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NON-POLYNOMIAL QUARTIC SPLINE METHOD FOR SOLVING TWELFTH ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, a non-polynomial quartic spline method is presented to obtain the approximate solution of twelfth-order boundary value problems with two point boundary conditions. For the employment of the method, the given problem is decomposed into a system of sixth order boundary value problems. Convergence analysis of the method for second and fourth order has been discussed. Numerical examples are given to demonstrate the accuracy and efficiency of the developed method. Also, the results obtained by this method have been compared with the other existing methods.

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1. Introduction

We consider the following twelfth-order boundary value problem of the form

$$y^{(12)}(x) + f(x)y(x) = g(x), \ a \leq x \leq b \tag{1}$$

subject to the boundary conditions

$$y^{(2k)}(a) = A_{2k}, y^{(2k)}(b) = B_{2k}, 0 \leq k \leq 5 \tag{2}$$

where A_{2k} and $B_{2k}(k = 0, 1, ..., 5)$ are finite real constants. The functions $f(x)$ and $g(x)$ are continuous on the interval $[a, b]$.

The boundary value problems (BVPs) of higher order have been arised due to their mathematical importance and applications in various problems of applied sciences. Chandrasekhar [12] showed that this type of boundary-value problem arises in a way when a uniform magnetic field is applied across the fluid in the same direction as gravity, instability will occur. When instability sets in as over stability, it is modeled by twelfth-order boundary-value problems. Such problems arise mostly in geophysics when studying core fluid adjacent to the core-mantle boundary. Conditions for existence and uniqueness of solutions of the twelfth-order boundary-value problems was discussed in Agarwal[10]. There are many authors who developed the methods to determine the approximate solutions of twelfth-order boundary-value problems. For example, Boutayeb and Twizell [1], Siddique and Akram

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[14], Siddique and Twizell [13], Twizell et al.[5] and Wazwaz[3] were developed the methods to solve twelfth-order BVPs. In Kumar and Srivastava[8], cubic, quintic and sextic splines were presented to solve differential equations. However, most of these methods were based on higher degree splines to solve twelfth-order boundary value problems. In this paper, we use a non-polynomial quartic spline to solve twelfth order BVPs. The presented method based on the decomposition of twelfth order into a system of sixth order boundary value problem as follows:

$$y^{(6)}(x) = u(x) \tag{3}$$

$$u^{(6)}(x) + f(x)y(x) = g(x), \; a \leq x \leq b \tag{4}$$

subject to modified boundary conditions;

$$\begin{aligned} y^{(2k)}(a) &= A_{2k}, y^{(2k)}(b) = B_{2k}, \\ u^{(2k)}(a) &= C_{2k}, u^{(2k)}(b) = D_{2k}, \; 0 \leq k \leq 2 \end{aligned} \tag{5}$$

There are various methods in literature to solve the sixth order boundary value problems. For example, a non-polynomial spline approach by Islam et al.[15], cubic spline finite difference scheme by Jha et al.[9], spline collocation method by Rashidinia et al.[7], parametric quintic spline by Khan and Sultana[2] and variational iteration by Ji-Huan[6] were developed to determine approximate solution of sixth order BVPs.

After implementation of the problem over the method, we get a system of septa-block-diagonal matrix which is solved by using LU decomposition method. The paper describing a non-polynomial quartic spline method is organized into six sections. Section 2 gives a brief derivation of the scheme and the truncation error. In section 3, boundary conditions have been developed for second and fourth order method. Application of the method for solving twelfth order BVPs is discussed in section 4. Convergence analysis of the method is discussed in section 5 and in section 6, numerical examples and comparison with the existing methods are given.

2. Derivation of the scheme

Let $a = x_0 < x_1 < x_2 < \;\; < x_n = b$, in order to develop the numerical method for approximating the solution of given problem, we first divide the interval [a,b] into n equal parts by introducing

$$x_i = a + ih, \; i = 0, 1, ..., n \; and \; h = (b - a)/n$$

Let

$$P_i(x) = a_i \sin k(x - x_i) + b_i e^{k(x-x_i)} + c_i(x - x_i)^2 + d_i(x - x_i) + e_i \tag{6}$$

be a non-polynomial quartic spline P_i is defined on [a, b] of class $C^3[a, b]$ which interpolates at the mesh points x_i depends on a parameter k and reduces to an ordinary quartic spline in [a,b] as $k \longrightarrow 0$ and $k > 0$. To determine the coefficients a_i, b_i, c_i, d_i and e_i, we define the following interpolatory conditions as

$$P_i(x_i) = y_i,\ P_i(x_{i+1}) = y_{i+1} \tag{7}$$

$$P_i^{(2)}(x_i) = D_i,\ P_i^{(2)}(x_{i+1}) = D_{i+1} \tag{8}$$

$$P_i^{(4)}(x_i) = \frac{1}{2}(S_i + S_{i+1}),\ i = 0, 1, ..., n \tag{9}$$

By using (7), (8) and (9) we calculated the coefficients as

$$\begin{aligned}
a_i &= \frac{h^2(D_i - D_{i+1})}{\theta^2 \sin\theta} + \frac{h^4(S_i + S_{i+1})(e^\theta - 1)}{2\theta^4 \sin\theta}\\
b_i &= \frac{h^4(S_i + S_{i+1})}{2\theta^4}\\
c_i &= \frac{h^2 D_i}{2} - \frac{(S_i + S_{i+1})}{4\theta^2}\\
d_i &= \frac{y_{i+1} - y_i}{h} - \frac{h(D_i - D_{i+1}) - D_i\theta^2}{\theta^2}\\
&\quad + \frac{h^3\big(S_i\big(\theta^2 - 4(e^\theta - 1)\big) + S_{i+1}\big(\theta^2 - 4(e^\theta - 1)\big)\big)}{4\theta^4}\\
e_i &= y_i - b_i
\end{aligned}$$

where, $\theta = kh$

Using the continuity of first and third derivatives, $P_{i-1}^m(x_i) = P_i^m(x_i), m = 1, 3$ the following consistency relations are derived as

