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C-α-($\mathcal{I}$)-COMPACT SPACES

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Abstract. Viglino introduced the family of C-compact spaces. The concept of compactness modulo an ideal was introduced by Newcomb. Gupta and Noiri investigated the concept of C-compactness modulo an ideal. Agrawal generalized C-compact spaces by using α-open set and named them as C-α-compact spaces. In the present paper, we study the concept of C-α-compactness modulo an ideal which generalizes C-α-compactness and C-compactness modulo an ideal and term it as C-α-($\mathcal{I}$)-compact space. We also characterize some of its fundamental properties

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1. Introduction

An ideal $\mathcal{I}$ on a set X is a nonempty subset of $P(X)$, the power set of X, which is closed for subsets and finite unions. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau)$ (called the adherence of A modulo an ideal $\mathcal{I}$) or $A^*(\mathcal{I})$ or just A^* is the set $\{x \in X : A \cap U \notin \mathcal{I}$ for every neighborhood U of $x\}$. $A^*(\mathcal{I}, \tau)$ has been called the local function of A with respect to $\mathcal{I}$ in [5]. The operator cl*: $P(X) \to P(X)$ defined by $\mathrm{cl}^*(A)=A\cup A^*$ is a Kuratowski closure operator on X and hence generates a topology $\tau^*(\mathcal{I})$ or just τ^* on X finer than τ. In 2006, Gupta and Noiri [4] generalized the concepts of C-compactness of Viglino [11] and compactness modulo an ideal due to Newcomb [8] and Rancin [10] and defined C($\mathcal{I}$)-compact spaces. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is said to be C($\mathcal{I}$)-compact if for every closed set A and every τ-open cover $\mathcal{U}$ of A, there is a finite subcollection $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$ such that $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \in \mathcal{I}$. In [1], Agrawal described C-α-compact spaces. A topological space X is said to be C-α-compact if for each closed subset A of X and for each α-open cover $\mathcal{U}$ of A, there is a finite sub collection $\{U_1, U_2, ..., U_n\}$ of $\mathcal{U}$ such that $A \subset \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_i)$.
In the present paper we introduce and explore C-α-($\mathcal{I}$)-compact spaces, by generalizing simultaneously the concepts of C($\mathcal{I}$)-compact spaces and C-α-compact spaces.

2. Preliminaries

In this section, we recall some definitions and results which we have used in this paper.

Definition 2.1. A space (X, τ) is said to be C-compact [11] if for each closed subset A of X and for each τ-open covering $\mathcal{U}$ of A, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$ such that $A \subset \bigcup_{i=1}^{n} \mathrm{cl}(U_i)$.

Definition 2.2. A subset A of a topological space X is called α-open set [9] if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. The complement of an α-open set is called an α-closed set.

Definition 2.3. A set U in a topological space X is called an α-neighborhood [1] of a point x if U contains an α-open set V containing x.

Definition 2.4. X is said to be an α-Hausdorff space [1] if for any pair of distinct points x and y in X, there exists α-open sets U and V in X such that $x \in U, y \in V$ and $U \cap V = \phi$.

Definition 2.5. The intersection of all α-closed sets containing a subset $A \subset X$ is called the α-closure [2] of A and is denoted by $\mathrm{cl}_\alpha(A)$. A subset$A$ is called α-regular open [3] if $\mathrm{int}_\alpha(\mathrm{cl}_\alpha(A)) = A$.

Definition 2.6. A topological space X is said to be α-regular [3] if for every closed set F and a point $x \notin F$, there exists disjoint α-open sets A and B such that $x \in A$ and $F \subset B$.

Definition 2.7. A map $f : X \to Y$ is said to be α-continuous [7] if the inverse image of every open subset of Y is α-open set in X.
Remark 1. Continuity implies α-continuity but not conversely.
Remark 2. Every open mapping (closed mapping) is α-open mapping (α-closed mapping) but the converse is not true.

