Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 4. pp. 627 - 636

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.4.627

TO SOLVE MATRIX GAMES WITH FUZZY GOALS USING PIECEWISE LINEAR MEMBERSHIP FUNCTIONS

SANDEEP KUMAR

ABSTRACT. In this paper, the solution concepts are presented for a two person zero-sum matrix game with fuzzy goals using piecewise linear membership functions. This paper shows that the solution of matrix game with fuzzy goals can be determined by solving two crisp linear programming problems which are dual to each other in fuzzy sense. The fuzziness in goals is characterized through piecewise linear membership functions. One numerical example is illustrated to show the effectiveness and advantage of the proposed technique.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 90C70, 91A05.

KEYWORDS AND PHRASES. Fuzzy matrix games, Fuzzy goals, Fuzzy duality, Piecewise linear membership function.

1. INTRODUCTION

Game theory usually represents those systems that involve decisions based on conflicting interests, to be taken by at least two decision makers (players). In most of situations which can be modeled as games, it often happens that players are not able to evaluate exactly the data of the game and their own preferences and choices. In such cases, fuzzy set theory provides an excellent tool that can be used to handle the imprecision in decision making problems.

The present work is devoted to two person zero-sum matrix games in fuzzy environment. In fact, matrix games with fuzzy concept have appeared more visibly rather than deterministic-stochastic games. As in crisp games, a two person zero-sum fuzzy matrix game can be equivalent to two linear programming problems that are dual to each other in fuzzy sense [3, 4, 5]. So, first we study some fuzzy duality results; see [2, 8] and references therein. Fuzzy set theory was introduced for the first time in two person zero-sum matrix games with fuzzy payoffs by Campos [12]. His approach was based on ranking of fuzzy numbers. Thereafter, Sakawa and Nishizaki [13] solved single and multi-objective matrix games with fuzzy goals and fuzzy payoffs by using max-min principle of game theory. Bector et al. [4, 5] used theory of duality in fuzzy linear programming problems to solve a two person zero-sum matrix game with fuzzy goals and fuzzy payoffs. Chen and Larbani [16] solved fuzzy multiple attributes decision making problem using two person zero-sum game approach. They defined fuzzy matrix with triangular membership function and proved that two person zero-sum game with fuzzy payoff matrices is equivalent to two linear programming problems. Cevikel and Ahlatcioglu [1] proposed new concepts of solutions for multi-objective two person zero-sum games with fuzzy goals and fuzzy payoffs using linear membership functions. Pandey and Kumar [7] introduced a modified approach to solve multi-objective matrix game with vague payoffs using a new order function. Nan et al. [11] gave a lexicographic methodology to determine the solution of matrix game with payoffs of triangular intuitionistic fuzzy numbers for both players. Li and Hong [6] proposed an approach for solving constrained matrix games with payoffs of triangular fuzzy numbers. In this approach, they introduced the concepts of alpha constrained matrix games. Bandyopadhyay and Nayak [14] introduced an approach for solving a matrix game whose payoffs are symmetric trapezoidal fuzzy numbers. In this approach, they transformed symmetric trapezoidal fuzzy numbers to interval fuzzy numbers so that lengths of all intervals were different.

In this paper, we propose a solution approach to solve matrix games with fuzzy goals using piecewise linear membership functions to make work more efficiently. This approach is based on duality results established in [8].

The layout of this paper is as follows: In section 2, some definitions and results are introduced which are relevant to this paper and in section 3, the solution concepts are given to a matrix game with fuzzy goals using piecewise linear membership functions. In section 4, one numerical example is considered to discuss the proposed approach.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative part. Let $A \in \mathbb{R}^{m\times n}$ be a $(m \times n)$ real matrix. Mathematically, a two person zero-sum matrix game is a triplet $G = (S^m, S^n, A)$ where $S^m = \{x \in \mathbb{R}^m_+, \sum_{i=1}^{m} x_i = 1\}$ and $S^n = \{y \in \mathbb{R}^n_+, \sum_{j=1}^{n} y_j = 1\}$. Here $S^m(S^n)$ is the strategy space for Player I (Player II) and A is called the payoff matrix. It is a convention to assume that Player I is a maximizing player and Player II is a minimizing player. Further for $x \in S^m, y \in S^n$, the scalar x^TAy is the payoff to Player I, G being a zero-sum game, the payoff for the Player II is $-x^TAy$.

