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APPLIED SYMMETRICAL PRINCIPLE TO SOLVE SCHWARZ-CHRISTOFFEL PARAMETER PROBLEM

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Abstract. This paper adopted symmetry theory to solve the Schwarz-Christoffel parameter problem for axisymmetric polygons. Numerical conformal mappings were performed to shift the upper half-plane onto polygonal domains. Once the constraint conditions of the problem were treated in a special way such as added or deleted a little area, it turns to be a solution of a singular integral.In this paper, an auxiliary point was suggested to attach to the polygon that obeyed the principle of symmetry, which can accelerate the solving process of the singular integral. After that, several numerical examples, along with an application related to electrostatics, are provided to verify its feasibility and simplification. When the distance from the auxiliary point to polygon is controlled under 1E-08, the accuracy can be controlled within 1E-09, accuracy and consequences of the calculation basically meet the ordinary requirement.

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1. Introduction

The Schwarz-Christoffel formula (represented as Form.1) originated from the Riemann mapping theorem , which was developed by Herman Amandus Schwarz and Elwin Bruno Christoffel independently, which enable a person to carry out it optionally. Schwarz-Christoffel mapping shifts an upper half-plane onto the interior of a simple polygon. Owing to its orthogonal invariance between the transformation, it is utilized to solve the Laplace and Poisson equations and related problems in two-dimensional domains with irregular or unbounded geometry (include multiply connected domains).

Let P be a polygon in the Z plane with vertices z_i and interior angles θ_i, where θ_i $\in(0,\pi) \cup (\pi,\ 2\pi]$. And $a_i(i \geq 3, i = 1, 2, 3, \ldots, i, \ldots, n-1, n)$ is a series of points distributed on the real axis of the upper half-plane of the complex plane $\mathbb{C}$.

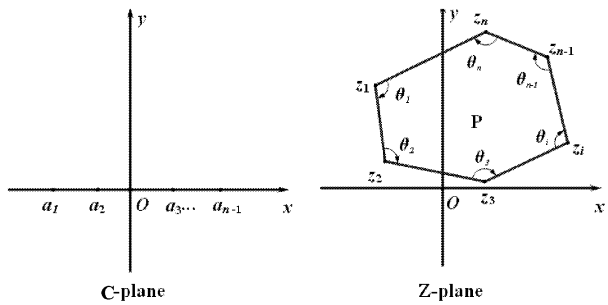

Figure 1. Sketch of Schwarz-Christoffel mapping

There exists a one-to-one conformal mapping $z = f(x)$ from the upper half plane $(\Pi + \{A \in \mathbb{C} : Im(a) \geq 0\})$ that satisfies the boundary conditions:

$$z_i = f(a_i), \quad i = 1, 2, 3, \ldots, i, \ldots, n-1; \quad z_n = f(\infty);$$

where: $a_1 < a_2 < a_{n-1} < a_n = \infty$; $\sum_{i=1}^{n} \theta_i = 2\pi$.

Then function $f(x)$ was shifted as an indefinite integral:

$$f(x) = Kg(x) + c = K \int_0^x \prod_{i=1}^{n} (x - a_i)^{\frac{\theta_i}{\pi} - 1} dx + c. \tag{1}$$

Here, a_i is the pre-images of the vertices z_i, K and c is undetermined complex constants that determined the size and position of P.

According to Riemann principle, when an upper half-plane was transformed onto the interior of a simple polygon P by Schwarz-Christoffel mapping, three of the pre-images could be assigned freely (usually specify $a_n = \infty$), the rest was determined by the geometrical shape [13] [14]. n is the total number of vertices on polygon P, when $n \geq 4$, the solution of pre-images becomes complicated. The determination of vertices after mapping is usually called as Schwarz-Christoffel parameter problem(Krantz 1999). The solution is the first stage in any SchwarzChristoffel mapping. Once the parameter problem is solved, the constant, function f and its inverse can be calculated numerically.

But Form.1 doesn't have a fixed pattern, the calculation contains parameters problem under constraint conditions and singular integral problem, which make the mapping complex that an effective numerical method must be found. L.N.Trefethen has developed a compound form of Gauss-Jacobi quadrature to evaluate the integral, which making possible high accuracy at reasonable cost and eliminate the constraints by a simple change of variables. Driscoll has expounded the numerical methodology on the solution and also developed a Schwarz-Christoffel toolbox in MATLAB for convenient use of other researchers [5] [6]. Moreover, some others also given their unique method to optimize the calculation. E.g, Mikko Nummelin prompt osculation algorithms to solve this problem at Helsinki Analysis Seminar. Calixto adopted a genetic algorithm to estimate the parameters of the Schwarz-Christoffel inverse transformation [7].And David M. Hough published an asymptotic Gauss-Jacobi quadrature error estimation method for Schwarz-Christoffel integral [8].

Symmetrization is a physical or mathematical procedure of the system which remains invariance under some transformation.Typically, the invariance is in line with a symmetry and symmetry creates a conservation laws. G.Polya and G.Szego have introduced the symmetrization of quadrilaterals, and they also consider continuous symmetrization of ring condensers with the additional property. V.N.Dubinin has discussed the change of conformal moduli of polygonal quadrilaterals under some geometric transformations. R.Kühnau has reviewed the conformal module of quadrilaterals and rings.

For the field that possesses axial character, due to its physical and geometrical symmetry, one can apply the Schwarz-Christoffel mapping.