Definition 2.8. A map $f : X \to Y$ is said to be α-irresolute [6] if the inverse image of every α-open subset of Y is α-open set in X.

Definition 2.9. Let X be a topological space and A be a subset of X. An element $x \in X$ is called α-adherent point [1] of A if every α-open set G containing x contains a point of A, that is $G \cap A \neq \phi$. Also a point x is said to be an α-adherent point of filter base $\mathcal{B}$ if for each α-neighborhood U of x and for each $B \in \mathcal{B}$, $U \cap B \neq \phi$. Clearly $\cap\{\mathrm{cl}_\alpha(B) : B \in \mathcal{B}\}$ is the α-adherent set of filter base $\mathcal{B}$.

Lemma 2.10. For each α-open set U of a topological space X, we have
$\mathrm{cl}_\alpha(\mathrm{int}_\alpha(\mathrm{cl}_\alpha(U))) = \mathrm{cl}_\alpha(U)$.
Proof: This is easy to be established.

3. C-$\boldsymbol{\alpha}$-($\mathcal{I}$)-compact spaces

Definition 3.1. A topological space X is said to be C-α-($\mathcal{I}$)-compact if for each closed subset A of X and for each α-open cover $\mathcal{U}$ of A, there exists a finite sub collection $\{U_1, U_2, ..., U_n\}$ of $\mathcal{U}$ such that $A - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_i) \in \mathcal{I}$.

Theorem 3.2. A topological space X is C-α-($\mathcal{I}$)-compact if and only if for each closed subset A of X and for each α-regular open cover $\{U_\lambda : \lambda \in \Lambda\}$ of A, there exists a finite subcollection $\{U_{\lambda_i} : i = 1, 2, ..., n\}$ such that $A - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_{\lambda_i}) \in \mathcal{I}$.
Proof : First, we assume that X is C-α-($\mathcal{I}$)-compact space. Let A be any closed subset of X and $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ any α-regular open cover of A. Since every α-regular open set is α-open. Therefore $\mathcal{U}$ is an α- open cover of A. By hypothesis, $A - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_{\lambda_i}) \in \mathcal{I}$.

Conversely, let $\{U_\lambda : \lambda \in \Lambda\}$ be an α-open cover of A. Then $\mathcal{U} = \{\text{int}_\alpha(\text{cl}_\alpha(U_\lambda))\}$ becomes an α-regular open cover of A. By assumption, there exists a finite subcollection $\{\text{int}_\alpha(\text{cl}_\alpha(U_{\lambda_i})) : i = 1, 2, ..., n\}$ of $\mathcal{U}$ such that $A - \bigcup_{i=1}^n \text{cl}_\alpha(\text{int}_\alpha(\text{cl}_\alpha(U_{\lambda_i}))) \in \mathcal{I}$. Since U_λ is α-open, therefore for each i, we have $\text{cl}_\alpha(\text{int}_\alpha(\text{cl}_\alpha(U_{\lambda_i}))) = \text{cl}_\alpha(U_{\lambda_i})$. Hence $A - \bigcup_{i=1}^n \text{cl}_\alpha(U_{\lambda_i}) \in \mathcal{I}$ implying that X is C-α-($\mathcal{I}$)-compact.

Theorem 3.3. Let $f : (X, \tau, \mathcal{I}) \to (Y, \varsigma, \mathcal{J})$ be an α-irresolute, continuous surjection, $(X, \tau, \mathcal{I})$ C-α-($\mathcal{I}$)-compact, and $f(\mathcal{I}) \subseteq \mathcal{J}$. Then $(Y, \varsigma, \mathcal{J})$ is C-α-($\mathcal{J}$)-compact.