Definition 2.1. *(Solution of Game) The triplet $(x^*, y^*, v^*) \in S^m \times S^n \times \mathbb{R}$ is called a solution of the game G if*

(i) $(x^*)^TAy \geq v^*$, *for all* $y \in S^n$ *and*

(ii) $x^TAy^* \leq v^*$, *for all* $x \in S^m$

where x^*/y^* is called the optimal strategy for Player I/Player II and v^* is called the value of the game G.

Let v_0, w_0 be scalars representing the aspiration levels of Player I and II with tolerance values p_0 and q_0 respectively. A two person zero-sum matrix game with fuzzy goals (FG) is defined as

$$FG = (S^m, S^n, A, v_0, \lesssim, p_0, w_0, \gtrsim, q_0),$$

where S^m, S^n and A have introduced as earlier in this section and " $\lesssim$ " and " $\gtrsim$ " are fuzzified versions for the usual symbols " $\leq$ " and " $\geq$ " respectively, having the linguistic interpretation as explained in [10].
The solution of the fuzzy matrix game FG is defined as following:

Definition 2.2. [5]*(Solution of Matrix Game with Fuzzy Goals) A point* $(x^*, y^*) \in S^m \times S^n$ *is called a solution of the fuzzy matrix game FG if*

(i) $(x^*)^T A y \gtrsim_{p_0} v_0$, *for all* $y \in S^n$ *and*
(ii) $x^T A y^* \lesssim_{q_0} w_0$, *for all* $x \in S^m$

where the symbols $\gtrsim_{p_0}$ *and* $\lesssim_{q_0}$ *mean 'fuzzily greater than with tolerance value* p_0*' and 'fuzzily less than with tolerance value* q_0*' respectively.*

2.1. **Auxiliary Duality Results with Piecewise Linear Membership Functions.** A fuzzy pair of primal-dual problems is introduced by Pandey and Kumar [8] in which they modified the fuzzy dual model of Bector and Chandra [2], using piecewise linear membership functions.
Consider the fuzzy versions of the usual primal and dual problems as defined in [2, 8]:

Find $z \in \mathbb{R}^n$ such that

$$\begin{aligned} c^T z &\gtrsim Z_0, \\ Az &\lesssim b, \\ z &\geq 0, \end{aligned} \tag{1}$$

and

Find $u \in \mathbb{R}^m$ such that

$$\begin{aligned} b^T u &\lesssim U_0, \\ A^T u &\gtrsim c, \\ u &\geq 0, \end{aligned} \tag{2}$$

where Z_0, U_0 respectively denote the aspiration levels of two objectives $c^T z, b^T u$.

In this work, the piecewise linear membership function for objective function of primal (respectively dual) is approximated by two line segments. Let $Z_{01}, Z_{02} = Z_0$ be the aspiration levels for objective function of primal problem and $U_{01}, U_{02} = U_0$ be aspiration levels for objective function of dual. Let $p_{01}, p_{02}, p_1, p_2, \ldots, p_m$ be subjectively chosen positive constants representing the admissible tolerance values associated with the objective function and m linear constraints of primal problem. Similarly, suppose that $q_{01}, q_{02}, q_1, q_2, \ldots, q_n$ are subjectively chosen positive constants representing the admissible tolerance values associated with objective function and n linear constraints of dual problem. Then taking the membership function in Zimmermann sense, the following pair of crisp linear programming problems is obtained for the fuzzy pair (1) and (2):

$$\max \lambda$$
subject to,
$$\lambda \leq 1 - \frac{Z_{01} - c^T z}{p_{01}},$$
$$\lambda \leq 1 - \frac{Z_{02} - c^T z}{p_{02}},$$
$$\lambda \leq 1 - \frac{A_i z - b_i}{p_i}, \quad i = 1, 2, \ldots, m,$$
$$\lambda \leq 1,$$
(3)
$$z \geq 0, \ \lambda \geq 0,$$

and
$$\min\ (-\eta)$$
subject to,
$$\eta \leq 1 + \frac{U_{01} - b^T u}{q_{01}},$$
$$\eta \leq 1 + \frac{U_{02} - b^T u}{q_{02}},$$
$$\eta \leq 1 + \frac{A_j^T u - c_j}{q_j}, \quad j = 1, 2, \ldots, n,$$
$$\eta \leq 1,$$
(4)
$$u \geq 0, \ \eta \geq 0.$$