(10)

$$A_{11}D_{i-1} + A_{12}D_i + A_{13}D_{i+1} = A_{14}(y_{i+1} - 2y_i + y_{i-1}) + A_{15}S_{i-1} + A_{16}S_i + A_{17}S_{i+1}$$

and

$$B_{11}D_{i-1} + B_{12}D_i + B_{13}D_{i+1} = B_{14}S_{i-1} + B_{15}S_i + B_{16}S_{i+1} \tag{11}$$

where,

$$\begin{aligned}
A_{11} &= 2\theta\cos\theta + (\theta^2 - 2)\sin\theta\\
A_{12} &= -2\theta\cos\theta - 2\theta + (4 + \theta^2)\sin\theta\\
A_{13} &= 2\theta - 2\sin\theta\\
A_{14} &= \frac{2\theta^2}{h^2\sin\theta}\\
A_{15} &= \frac{h^2\big((\theta^2 + 4e^\theta - 2\theta e^\theta - 4)\sin\theta + 2\theta(1 - e^\theta)\cos\theta\big)}{2\theta^2}\\
A_{16} &= \frac{h^2\big(2\theta(\theta - e^\theta + 1)\sin\theta + 2\theta(1 - e^\theta)\cos\theta + 2\theta e^\theta - 2\theta\big)}{2\theta^2}\\
A_{17} &= \frac{h^2\big(2\theta e^\theta - 2\theta + (4 + 2\theta + \theta^2 - 4e^\theta)\sin\theta\big)}{2\theta^2}
\end{aligned}$$

$$\begin{aligned}
B_{11} &= -\cos\theta \\
B_{12} &= 1+\cos\theta \\
B_{13} &= -1 \\
B_{14} &= \frac{h^2\big(-e^\theta\sin\theta+(e^\theta-1)\cos\theta\big)}{2\theta^2} \\
B_{15} &= \frac{h^2(\sin\theta-\cos\theta+1)(1-e^\theta)}{2\theta^2} \\
B_{16} &= \frac{h^2\big(\sin\theta-(e^\theta-1)\big)}{2\theta^2}
\end{aligned}$$

Using (10) and (11), we obtain the following relation in terms of y_i and D_i

(12)

$$\gamma y_{i-2}+\alpha y_{i-1}+\beta y_i+\alpha y_{i+1}+y_{i+2}=h^4(\alpha_2 S_{i-2}+\beta_2 S_{i-1}+\gamma_1 S_i+\beta_1 S_{i+1}+\alpha_1 S_{i+2}),$$

where,

$$\begin{aligned}
\alpha &= -3-\cos\theta \\
\beta &= 4+2\cos\theta \\
\gamma &= \cos\theta \\
\alpha_1 &= \frac{2\theta+1+\frac{1}{2}\theta^2-\sin\theta-e^\theta}{2\theta^4} \\
\beta_1 &= \frac{e^\theta(\sin\theta+\cos\theta)-\cos\theta-2\theta e^\theta\cos\theta}{2\theta^4}+\frac{e^\theta(\cos\theta+\sin\theta)}{4\theta^2} \\
\gamma_1 &= \frac{\theta^2e^\theta+1+2\theta+(e^\theta-\theta^2-1)\sin\theta+e^\theta\cos\theta-\theta^2-e^\theta-2\theta e^\theta\cos\theta}{2\theta^4}+ \\
&\quad \frac{e^\theta(\sin\theta-\cos\theta)}{4\theta^2} \\
\alpha_2 &= \frac{-1+(1-2\theta)e^\theta+(1+e^\theta+\theta^2e^\theta)\sin\theta+(-1-2\theta+e^\theta+2\theta e^\theta)\cos\theta}{2\theta^4}+ \\
&\quad \frac{2\theta^2-\sin\theta+(1-2e^\theta)\cos\theta+e^\theta}{4\theta^2} \\
\beta_2 &= \frac{(1+2\theta-e^\theta)\cos\theta-2e^\theta+2\theta e^\theta+1-e^\theta\sin\theta-\sin\theta+e^\theta-\theta^2\sin\theta-\theta^2+\theta^2e^\theta}{2\theta^4} \\
&\quad +\frac{\cos\theta+\sin\theta-e^\theta}{4\theta^2}
\end{aligned}$$

Remark: Our method reduces to Al-Said and Noor[4] based on quartic spline when

$$(\alpha_1,\beta_1,\gamma_1,\beta_2,\alpha_2) = \frac{1}{48}(1,12,22,12,1)$$

For making the system consistent with the given boundary conditions, we use finite difference formula of $O(h^4)$

(13)

$$-S_{i-2}+16S_{i-1}-30S_i+16S_{i+1}-S_{i+2}=12h^2y_i^{(6)}+O(h^6),\ i=2,3,...,n-2$$

Using (12)and(13), we obtained the following relation

(14)

$$C_{11}S_{i-2}+C_{12}S_{i-1}+C_{13}S_i+C_{14}S_{i+1}=\frac{\gamma y_{i-2}+\alpha y_{i-1}+\beta y_i+\alpha y_{i+1}+y_{i+2}}{h^4}, i=2,3,...,n-2$$

where,

$$\begin{aligned}C_{11} &= \alpha_2-\alpha_1\\ C_{12} &= \beta_2+16\alpha_1\\ C_{13} &= \gamma_1-30\alpha_1\\ C_{14} &= \beta_1+16\alpha_1\end{aligned}$$