Proof: Let A be a closed subset of (Y, ς) and $\mathcal{V}$ any α-open cover of A in Y. By continuity of f, $f^{-1}(A)$ is a closed subset of X and by α-irresoluteness of f, $\mathcal{W} = \{ f^{-1}(V) : V \in \mathcal{V}\}$ is a cover of $f^{-1}(A)$ by α-open sets in X. Hence, by the C-α-($\mathcal{I}$)-compactness of X, there exists a finite subcollection $\{f^{-1}(V_i) : i = 1, 2, ..., n\}$ of $\mathcal{W}$ such that $f^{-1}(A) - \bigcup_{i=1}^n \text{cl}_\alpha(f^{-1}(V_i)) \in \mathcal{I}$. By continuity of f, $\text{cl}_\alpha(f^{-1}(B)) \subset f^{-1}(\text{cl}_\alpha(B))$ for every subset B of Y. Hence we have $f^{-1}(A) - \bigcup_{i=1}^n f^{-1}(\text{cl}_\alpha(V_i)) = f^{-1}(A - \bigcup_{i=1}^n \text{cl}_\alpha(V_i)) \in \mathcal{I}$. Since f is surjective, $A - \bigcup_{i=1}^n \text{cl}_\alpha(V_i) \in f(\mathcal{I}) \subseteq \mathcal{J}$. Hence, Y is C-α-($\mathcal{J}$)-compact.

Theorem 3.4. For a topological space X, the following are equivalent:

(a) X is C-α-($\mathcal{I}$)-compact;

(b) For each closed subset A of X and for each family $\mathcal{F}$ of α-closed subset of X with $\cap\{F \cap A : F \in \mathcal{F}\} = \phi$ there exists a finite subfamily $\{F_1, F_2, F_3, ..., F_n\}$ such that $\bigcap_{i=1}^n(\text{int}_\alpha(F_i)) \cap A \in \mathcal{I}$;

(c) If A is a closed subset of X and $\mathcal{B}$ is an α-open filter base on X such that$\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$, then $\cap\{\text{cl}_\alpha(B) : B \in \mathcal{B}\} \cap A \neq \phi$.

Proof: (a)$\Rightarrow$(b) Let A be a closed subset of a C-α-($\mathcal{I}$)-compact space X and $\mathcal{F}$ be a family of α-closed subset of X with $\cap\{F \cap A : F \in \mathcal{F}\} = \phi$. This implies $\cap\{F : F \in \mathcal{F}\} \subset X - A \Rightarrow A \subset X - \cap\{F : F \in \mathcal{F}\} = \cup\{X - F : F \in \mathcal{F}\}$. Hence $\{X - F : F \in \mathcal{F}\}$ is an α-open cover of A and so by C-α-($\mathcal{I}$)-compactness of X, there exists a finite subfamily $\{X - F_i : i = 1, 2, ..., n\}$ such that $A - \bigcup_{i=1}^n \text{cl}_\alpha(X - F_i) \in \mathcal{I}$. This set in $\mathcal{I}$ is easily seen to be $\bigcap_{i=1}^n(\text{int}_\alpha(F_i)) \cap A$.

(b)$\Rightarrow$(c) Suppose A is a closed set and $\mathcal{B}$ is an α-open filter base on X such that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$. Suppose if possible, $\cap\{\text{cl}_\alpha(B) : B \in \mathcal{B}\} \cap A = \phi$. Then $\{\text{cl}_\alpha(B) : B \in \mathcal{B}\}$ is a family of α-closed sets such that $\cap\{(\text{cl}_\alpha B) : B \in \mathcal{B}\} \cap A = \phi \Rightarrow \cap\{(\text{cl}_\alpha B) \cap A : B \in \mathcal{B}\} = \phi$, so by (b) there is a finite subfamily, say $\{F_i = \text{cl}_\alpha B_i : i = 1, 2, ..., n\}$of $\mathcal{F}$ with $\bigcap_{i=1}^n(\text{int}_\alpha(F_i)) \cap A \in \mathcal{I}$. However, this set is $\bigcap_{i=1}^n(\text{int}_\alpha(\text{cl}_\alpha(B_i))) \cap A$ and $(\bigcap_{i=1}^n B_i) \cap A$ is a subset of it. Therefore, $(\bigcap_{i=1}^n B_i) \cap A \in \mathcal{I}$. Since $\mathcal{B}$ is a filter base, we have a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^n B_i$. But then $B \cap A \in \mathcal{I}$, which contradicts the fact that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$.