Here A_i(respectivelyA_j) denotes the i^{th} row (respectively j^{th} column) of A.
Modified weak duality theorem: [2] Let (z, λ) be feasible solution for (3) and (u, η) be feasible solution for (4), then $(\lambda - 1)p^T u + (\eta - 1)q^T z \leq (b^T u - c^T z)$, where $p^T = (p_1, p_2, \ldots, p_m)$ and $q^T = (q_1, q_2, \ldots, q_n)$.
Obviously modified weak duality theorem reduces to standard crisp weak duality theorem for $\lambda = \eta = 1$.

Theorem 2.1. [8] *Let (z, λ) be the feasible solution for non-linear primal (3) and (u, η) be the feasible solution for non-linear dual (4), then*

$$(\lambda-1)(p_{01}+p_{02})+(\eta-1)(q_{01}+q_{02}) \leq 2(c^T z - b^T u) + [(U_{01}+U_{02})-(Z_{01}+Z_{02})]$$

3. Solution Concepts for Two Person Zero-Sum Matrix Game with Fuzzy Goals

For a given two person zero-sum matrix game $G = (S^m, S^n, A)$, the optimization problems for players I and II can be constructed as follows:
$$\max v$$
subject to,
$$\sum_{i=1}^{m} a_{ij} x_i \geq v, \quad (j = 1, 2, \ldots, n),$$
$$\sum_{i=1}^{m} x_i = 1,$$
(5)
$$x \geq 0,$$

and

$$\begin{aligned} &\min w \\ &\text{subject to,} \\ \sum_{j=1}^{n} a_{ij} y_j &\leq w, \quad (i = 1, 2, \ldots, m), \\ \sum_{j=1}^{n} y_j &= 1, \\ y &\geq 0. \end{aligned} \tag{6}$$

respectively.

Model (5) and (6) are primal-dual linear programming problems and their optimal solutions give the optimal strategy for Player I and II respectively, see [9].

For a fuzzy matrix game FG and its solution as defined in definition 2.2, the fuzzified versions of model (5) and (6) can be expressed as the following fuzzy linear programming problems for Player I and II respectively:

$$\begin{aligned} &\text{Find } x \in \mathbb{R}^m \text{ such that} \\ \sum_{i=1}^{m} a_{ij} x_i &\gtrsim_{p_0} v_0, \quad (j = 1, 2, \ldots, n), \\ \sum_{i=1}^{m} x_i &= 1, \\ x &\geq 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\text{Find } y \in \mathbb{R}^n \text{ such that} \\ \sum_{j=1}^{n} a_{ij} y_j &\lesssim_{q_0} w_0, \quad (i = 1, 2, \ldots, m), \\ \sum_{j=1}^{n} y_j &= 1, \\ y &\geq 0. \end{aligned} \tag{8}$$

Let for $j = 1, 2, \ldots, n; A_j$ denote the j^{th} column of matrix A. Then the j^{th} constraint of model (7) can be written as $A_j^T x \gtrsim_{p_0} v_0$. Similarly the i^{th} constraint of model (8) can be written as $A_i y \lesssim_{q_0} w_0$ for $i = 1, 2, \ldots, m$.