From Lucas[16] for quartic spline, it can be written as

$$(15)\qquad S_{i-1}-2S_i+S_{i+1} = h^2y_i^{(6)}+O(h^8),\ i=1,2,...,n-1$$

By using (14) and (15), we obtained the following relation as

$$\begin{aligned}D_{11}y_{i-2}^{(6)}+D_{12}y_{i-1}^{(6)}+D_{13}y_i^{(6)}+D_{14}y_{i+1}^{(6)} &= E_{11}y_{i-3}+E_{12}y_{i-2}+E_{13}y_{i-1}+E_{14}y_i+\\ (16)\qquad & E_{15}y_{i+1}+E_{16}y_{i+2}+y_{i+3},\ i=3,4,...,n-3\end{aligned}$$

where,

$$\begin{aligned}D_{11} &= h^6(\alpha_2-\alpha_1)\\ D_{12} &= h^6(4\alpha_1+\beta_2)\\ D_{13} &= h^6(-6\alpha_1+\gamma_1)\\ D_{14} &= h^6(4\alpha_1+\beta_1)\\ E_{11} &= \gamma; E_{12}=\alpha-2\\ E_{13} &= \gamma-2\alpha+\beta; E_{14}=2\alpha-2\beta\\ E_{15} &= -2\alpha+\beta+1; E_{16}=\alpha-2\end{aligned}$$

Truncation error. The local truncation error $t_i, i=3,4,...,n-3$ associated with presented method is obtained by expanding (16) using Taylor series

$$\begin{aligned}t_i =\ & (-2+2\gamma)y_i+h^2(-3+2\alpha+\beta+5\gamma)y_i^{(2)}+\frac{h^4(18+28\alpha+2\beta+82\gamma)}{4!}y_i^{(4)}+\\ & \frac{h^5(81-81\gamma)}{40}y_i^{(5)}+h^6\left(\frac{474+124\alpha+2\beta+730\gamma}{6!}-(\beta_1+\beta_2+\alpha_1+\alpha_2+\gamma_1)\right)y_i^{(6)}+\\ & h^7(-2\alpha_1+2\alpha_2+\beta_2-\beta_1)y_i^{(7)}+h^8\left(\frac{6238+508\alpha+2\beta+\gamma}{8!}-\right.\\ & \left.\frac{4\alpha_1+4\alpha_2+\beta_1+\beta_2}{2!}\right)y_i^{(8)}+h^9(-8\alpha_1+16\alpha_2+\beta_2-\beta_1)y_i^{(9)}+\\ & h^{10}\left(\frac{59454+2044\alpha+2\beta+59050\gamma}{10!}-\frac{4\alpha_1+4\alpha_2+\beta_1+\beta_2}{4!}\right)y_i^{(10)}+\\ & h^{11}(-32\alpha_1+32\alpha_2+\beta_2-\beta_1)y_i^{(11)}+h^{12}\left(\frac{515058+8188\alpha+2\beta+531442\gamma}{12!}-\right.\\ & \left.\frac{56\alpha_1+64\alpha_2+\beta_1+\beta_2}{6!}\right)y_i^{(12)}+O(h^{13}), i=3,4...,n-3\end{aligned}$$

(17)

For different values of parameters, we get method of different orders.

3. Development of boundary conditions

The recurrence relation (16) gives (n-5) linear equations in (n-1) unknowns $y_i, i = 1, 2, ..., n-1$. We need four more equations, two at each end of the range of integration. These four equations are defined as

$$\begin{aligned}
\sum_{k=0}^{4}\delta_k y_k + a_1 h^2 y_0'' + b_1 h^4 y_0^{(4)} + h^6 \sum_{k=0}^{5} w_k y_k^{(6)} + t_1 &= 0;\ i=1\\
\sum_{k=0}^{5}\eta_k y_k + a_2 h^2 y_0'' + b_2 h^4 y_0^{(4)} + h^6 \sum_{k=1}^{6} \lambda_k y_k^{(6)} + t_2 &= 0;\ i=2\\
\sum_{k=n-5}^{n}\eta_k y_k + a_2 h^2 y_n'' + b_2 h^4 y_n^{(4)} + h^6 \sum_{k=n-6}^{n-1} \lambda_k y_k^{(6)} + t_{n-2} &= 0;\ i=n-2\\
\sum_{k=n-4}^{n}\delta_k y_k + a_1 h^2 y_n'' + b_1 h^4 y_n^{(4)} + h^6 \sum_{k=n-5}^{n} w_k y_k^{(6)} + t_{n-1} &= 0;\ i=n-1
\end{aligned}$$

where $\delta_k's, \eta_k's, a_1, b_1, a_2, b_2, \lambda_k's$ and $w_k's$ are arbitrary parameters to be determined at $i = 1, 2, n-2, n-1$ by using Taylor series expansion. The values of these parameters for second order method is

$$\begin{aligned}
(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, a_1, b_1) &= (5, -14, 14, -6, 1, -2, \frac{5}{6})\\
(w_0, w_1, w_2, w_3, w_4, w_5) &= (\frac{29}{180}, -1, 0, 0, 0, 0)\\
(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, a_2, b_2) &= (-4, 14, -20, 15, -6, 1, 1, \frac{1}{12})\\
(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) &= (\frac{1}{180}, -\frac{361}{360}, 0, 0, 0, 0)\\
(\eta_{n-5}, \eta_{n-4}, \eta_{n-3}, \eta_{n-2}, \eta_{n-1}, \eta_n, a_2, b_2) &= (1, -6, 15, -20, 14, -4, 1, \frac{1}{12})\\
(\lambda_{n-6}, \lambda_{n-5}, \lambda_{n-4}, \lambda_{n-3}, \lambda_{n-2}, \lambda_{n-1}) &= (0, 0, 0, 0, -\frac{361}{360}, \frac{1}{180})\\
(\delta_{n-4}, \delta_{n-3}, \delta_{n-2}, \delta_{n-1}, \delta_n, a_1, b_1) &= (1, -6, 14, -14, 5, -2, \frac{5}{6})\\
(w_{n-5}, w_{n-4}, w_{n-3}, w_{n-2}, w_{n-1}, w_n) &= (0, 0, 0, 0, -1, \frac{29}{180})
\end{aligned}$$

and the local truncation error is

$$t_i = \begin{cases} \frac{9580}{8!} h^8 y_i^{(8)} + O(h^9), & \text{i=1,n-1,}\\ \frac{9966}{8!} h^8 y_i^{(8)} + O(h^9), & \text{i=2,n-2.} \end{cases}$$