(c)$\Rightarrow$(a) Assume that X is not C-α-($\mathcal{I}$)-compact. Then there exist a closed subset A of X and an α-open cover $\mathcal{U}$ of A such that for any finite subcollection $\{U_1, U_2, ..., U_n\}$ of $\mathcal{U}$, $A - \bigcup_{i=1}^n \text{cl}_\alpha(U_i) \notin \mathcal{I}$. We may assume that $\mathcal{U}$ is closed under finite unions. Then the family $\mathcal{B} = \{X - \text{cl}_\alpha U : U \in \mathcal{U}\}$ is an α-open filter base on X such that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$. So by the hypothesis, $\cap\{\text{cl}_\alpha(X - \text{cl}_\alpha(U)) : U \in \mathcal{U}\} \cap A \neq \phi$. Let x be a point in the intersection. Then $x \in A$ and $x \in \text{cl}_\alpha(X - \text{cl}_\alpha(U)) = X - \text{int}_\alpha(\text{cl}_\alpha(U)) \subset X - U$ for each U in $\mathcal{U}$. This implies $x \notin U$ for any $U \in \mathcal{U}$. But this contradicts the fact that $\mathcal{U}$ is a cover of A. Hence X is C-α-($\mathcal{I}$)-compact.

Definition 3.5. A filter base $\mathcal{B}$ is said to be α-($\mathcal{I}$) *adherent convergent* if for every neighborhood G of the α-adherent set of $\mathcal{B}$, there exists an element $B \in \mathcal{B}$ such that $(X - G) \cap B \in \mathcal{I}$.

Theorem 3.6. A topological space X is C-α-($\mathcal{I}$)-compact if and only if every α-open filter base on $P(X)-\mathcal{I}$ is α-($\mathcal{I}$) adherent convergent.

Proof: Let X be C-α-($\mathcal{I}$)-compact and $\mathcal{B}$ an α-open filter base on $P(X)-\mathcal{I}$ with A as its α-adherent set. Let G be an open neighborhood of A. Since A is the α-adherent set of $\mathcal{B}$, we have $A = \cap\{\mathrm{cl}_\alpha B : B \in \mathcal{B}\}$. Since G is an open neighborhood of A, we have $A \subset G$ and $X - G$ is closed subset of X. Now $\{X - \mathrm{cl}_\alpha(B) : B \in \mathcal{B}\}$ is an α-open cover of $X - G$ and so by the hypothesis, it admits a finite subfamily$\{X - \mathrm{cl}_\alpha(B_i) : i = 1, 2, ..., n\}$ such that $(X - G) - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(X - \mathrm{cl}_\alpha(B_i)) \in \mathcal{I} \Rightarrow (X - G) \cap (\bigcap_{i=1}^{n} \mathrm{int}_\alpha(\mathrm{cl}_\alpha(B_i))) \in \mathcal{I}$. However, $B_i \subset \mathrm{int}_\alpha(\mathrm{cl}_\alpha(B_i))$ implies $(X-G)\cap(\bigcap_{i=1}^{n} B_i) \in \mathcal{I}$. Since $\mathcal{B}$ is a filter base and $B_i \in \mathcal{B}$, there is a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^{n} B_i$. But then $(X - G) \cap B \in \mathcal{I}$ as required.