Now, the membership function $\mu_j^I(A_j^T x)(j = 1, 2, \ldots, n)$, that indicates the degree to which x satisfies the fuzzy constraint $A_j^T x \gtrsim_{p_0} v_0$, is defined as following:

$$\mu_j^I(A_j^T x) = \begin{cases} 1 & , A_j^T x \geq v_0, \\ 1 - \frac{v_0 - A_j^T x}{p_0} & , (v_0 - p_0) < A_j^T x \leq v_0, \\ 0 & , A_j^T x < (v_0 - p_0). \end{cases} \tag{9}$$

We will now approximate each $\mu_j^I(A_j^T x)(j = 1, 2, \ldots, n)$, by two linear segments and define the membership functions $\mu_{1j}^I(A_j^T x), \mu_{2j}^I(A_j^T x)$ corresponding to each segment as given:

$$\mu_{1j}^I(A_j^T x) = \begin{cases} 1 & , A_j^T x \geq v_{01}, \\ 1 - \frac{v_{01} - A_j^T x}{p_{01}} & , (v_{01} - p_{01}) \leq A_j^T x < v_{01}, \\ 0 & , \text{otherwise}, \end{cases} \tag{10}$$

and

$$\mu_{2j}^I(A_j^T x) = \begin{cases} 1 & , A_j^T x \geq v_{02}, \\ 1 - \frac{v_{02} - A_j^T x}{p_{02}} & , (v_{02} - p_{02}) \leq A_j^T x < v_{02}, \\ 0 & , \text{otherwise}, \end{cases} \tag{11}$$

where $p_{01} + p_{02} > p_0$ and $v_{01} < v_{02} = v_0$.
Thus, we have the crisp formulation of the fuzzy linear programming (7) as:

$$\max \lambda$$

subject to,

$$\begin{aligned} \lambda &\leq 1 - \frac{v_{01} - A_j^T x}{p_{01}}, \\ \lambda &\leq 1 - \frac{v_{02} - A_j^T x}{p_{02}}, \quad (j = 1, 2, \ldots, n), \\ \sum_{i=1}^{m} x_i &= 1, \\ \lambda &\leq 1, \\ x &\geq 0, \ \lambda \geq 0. \end{aligned} \tag{12}$$

Further, the fuzzy constraints of above crisp linear programming are:

$$\sum_{i=1}^{m} a_{ij} x_i \gtrsim_{p_{01}} v_{01}, \tag{13}$$

$$\sum_{i=1}^{m} a_{ij} x_i \gtrsim_{p_{02}} v_{02}, \quad (j = 1, 2, \ldots, n), \tag{14}$$

$$\sum_{i=1}^{m} x_i = 1, \tag{15}$$

$$x \geq 0. \tag{16}$$

In a similar manner, the membership functions $\mu_i^{II}(A_i y)(i = 1, 2, \ldots, m)$, which define the degree to which y satisfies the constraint $A_i y \lesssim_{q_0} w_0$, is approximated by following membership functions:

$$\mu_{1i}^{II}(A_i y) = \begin{cases} 1, & A_i y \leq w_{01}, \\ 1 - \frac{A_i y - w_{01}}{q_{01}}, & w_{01} < A_i y \leq w_{01} + q_{01}, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

and

$$(18)\qquad \mu_{2i}^{II}(A_iy) = \begin{cases} 1, & A_iy \leq w_{02}, \\ 1-\frac{A_iy-w_{02}}{q_{02}}, & w_{02} < A_iy \leq w_{02}+q_{02}, \\ 0, & \text{otherwise,} \end{cases}$$

where $q_{01}+q_{02} > q_0$ and $w_{01} > w_{02} = w_0$.
The crisp formulation of (8) as:

$$\begin{aligned} &\max \eta \\ &\text{subject to,} \\ &\eta \leq 1-\frac{A_iy-w_{01}}{q_{01}}, \\ &\eta \leq 1-\frac{A_iy-w_{02}}{q_{02}}, \quad (i=1,2,\dots,n), \\ &\sum_{j=1}^{m} y_j = 1, \\ &\eta \leq 1, \\ (19)\qquad &y \geq 0,\ \eta \geq 0. \end{aligned}$$

The fuzzy constraints of above crisp linear programming are:

$$(20)\qquad \sum_{j=1}^{n} a_{ij}y_j \lesssim_{q_{01}} w_{01},$$

$$(21)\qquad \sum_{j=1}^{n} a_{ij}y_j \lesssim_{q_{02}} w_{02}, \quad (i=1,2,\dots,m),$$

$$(22)\qquad \sum_{j=1}^{n} y_j = 1,$$

$$(23)\qquad y \geq 0.$$

We now define the generalized model of two person zero-sum matrix game with fuzzy goals, denoted by GFG, as following:

$$GFG = (S^m, S^n, A, v_{01}, v_{02}, \gtrsim, p_{01}, p_{02}, w_{01}, w_{02}, \lesssim, q_{01}, q_{02})$$

We are assuming here that $v_{02} = v_0$ and $w_{02} = w_0$ are the aspiration levels for Player I and Player II respectively.