For $\left(\alpha_1, \beta_1, \gamma_1 = \frac{1}{48}, \frac{9}{48}, \frac{28}{48}\right)$ the local truncation error is

$$t_i = \left(-\frac{1}{48}\right) h^8 y_i^{(8)} + O(h^{10}), i = 3, ..., n-3$$

The values of these parameters for fourth order method is

$$\begin{aligned}
(\delta_0,\delta_1,\delta_2,\delta_3,\delta_4,a_1,b_1) &= (5,-14,14,-6,1,-2,\frac{5}{6})\\
(w_0,w_1,w_2,w_3,w_4,w_5) &= (-\frac{323}{5040},-\frac{1133}{2016},-\frac{101}{504},-\frac{25}{2016},0,0)\\
(\eta_0,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,a_2,b_2) &= (-4,14,-20,15,-6,1,1,\frac{1}{12})\\
(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6) &= (-\frac{729}{3013},-\frac{444}{875},-\frac{1735}{6991},\frac{29}{86688},0,0)\\
(\eta_{n-5},\eta_{n-4},\eta_{n-3},\eta_{n-2},\eta_{n-1},\eta_n,a_2,b_2) &= (1,-6,15,-20,14,-4,1,\frac{1}{12})\\
(\lambda_{n-6},\lambda_{n-5},\lambda_{n-4},\lambda_{n-3},\lambda_{n-2},\lambda_{n-1}) &= (0,0,\frac{29}{86688},-\frac{1735}{6991},-\frac{444}{875},-\frac{729}{3013})\\
(\delta_{n-4},\delta_{n-3},\delta_{n-2},\delta_{n-1},\delta_n,a_1,b_1) &= (1,-6,14,-14,5,-2,\frac{5}{6})\\
(w_{n-5},w_{n-4},w_{n-3},w_{n-2},w_{n-1},w_n) &= (0,0,-\frac{25}{2016},-\frac{101}{504},-\frac{1133}{2016},-\frac{323}{5040})
\end{aligned}$$

and the local truncation error is

$$t_i = \begin{cases} -\frac{13046}{10!}h^{10}y_i^{(10)} + O(h^{11}), & \text{i=1,n-1,}\\ \frac{146306/3}{10!}h^{10}y_i^{(10)} + O(h^{11}), & \text{i=2,n-2.}\end{cases}$$

For $\left(\alpha_1,\beta_1,\gamma_1 = 1,-\frac{15}{4},\frac{13}{2}\right)$ the local truncation error is

$$t_i = \left(\frac{1}{120}\right)h^{10}y_i^{(10)} + O(h^{12}), i = 3,...,n-3$$

4. Application to twelfth order boundary value problem

To illustrate the application of the presented method, we consider a twelfth order linear boundary value problem of the form

$$y^{(12)}(x) + f(x)y(x) = g(x),\ \ a \le x \le b \tag{18}$$

subject to the boundary conditions;

$$y^{(2k)}(a) = A_{2k}, y^{(2k)}(b) = B_{2k},\ \ 0 \le k \le 5 \tag{19}$$

where A_{2k} and B_{2k} $(k = 0,1,...,5)$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$. We rewrite the given problem into the system of sixth order boundary value problems as follows:

$$y^{(6)}(x) = u(x) \tag{20}$$

$$u^{(6)}(x) + f(x)y(x) = g(x),\ \ a \le x \le b \tag{21}$$

subject to modified boundary conditions;

$$\begin{aligned} y^{(2k)}(a) &= A_{2k}, y^{(2k)}(b) = B_{2k},\\ u^{(2k)}(a) &= C_{2k}, u^{(2k)}(b) = D_{2k}\ 0 \le k \le 2. \end{aligned} \tag{22}$$

After applying the scheme (16) to the sixth order boundary value problems (20) and (21), we get the following schemes:

$$\begin{aligned} D_{11}y_{i-2}^{(6)} + D_{12}y_{i-1}^{(6)} + D_{13}y_i^{(6)} + D_{14}y_{i+1}^{(6)} &= E_{11}y_{i-3} + E_{12}y_{i-2} + E_{13}y_{i-1} + E_{14}y_i + \\ &\quad E_{15}y_{i+1} + E_{16}y_{i+2} + y_{i+3}, i = 3, 4, ..., n-3 \end{aligned}$$

$$\begin{aligned} D_{11}u_{i-2}^{(6)} + D_{12}u_{i-1}^{(6)} + D_{13}u_i^{(6)} + D_{14}u_{i+1}^{(6)} &= E_{11}u_{i-3} + E_{12}u_{i-2} + E_{13}u_{i-1} + E_{14}u_i + \\ &\quad E_{15}u_{i+1} + E_{16}u_{i+2} + u_{i+3}, i = 3, 4, ..., n-3 \end{aligned}$$

Finally, we obtain the vector difference equation for boundary value problem (18-19) as follows:

