0.00	-0.52
	10	0.00	0.78	-0.55	0.00	-0.27
	-10	0.00	-0.58	0.50	0.00	0.27
	-20	0.00	-1.27	1.11	0.00	0.52

Table 2: Sensitivity analysis

The following observations made from sensitivity analysis:

Optimal value of ξ^* changes slightly with the change in the value of parameters α_2 , moderately with θ whereas highly with $D_1 \& D_2$.

Optimal value of t_1^* changes slightly with the change in the value of parameters $D_1 \& D_2$ and moderately with $\alpha_1, \alpha_2, c_d \& s_c$.

Optimal value of T^* changes slightly with the change in the value of parameters $\alpha_1, \alpha_2 \& D_2$, moderately with $c_d \& s_c$.

Optimal value of Q^* changes slightly to the change in the value of parameters c_d highly with $\alpha_2 \& \theta$.

Optimal value of TC^* changes highly to the change in the value of parameters $c_d, \theta, D_2 \& s_c$ moderately to the value of $\alpha_1, \alpha_2 \& d_1$ and highly with D_1.

6. Conclusions

The classical optimization technique is used to derive the optimal order quantity and optimal average total cost. For the practical use of this model, we consider a numerical example to validate the proposed model with sensitivity analysis. Sensitivity analysis and numerical analysis analysis is simplified by the help of MATHEMATICA Software-9.0. The proposed model can assist the manufacturer and retailer in accurately determining the optimal cycle, preservation cost, time order quantity, and total cost. Moreover, in the market, there are certain items where during the season period, the demand increases with time, and when the season is off, the demand sharply decreases and then becomes constant. Therefore, the proposed model can also use in inventory control of seasonal items. This paper can further extend by considering the time-dependent deterioration rate, quadratic demand ramp type demand function. Further, the fuzzy or stochastic uncertainty in inventory parameters may be considered.

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