Definition 3.1. *(Solution of GFG): A point $(x^*, y^*) \in S^m \times S^n$ is called a solution of the generalized fuzzy matrix game GFG if*

(i) $(x^*)^T Ay \gtrsim_{p_{01}} v_{01}$,
$(x^*)^T Ay \gtrsim_{p_{02}} v_{02}$, *for all* $y \in S^n$,
and
(ii) $x^T Ay^* \lesssim_{q_{01}} w_{01}$,
$x^T Ay^* \lesssim_{q_{02}} w_{02}$, *for all* $x \in S^m$.

Further this solution of GFG is equivalent to solution of crisp linear programming problems (12) and (19) for Players I and II respectively, and that if (x^*, λ^*) is an optimal solution of problem (12), then x^* is optimal strategy for Player I and λ^* is the degree to which the actual aspiration level

$v_0 (= v_{02})$ of Player I can be achieved by selecting to play the strategy x^*. Similar interpretation can also be given to an optimal solution (y^*, η^*) of problem (19). Thus, as in classical game theory, we have following theorem in fuzzy environment.

Theorem 3.1. *The pair (12) and (19) constitutes a fuzzy primal-dual pair in the sense of modified weak duality theorem.*

Proof. The proof follows by noting that model (19) can be written as,

$$-\min\ (-\eta)$$

subject to,

$$\begin{aligned} \eta &\leq 1 - \frac{A_i y - w_{01}}{q_{01}}, \\ \eta &\leq 1 - \frac{A_i y - w_{02}}{q_{02}}, \quad (i = 1, 2, \ldots, n), \\ \sum_{j=1}^{m} y_j &= 1, \\ \eta &\leq 1, \\ y &\geq 0,\ \eta \geq 0. \end{aligned} \tag{24}$$

□

4. Numerical Example

Consider a two person zero-sum crisp matrix game G whose payoff matrix A is

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix}$$

The solution of the game G is (x^*, y^*, v^*) where $x^* = (0.67, 0.33)^T$, $y^* = (0.50, 0.50)^T$ and $v^* = 2$.

Here we choose the aspiration levels as $v_0 = 2.50, w_0 = 3, p_0 = 1, q_0 = 0.50$. Solution of this fuzzy game FG as obtained in [5] is:

Optimal solution for Player I: $(x^* = (0.67, 0.33)^T, \lambda^* = 0.50)$ and,

Optimal solution for Player II: $(y^* = (0.75, 0.25)^T, \eta^* = 1)$.

Now for the solution of this game by GFG, taking $v_{01} = 2, v_{02} = 2.50$ with $p_{01} = 0.50, p_{02} = 2$ for Player I. Then crisp equivalent of (12) becomes

$$\max\ \lambda$$

subject to,

$$\begin{aligned} -0.5\lambda + x_1 + 4x_2 &\geq 1.5, \\ -2\lambda + x_1 + 4x_2 &\geq 0.5, \\ -0.5\lambda + 3x_1 &\geq 1.5, \\ -2\lambda + 3x_1 &\geq 0.5, \\ \lambda &\leq 1, \\ x_1 + x_2 &= 1, \\ \lambda, x_1, x_2 &\geq 0. \end{aligned} \tag{25}$$

Taking $w_{01} = 3.25, w_{02} = 3$ with $q_{01} = 0.25$ and $q_{02} = 1$ for Player II.