(23) $AY_{i-3} + BY_{i-2} + CY_{i-1} + DY_i + EY_{i+1} + FY_{i+2} + GY_{i+3} = H$

which can be written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{i-3} \\ u_{i-3} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_{i-2} \\ u_{i-2} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} y_{i-1} \\ u_{i-1} \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} y_i \\ u_i \end{bmatrix} +$$
$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} y_{i+1} \\ u_{i+1} \end{bmatrix} + \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} y_{i+2} \\ u_{i+2} \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} y_{i+3} \\ u_{i+3} \end{bmatrix} = \begin{bmatrix} h_{i1} \\ h_{i2} \end{bmatrix}$$
$$i = 3, 4, ..., n-3$$

where,

$$\begin{aligned} a_{11} &= \gamma\ ,\ a_{12} = 0\ ,\ a_{21} = 0\ ,\ a_{22} = \gamma\ ; \\ b_{11} &= \alpha - 2, b_{12} = h^6(\alpha_1 - \alpha_2), b_{21} = h^6(\alpha_1 - \alpha_2)g_{i-2}, b_{22} = \alpha - 2\ ; \\ c_{11} &= \gamma - 2\alpha + \beta, c_{12} = h^6(-4\alpha_1 - \beta_2), c_{21} = h^6(-4\alpha_1 - \beta_2)g_{i-1}, c_{22} = \gamma - 2\alpha + \beta \\ d_{11} &= 2\alpha - 2\beta, d_{12} = h^6(6\alpha_1 - \gamma_1), d_{21} = h^6(6\alpha_1 - \gamma_1)g_i, d_{22} = 2\alpha - 2\beta\ ; \\ e_{11} &= 1 - 2\alpha + \beta, e_{12} = h^6(-4\alpha_1 - \beta_1), e_{21} = h^6(-4\alpha_1 - \beta_1)g_{i+1}, e_{22} = 1 - 2\alpha + \beta \\ f_{11} &= \alpha - 2, f_{12} = 0, f_{21} = 0, f_{22} = \alpha - 2\ ; \\ g_{11} &= 1, g_{12} = 0, g_{21} = 0, g_{22} = 1\ ; \\ h_{i1} &= 0, h_{i2} = h^6[(\alpha_2 - \alpha_1)f_{i-2} + (4\alpha_1 + \beta_2)f_{i-1} + (-6\alpha_1 + \gamma_1)f_i + (4\alpha_1 + \beta_1)f_{i+1} \end{aligned}$$

$i = 3, ..., n-3$

We require four more equations at $i = 1, 2, n-2$ and $n-1$ which are given as follows:
For $i = 1$

$$\begin{bmatrix} \delta_1 & w_1h^6 \\ -w_1g_1h^6 & \delta_1 \end{bmatrix} \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} \delta_2 & w_2h^6 \\ -w_2g_2h^6 & \delta_2 \end{bmatrix} \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} + \begin{bmatrix} \delta_3 & w_3h^6 \\ -w_3g_3h^6 & \delta_3 \end{bmatrix} \begin{bmatrix} y_3 \\ u_3 \end{bmatrix} +$$
$$\begin{bmatrix} \delta_4 & w_4h^6 \\ -w_4g_4h^6 & \delta_4 \end{bmatrix} \begin{bmatrix} y_4 \\ u_4 \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{12} \end{bmatrix}$$

where,

$$\begin{aligned} h_{11} &= -\delta_0 y_0 - a_1 h^2 y_0^{(2)} - b_1 h^4 y_0^{(4)} - w_0 u_0 h^6 \\ h_{12} &= -\delta_0 u_0 - a_1 h^2 u_0^{(2)} - b_1 h^4 u_0^{(4)} - w_0 y_0 g_0 h^6 - h^6(w_0 f_0 + w_1 f_1 + w_2 f_2 + w_3 f_3 + w_4 f_4) \end{aligned}$$

for $i = 2$

$$\begin{bmatrix} \eta_1 & \lambda_1 h^6 \\ -\lambda_1 g_1 h^6 & \eta_1 \end{bmatrix} \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} \eta_2 & \lambda_2 h^6 \\ -\lambda_2 g_2 h^6 & \eta_2 \end{bmatrix} \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} + \begin{bmatrix} \eta_3 & \lambda_3 h^6 \\ -\lambda_3 g_3 h^6 & \eta_3 \end{bmatrix} \begin{bmatrix} y_3 \\ u_3 \end{bmatrix} + \begin{bmatrix} \eta_4 & \lambda_4 h^6 \\ -\lambda_4 g_4 h^6 & \eta_4 \end{bmatrix} \begin{bmatrix} y_4 \\ u_4 \end{bmatrix} + \begin{bmatrix} \eta_5 & \lambda_5 h^6 \\ -\lambda_5 g_5 h^6 & \eta_5 \end{bmatrix} \begin{bmatrix} y_5 \\ u_5 \end{bmatrix} = \begin{bmatrix} h_{21} \\ h_{22} \end{bmatrix}$$

where,

$$\begin{aligned} h_{21} &= -\eta_0 y_0 - a_2 h^2 y_0^{(2)} - b_2 h^4 y_0^{(4)} \\ h_{22} &= -\eta_0 u_0 - a_2 h^2 u_0^{(2)} - b_2 h^4 u_0^{(4)} - h^6(\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 + \lambda_5 f_5) \end{aligned}$$

for $i = n-2$

$$\begin{bmatrix} \eta_{n-5} & \lambda_{n-5} h^6 \\ -\lambda_{n-5} g_{n-5} h^6 & \eta_{n-5} \end{bmatrix} \begin{bmatrix} y_{n-5} \\ u_{n-5} \end{bmatrix} + \begin{bmatrix} \eta_{n-4} & \lambda_{n-4} h^6 \\ -\lambda_{n-4} g_{n-4} h^6 & \eta_{n-4} \end{bmatrix} \begin{bmatrix} y_{n-4} \\ u_{n-4} \end{bmatrix} + \begin{bmatrix} \eta_{n-3} & \lambda_{n-3} h^6 \\ -\lambda_{n-3} g_{n-3} h^6 & \eta_{n-3} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ u_{n-3} \end{bmatrix} + \begin{bmatrix} \eta_{n-2} & \lambda_{n-2} h^6 \\ -\lambda_{n-2} g_{n-2} h^6 & \eta_{n-2} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ u_{n-2} \end{bmatrix} + \begin{bmatrix} \eta_{n-1} & \lambda_{n-1} h^6 \\ -\lambda_{n-1} g_{n-1} h^6 & \eta_{n-1} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} h_{n-2\mathbf{1}} \\ h_{n-2\mathbf{2}} \end{bmatrix}$$