Conversely, let X be not C-α-($\mathcal{I}$)-compact, A any closed set and $\mathcal{U}$ an α-open cover of A such that for no finite subfamily $\{U_1, U_2, U_3, ..., U_n\}$of $\mathcal{U}$, $A - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_i) \in \mathcal{I}$. Without loss of generality, we may assume that $\mathcal{U}$ is closed for finite unions. Therefore, $\mathcal{B}$=$\{X - \mathrm{cl}_\alpha(U) : U \in \mathcal{U}\}$ becomes an α-open filter base on $P(X)-\mathcal{I}$. If x is an α-adherent point of $\mathcal{B}$, that is, if $x \in \cap\{\mathrm{cl}_\alpha(X - \mathrm{cl}_\alpha(U)) : U \in \mathcal{U}\} = X - \cup\{\mathrm{int}_\alpha(\mathrm{cl}_\alpha(U)) : U \in \mathcal{U}\}$, then $x \notin A$, because $\mathcal{U}$ is an open cover of A and for $U \in \mathcal{U}$, $U \subset \mathrm{int}_\alpha(\mathrm{cl}_\alpha(U))$. Therefore the α-adherent set of $\mathcal{B}$ is contained in $X-A$, which is an open set. By the hypothesis, there exists an element $B \in \mathcal{B}$ such that $(X - (X - A)) \cap B \in \mathcal{I}$, that is, $A \cap B \in \mathcal{I}$, that is, $A \cap (X - \mathrm{cl}_\alpha(U)) \in \mathcal{I}$, that is $A - \mathrm{cl}_\alpha(U) \in \mathcal{I}$, for some $U \in \mathcal{U}$. This contradicts our assumption.

Definition 3.7. A function $f : (X, \tau)\to(Y, \varsigma)$ is said to be *θ-α-continuous* at a point $x \in X$ if for every α-open set V of Y containing $f(x)$, there exists an α-open set U of X containing x such that $f(\mathrm{cl}_\alpha(U)) \subseteq \mathrm{cl}_\alpha(V)$.

Theorem 3.8. Let $f : (X, \tau, \mathcal{I})\to(Y, \varsigma, \mathcal{J})$ be a θ-α-continuous function, $(X, \tau, \mathcal{I})$ C-α-($\mathcal{I}$)-compact, (Y, ς) α-Hausdorff, and $f(\mathcal{I})\subseteq \mathcal{J}$. Then $f(A)$ is $\varsigma^*(\mathcal{J})$-closed for each closed set A of X.

Proof: Let A be any closed set in X and $a \notin f(A)$. For each $x \in A$, there exists a ς-α-open set V_y containing $y = f(x)$ such that $a \notin \mathrm{cl}_\alpha(V_y)$. Now because f is θ-α-continuous, there exists an α-open set U_x containing x such that $f(\mathrm{cl}_\alpha(U_x)) \subseteq \mathrm{cl}_\alpha(V_y)$. Now the family $\{U_x : x \in A\}$ is an α-open cover of A. Since X is C-α-($\mathcal{I}$)-compact, therefore there exists a finite subfamily $\{U_{x_i} : i = 1, 2, ..., n\}$ such that $A - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_{x_i}) \in \mathcal{I}$. But then $f(A - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_{\lambda_i})) \in f(\mathcal{I}) \subseteq \mathcal{J}$, that is, $f(A) - f(\bigcup_{i=1}^{n} \mathrm{cl}_\alpha(U_{\lambda_i})) \in f(\mathcal{I}) \subseteq \mathcal{J}$ because $f(\mathcal{I})$ is also an ideal. Hence,$f(A) - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(V_y) \in f(\mathcal{I}) \subseteq \mathcal{J}$. Now $a \notin \mathrm{cl}_\alpha(V_y)$ for any i implies that $a \in Y - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(V_{y_i})$ which is open in (Y, ς). That is $Y - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(V_{y_i})$ is an α-neighborhood of a in (Y, ς). $(Y - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(V_{y_i})) \cap f(A) = f(A) - \bigcup_{i=1}^{n} \mathrm{cl}_\alpha(V_{y_i}) \in f(\mathcal{I}) \subseteq \mathcal{J}$. Hence, $a \notin (f(A))^*$ $(\varsigma, \mathcal{J})$. Thus $(f(A))^*(\varsigma, \mathcal{J}) \subset f(A)$. This implies $f(A)$ is $\varsigma^*(\mathcal{J})$-closed.

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