Then crisp equivalent of (19) becomes

$$\begin{aligned}
&\max \eta\\
&\text{subject to,}\\
&\eta + y_1 + 3y_2 \le 4,\\
&0.25\eta + y_1 + 3y_2 \le 3.50,\\
&\eta + 4y_1 \le 4,\\
&0.25\eta + 4y_1 \le 3.50,\\
&\eta \le 1,\\
&y_1 + y_2 = 1,\\
&\eta, y_1, y_2 \ge 0. \qquad (26)
\end{aligned}$$

The optimal solutions of linear programming problems (25) and (26) by proposed approach of GFG are

Optimal solution for Player I: $(x^* = (0.67, 0.33)^T, \lambda^* = 0.75)$ and,

Optimal solution for Player II: $(y^* = (0.75, 0.25)^T, \eta^* = 1)$.

It is observed that the degree of attainment of fuzzy goal in GFG is better than that of [5] for the Player I, while for other player it is same.

Conclusions

In this paper, a two person zero-sum matrix game with fuzzy goals is studied using piecewise linear membership functions. The proposed method is a generalization of the linear membership function approach used in [5] to solve a matrix game with fuzzy goals. The present methodology is tested on a numerical example. In this example, the optimal solution result 0.75 as the degree of attainment of fuzzy goal by Player I as against 0.50 obtained by using [5]. However results for Player II are same in both the models.

References

[1] A. C. Cevikel and M. Ahlatcioglu, *A linear interactive solution concept for fuzzy multiobjective games*, European Journal of Pure and Applied Mathematics 35 (2010), 107-117.

[2] C. R. Bector and S. Chandra, *On duality in linear programming under fuzzy environment*, Fuzzy Sets and Systems 125 (2002), 317-325.

[3] C. R. Bector and S. Chandra, *Fuzzy Mathematical Programming and Fuzzy Matrix Games*, Springer, Berlin, Heidelberg, 2004.

[4] C. R. Bector, S. Chandra and V. Vijay, *Duality in linear programming with fuzzy parameters and matrix games with fuzzy payoffs*, Fuzzy Sets and Systems 146 (2004), 253-269.

[5] C. R. Bector, S. Chandra and V. Vijay, *Matrix games with fuzzy goals and fuzzy linear programming duality*, Fuzzy Optimization and Decision Making 3 (2004), 255-269.

[6] D. F. Li and F. X. Hong, *Solving constrained matrix games with payoffs of triangular fuzzy numbers*, Computers & Mathematics with Applications 64 (2012), 432-448.

[7] D. Pandey and S. Kumar, *Modified approach to multiobjective matrix game with vague payoffs*, Journal of International Academy of Physical Sciences 14 (2010), 149-157.

[8] D. Pandey and S. Kumar, *Fuzzy optimization of primal-dual pair using piecewise linear membership functions*, Yugoslav Journal of Operations Research 22 (2012), 97-106.

[9] G. Owen, *Game Theory*, Academic Press, San Diego, 1995.

[10] H. J. Zimmermann, *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets and Systems 1 (1978), 45-55.

[11] J. X. Nan, D. F. Li and M. J. Zhang, *A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers*, International Journal of Computational Intelligence Systems 3 (2010), 280-289.
[12] L. Campos, *Fuzzy linear programming models to solve fuzzy matrix games*, Fuzzy Sets and Systems 32 (1989), 275-289.
[13] M. Sakawa and I. Nishizaki, *Max-min solutions for fuzzy multiobjective matrix games*, Fuzzy Sets and Systems 67 (1994), 53-69.
[14] S. Bandyopadhyay and P. K. Nayak, *Matrix games with trapezoidal fuzzy pay offs*, International Journal of Engineering Research and Development 5 (2013), 21-29.
[15] T. Yang, J. P. Ignizio and H. J. Kim, *Fuzzy programming with nonlinear membership functions: Piecewise linear approximation*, Fuzzy Sets and Systems 41 (1991), 39-53.
[16] Y. W. Chen and M. Larbani, *Two-person zero-sum game approach for fuzzy multiple attribute decision making problems*, Fuzzy Sets and Systems 157 (2006), 34–51.

Department of Mathematics,, Ch. Charan Singh University, Meerut-250004, Uttar Pradesh (India)
E-mail address: drsandeepmath@gmail.com