where,

$$\begin{aligned} h_{n-2\mathbf{1}} &= -\eta_n y_n - a_2 h^2 y_n^{(2)} - b_2 h^4 y_n^{(4)} \\ h_{n-2\mathbf{2}} &= -\eta_n u_n - a_2 h^2 u_n^{(2)} - b_2 h^4 u_n^{(4)} - h^6(\lambda_{n-5} f_{n-5} + \lambda_{n-4} f_{n-4} + \lambda_{n-3} f_{n-3} \\ &\quad + \lambda_{n-2} f_{n-2} + \lambda_{n-1} f_{n-1}) \end{aligned}$$

for $i = n-1$

$$\begin{bmatrix} \delta_{n-4} & w_{n-4} h^6 \\ -w_{n-4} g_{n-4} h^6 & \delta_{n-4} \end{bmatrix} \begin{bmatrix} y_{n-4} \\ u_{n-4} \end{bmatrix} + \begin{bmatrix} \delta_{n-3} & w_{n-3} h^6 \\ -w_{n-3} g_{n-3} h^6 & \delta_{n-3} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ u_{n-3} \end{bmatrix} + \begin{bmatrix} \delta_{n-2} & w_{n-2} h^6 \\ -w_{n-2} g_{n-2} h^6 & \delta_{n-2} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ u_{n-2} \end{bmatrix} + \begin{bmatrix} \delta_{n-1} & w_{n-1} h^6 \\ -w_{n-1} g_{n-1} h^6 & \delta_{n-1} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} h_{n-1\mathbf{1}} \\ h_{n-1\mathbf{2}} \end{bmatrix}$$

where,

$$\begin{aligned} h_{n-1\mathbf{1}} &= -\delta_n y_n - a_1 h^2 y_n^{(2)} - b_1 h^4 y_n^{(4)} - w_n u_n h^6 \\ h_{n-1\mathbf{2}} &= -\delta_n u_n - a_1 h^2 u_n^{(2)} - b_1 h^4 u_n^{(4)} - w_n y_n g_n h^6 - h^6(w_{n-4} f_{n-4} + w_{n-3} f_{n-3} \\ &\quad + w_{n-2} f_{n-2} + w_{n-1} f_{n-1} + w_n f_n) \end{aligned}$$

where the coefficients δ_i's, w_i's, η_i's and λ_i's are different for second and fourth order method which are given in section 3.

5. Convergence analysis

The developed method leads to the following septa-block-diagonal matrix of the form

$$LZ = H \tag{24}$$

where,

$$L = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & & & & \\ B_1 & B_2 & B_3 & B_4 & B_5 & & & \\ B & C & D & E & F & G & & \\ A & B & C & D & E & F & G & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \\ & A & B & C & D & E & F & G \\ & & A & B & C & D & E & F \\ & & & C_{n-5} & C_{n-4} & C_{n-3} & C_{n-2} & C_{n-1} \\ & & & & D_{n-4} & D_{n-3} & D_{n-2} & D_{n-1} \end{bmatrix} \tag{25}$$

where, $A_i's, B_i's...G_i's$ are matrices of order 2×2 , $Z = [z_1, z_2, ..., z_{n-1}]^T$ where, $z_i = [y_i, u_i]^T$ and the right hand side vector $H = [h_1, h_2, ..., h_{n-1}]^T$ where, $h_i = [h_{i1}, h_{i2}]^T$.
We also have,

$$L\tilde{Z} = H + T \tag{26}$$

where $\tilde{Z} = [\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_{n-1}]^T$ where, $\tilde{z}_i = [\tilde{y}_i, \tilde{u}_i]^T$ be the exact solution and $T = [t_1, t_2, ..., t_{n-1}]^T$ where, $t_i = [\tilde{y}_i - y_i, \tilde{u}_i - u_i]^T$ be the local truncation error. From (24) and (26) we have,

$$L(\tilde{Z} - Z) = T$$
$$LE = T,$$
$$E = \tilde{Z} - Z = [e_1, e_2, ..., e_{n-1}]^T$$

Now we calculate sum of each row of the matrix L

$$S_{1j} = \begin{cases} \delta_1+\delta_2+\delta_3+\delta_4+h^6(w_1+w_2+w_3+w_4), & \text{j=1} \\ \delta_1+\delta_2+\delta_3+\delta_4-h^6(w_1g_1+w_2g_2+w_3g_3+w_4g_4), & \text{j=2} \end{cases}$$

$$S_{2j} = \begin{cases} \eta_1+\eta_2+\eta_3+\eta_4+\eta_5+h^6(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5), & \text{j=1} \\ \eta_1+\eta_2+\eta_3+\eta_4+\eta_5-h^6(\lambda_1g_1+\lambda_2g_2+\lambda_3g_3+\lambda_4g_4+\lambda_5g_5), & \text{j=2} \end{cases}$$

$$S_{ij} = \begin{cases} -2+2\gamma-h^6(\alpha_1+\beta_1+\gamma_1+\beta_2+\alpha_2) & \text{j=1,i=3,5,...,n-4} \\ -2+2\gamma+h^6\big((\alpha_1-\alpha_2)g_{i-2}+(-4\alpha_1-\beta_2)g_{i-1}+ & \\ (6\alpha_1-\gamma_1)g_i+(-4\alpha_1-\beta_1)g_{i+1}\big) & \text{j=2,i=4,...,n-3} \end{cases}$$

$$S_{n-2\mathbf{j}} = \begin{cases} \eta_{n-5}+\eta_{n-4}+\eta_{n-3}+\eta_{n-2}+\eta_{n-1}+h^6(\lambda_{n-5}+\lambda_{n-4}+\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1}), & \text{j=1} \\ \eta_{n-5}+\eta_{n-4}+\eta_{n-3}+\eta_{n-2}+\eta_{n-1}-h^6(\lambda_{n-5}g_{n-5}+\lambda_{n-4}g_{n-4}+\lambda_{n-3}g_{n-3}+ & \\ \lambda_{n-2}g_{n-2}+\lambda_{n-1}g_{n-1}), & \text{j=2} \end{cases}$$

$$S_{n-1\mathbf{j}} = \begin{cases} \delta_{n-4}+\delta_{n-3}+\delta_{n-2}+\delta_{n-1}+h^6(w_{n-4}+w_{n-3}+w_{n-2}+w_{n-1}), & \text{j=1} \\ \delta_1+\delta_2+\delta_3+\delta_4-h^6(w_{n-4}g_{n-4}+w_{n-3}g_{n-3}+w_{n-2}g_{n-2}+w_{n-1}g_{n-1}), & \text{j=2} \end{cases}$$

Let $0 < M \in Z^+$ is the minimum of $\mid g_i \mid$. For sufficiently small h, we can say that

$$S_{1j} \geq \begin{cases} h^6(w_1+w_2+w_3+w_4), & \text{j=1} \\ h^6(w_1+w_2+w_3+w_4)M, & \text{j=2} \end{cases}$$

$$S_{2j} \geq \begin{cases} h^6(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5), & \text{j=1} \\ h^6(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)M, & \text{j=2} \end{cases}$$

$$S_{ij} \geq \begin{cases} h^6(\alpha_1+\beta_1+\gamma_1+\beta_2+\alpha_2) & \text{j=1, } i=3,5,...,n-4 \\ h^6(\alpha_1+\beta_1+\gamma_1+\beta_2+\alpha_2)M & \text{j=2, } i=4,6,...,n-3 \end{cases}$$

$$S_{n-2\mathbf{j}} \geq \begin{cases} h^6(\lambda_{n-5}+\lambda_{n-4}+\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1}), & \text{j=1} \\ h^6(\lambda_{n-5}+\lambda_{n-4}+\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1})M, & \text{j=2} \end{cases}$$

$$S_{n-1\mathbf{j}} \geq \begin{cases} h^6(w_{n-4}+w_{n-3}+w_{n-2}+w_{n-1}), & \text{j=1} \\ h^6(w_{n-4}+w_{n-3}+w_{n-2}+w_{n-1})M, & \text{j=2} \end{cases}$$

$$\begin{aligned} S_1 \geq max[&h^6(w_1+w_2+w_3+w_4), h^6(w_1+w_2+w_3+w_4)M] \\ &= h^6(w_1+w_2+w_3+w_4)M, \ i=1 \end{aligned}$$

$$\begin{aligned} S_2 \geq max[&h^6(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5), h^6(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)M] \\ &= h^6(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)M, \ i=2 \end{aligned}$$

$$\begin{aligned} S_i \geq max[&h^6(\alpha_1+\beta_1+\gamma_1+\beta_2+\alpha_2), h^6(\alpha_1+\beta_1+\gamma_1+\beta_2+\alpha_2)M] \\ &= h^6(\alpha_1+\beta_1+\gamma_1+\beta_2+\alpha_2)M, \ i=3,4,...,n-3 \end{aligned}$$

$$\begin{aligned} S_{n-2} \geq max[&h^6(\lambda_{n-5}+\lambda_{n-4}+\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1}), h^6(\lambda_{n-5}+\lambda_{n-4} \\ &+\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1})M] \\ &= h^6(\lambda_{n-5}+\lambda_{n-4}+\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1})M, \ i=n-2 \end{aligned}$$

$$S_{n-1} \geq max[h^6(w_{n-4}+w_{n-3}+w_{n-2}+w_{n-1}), h^6(w_{n-4}+w_{n-3}+w_{n-2}+w_{n-1})M]$$
$$= h^6(w_{n-4}+w_{n-3}+w_{n-2}+w_{n-1})M, \; i = n-1$$

Therefore, we get

$$\frac{1}{S_i} \leq \begin{cases} \frac{1}{h^6(w_1+w_2+w_3+w_4)M}, & \text{i=1} \\ \frac{1}{h^6(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)M}, & \text{i=2} \\ \frac{1}{h^6(\alpha_1+\beta_1+\gamma_1+\beta_2+\alpha_2)M}, & \text{i=3,4,...,n-3} \\ \frac{1}{h^6(\lambda_{n-5}+\lambda_{n-4}+\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1})M}, & \text{i=n-2} \\ \frac{1}{h^6(w_{n-4}+w_{n-3}+w_{n-2}+w_{n-1})M}, & \text{i=n-1} \end{cases}$$

For sufficiently small h, we can easily show that the matrix L is irreducible and monotone. Therefore, L^{-1} exist and $L^{-1} \geq 0$.
Hence,

$$\|E\| = \|L^{-1}\|\|T\|.$$

Let $L^{-1} = (l^*_{i,j})$, then by theory of matrices Varga[11], we get

$$\sum_{i=1}^{n-1} l^*_{i,j} S_i = 1, \; j = 1, ..., n-1$$

Therefore,

$$\begin{aligned} l^*_{i,j} &\leq \frac{1}{S_i} \\ \|L^{-1}\| &= \max_{1\leq i\leq n-1} \sum_{j=1}^{n-1} |l^*_{i,j}| \leq \sum_{i=1}^{n-1} \frac{1}{S_i} = \frac{1}{h^6 M}\Bigg(\frac{2}{w_1+w_2+w_3+w_4} \\ &\quad + \frac{2}{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5} + \frac{1}{2\alpha_1+2\beta_1+\gamma_1}\Bigg), \\ &\quad i = 1, ..., n-1 \; and \\ \|T_i\| &= \max_{1\leq i\leq n-1} \sum_{i=1}^{n-1} |T_i| \end{aligned}$$

The error is given by

$$\begin{aligned} \|E\| &= \|L^{-1}\|\|T\| \leq \frac{1}{h^6 M}\Bigg(\frac{2}{w_1+w_2+w_3+w_4} + \frac{2}{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5} \\ &\quad + \frac{1}{2\alpha_1+2\beta_1+\gamma_1}\Bigg) \end{aligned}$$

From (17), we get $\|T\| = O(h^8)$ for second order method.

$$\begin{aligned} \|E\| &\leq \frac{1}{h^6 M}\Bigg(\frac{2}{w_1+w_2+w_3+w_4} + \frac{2}{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5} + \frac{1}{2\alpha_1+2\beta_1+\gamma_1}\Bigg) O(h^8) \\ &= O(h^2) \end{aligned}$$

Hence, the scheme is second order convergent. Similarly, we can prove the fourth order convergence of the scheme.
For fourth order method, $\|T\| = O(h^{10})$. Then $\|E\| = O(h^4)$.

Theorem. The method given by equation (16) for solving the given boundary value problem (1) for sufficiently small h has a second and fourth order convergent solution depending upon the parameters.

6. Numerical Examples

To illustrate the comparative performance of our method, we consider two examples of twelfth order linear boundary value problems with two point boundary conditions of the form (1-2). The numerical results are carried out using different values of h.

Example 1: Consider the twelfth order linear boundary value problem from Siddiqi and Twizell[13]

$$y^{(12)}(x) + xy(x) = -(120 + 23x + x^3)\exp(x)\ , 0 \le x \le 1 \tag{27}$$

with

$$\begin{aligned} y(0) &= 0,\ y(1) = 0 \\ y^{(2)}(0) &= 0,\ y^{(2)}(1) = -4e \\ y^{(4)}(0) &= -8,\ y^{(4)}(1) = -16e \\ y^{(6)}(0) &= -24,\ y^{(6)}(1) = -36e \\ y^{(8)}(0) &= -48,\ y^{(8)}(1) = -64e \\ y^{(10)}(0) &= -80,\ y^{(10)}(1) = -100e \end{aligned}$$

The exact solution is $y(x) = x(1-x)\exp(x)$. The maximum absolute errors in y_i for the problem (27) are summarized in Table 1.

Example 2: Consider the following boundary value problem from Siddiqi and Twizell[13]

$$y^{(12)}(x) + y(x) = -12(2x\cos(x) + 11\sin(x))\ , -1 \le x \le 1 \tag{28}$$

with

$$\begin{aligned} y(-1) &= 0,\ y(1) = 0 \\ y^{(2)}(-1) &= -4\cos(-1) + 2\sin(-1),\ y^{(2)}(1) = 4\cos(1) + 2\sin(1) \\ y^{(4)}(-1) &= 8\cos(-1) - 12\sin(-1),\ y^{(4)}(1) = -8\cos(1) - 12\sin(1) \\ y^{(6)}(-1) &= -12\cos(-1) + 30\sin(-1),\ y^{(6)}(1) = 12\cos(1) + 30\sin(1) \\ y^{(8)}(-1) &= 16\cos(-1) - 56\sin(-1),\ y^{(8)}(1) = -16\cos(1) - 56\sin(1) \\ y^{(10)}(-1) &= -20\cos(-1) + 90\sin(-1),\ y^{(10)}(1) = 20\cos(1) + 90\sin(1) \end{aligned}$$

The exact solution is $y(x) = (x^2 - 1)\sin(x)$. The maximum absolute errors in y_i for the problem (28) are summarized in Table 2.

TABLE 1. Maximum absolute errors for example 1

Method	Different parameters α_1,β_1,γ_1	$h = 1/11$	$h = 1/22$	$h = 1/44$
Fourth order method	$\left(1, -\frac{15}{4}, \frac{13}{2}\right)$	8.093×10^{-8}	1.175×10^{-8}	3.449×10^{-9}
Second order method	$\left(\frac{1}{120}, \frac{25}{120}, \frac{17}{30}\right)$	7.052×10^{-5}	6.297×10^{-6}	8.099×10^{-7}
Siddiqi and Twizell[13]		-	5.582×10^{-3}	-
Siddiqi and Akram[14]		-	4.72×10^{-6}	-

TABLE 2. Maximum absolute errors for example 2

Method	Different parameters α_1,β_1,γ_1	$h = 1/11$	$h = 1/22$	$h = 1/44$
Fourth order method	$\left(1, -\frac{15}{4}, \frac{13}{2}\right)$	1.538×10^{-6}	8.323×10^{-7}	7.998×10^{-7}
Second order method	$\left(\frac{1}{48}, \frac{9}{48}, \frac{28}{48}\right)$	1.950×10^{-4}	9.263×10^{-6}	1.508×10^{-6}
	$\left(\frac{1}{120}, \frac{25}{120}, \frac{17}{30}\right)$	2.138×10^{-4}	1.793×10^{-5}	1.012×10^{-6}
Siddiqi and Twizell[13]		-	1.366×10^{-4}	-

Conclusion. In this paper, we developed a non-polynomial quartic spline method for solving twelfth order boundary value problems with two point boundary conditions. We reduced the given problem into a system of sixth order boundary value problems. Then implement our method on the system of sixth order boundary value problems. The presented scheme (16) is second and fourth order convergent. Our main aim in this paper was to solve a twelfth order boundary value problem by using lower degree splines. However, in most of the previous papers, higher degree splines were used to solve the twelfth order boundary value problems. Comparison of our method with the existing methods Siddiqi and Akram[14] and Siddiqi and Twizell[13] in Tables 1 and 2 show that our method is better in accuracy, applicability and efficiency.

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