2. Main Method on The Calculation of Schwarz-Chrisstofel Mapping

2.1. Special Way of Dealing with the Constraint Conditions.

For the purpose to simplify the calculation of the problem, initial value of a_n was set as ∞ in general, thus Form.1 could be reduced as:

$$f(x) = Kg(x) + c = K \int_0^{\varepsilon} \prod_{i=1}^{n-1} (x - a_i)^{\frac{\theta_i}{\pi} - 1} dx + c. \tag{2}$$

In Form.2, initial values could be set as :

$$a_1 = -1, \quad a_2 = 0.$$

Then $n-3$ residual values need to be calculated, which satisfied the constraint conditions :

(3) $$0 < a_3 < a_4 < \ldots < a_i < \ldots < a_{n-1} < \infty.$$

In order to speed up the procedure of solution, integral functions that fulfilled the algebraic similar conditions according to Form.1 were introduced as below:

(4)
$$g(a_i) - g(a_{i-1}) = \frac{f(a_i) - f(a_{i-1})}{K} = \int_{a_{i-1}}^{a_i} \prod_{i=1}^{n-1} (x - a_i)^{\frac{\theta_i}{\pi} - 1} dx.$$

For any group of a_i that satisfied conditions (3), through function $g(x)$ derived from Form.1, real axis on complex plane $\mathbb{C}$ were transformed to be polygon p' on complex plane Z' , the inner angles were consistent with polygon P, and polygon p' keeps similarity to polygon P in appearance. Adjusting the constants K and c, two polygons could be made congruent. And the following $n-3$ equations were established as:

(5)
$$\frac{|g(a_i) - g(a_{i-1})|}{|g(a_2) - g(a_1)|} - \frac{|z_i - z_{i-1}|}{|z_2 - z_1|} = 0, \quad i = 3, 4, \ldots, n-1.$$

In Form.5, the second quotient is the quotient of edge divided by the first edge(both belongs to polygon P), the first quotient was the quotient of edge divided by the first edge(both belongs to polygon p'). Form.5 gives the analogy between polygon P and polygon p'.

2.2. **A Solution for Singular Integral Problem.**

When function $f(x)$ was calculated every time, Form.4 must be calculated $n-2$ times.When Schwarz-Christoffel mapping parameters problem was solved , function $G(\bar{a})$

must be calculated repeatedly. Thus,the numerical integration of Form.4 is crux in the solution. Because $a_i$and a_{i-1} are singular points in the integral, so the parameters problem is also a singular integral problem. Since θ_i were interior angles of polygon P and $\theta_i \in(0,\pi) \cup (\pi, 2\pi]$, so $0 < \left|\frac{\theta_i}{\pi} - 1\right| < 1$, which indicated that the problem is a weak singular integral problem [15]. Numerical solution for this kind of singular integral usually includes three methods: 1) remove (or increase) a little area which including the singularity ; 2) use function transformation firstly and then using Gaussian quadrature formulas to calculate it; 3) calculate directly through Gaussian integral formulas (such as Gauss - Jacobi integration method).

2.3. **Attaching an Auxiliary Point to the Polygon.** Formally, the latest pre-images was setting at ∞, this may change the original symmetry and let subsequent calculations to be more complicated. Supposing an auxiliary point were added on the symmetrical axis and transformed as ∞, the amount of pre-images and vertices would correspond one by one on in graphic. After a series transformation as panning and zooming, geometry still keeps symmetrical between the Schwarz-Christoffel mapping, and that may be benefited for the follow-up physical analysis.

In order to explain the method of attaching an auxiliary point on the axial geometry that transformed by Schwarz-Christoffel mapping, an example as Fig.2 was constructed, and differentiate methods were enumerated to verify the numerical result.

In Fig.2, points a, b, c, d are vertices of the polygon on plane Z with the coordinates as : (3.5,0), (3.5,2.5), (-3.5,2.5), (-3.5,0). Edge $'ab'$ and $'cd'$ are symmetry with y axis.An auxiliary point e were setting on y axis which approaches to the midpoint of Edge $'cb'$(not on the line,

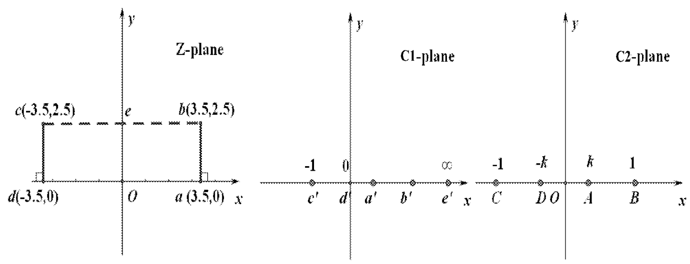

FIGURE 2. an example of axisymmetric polygon for Schwarz-Christoffel mapping

i.e, $e \to (0, 2.5))$). Through Schwarz-Christoffel mapping, the four vertices were converted onto the half-plane of $\mathbb{C}1$ plane at point $a'b'c'd'e'$ .Within it,three points were set as the initial value like:

$$c' = -1, \quad d' = 0, \quad e' = \infty.$$

In order to maintain the symmetry ,pre-images $c'd'a'b'$ were shift as points $CDAB$ on $\mathbb{C}2$ plane after a series of panning an zooming, and three points of which were setting as:

$$C = -1, \quad B = 1, \quad E = \infty. \tag{6}$$

Obviously, edge length: $L_{cd} = L_{ab}$. Due to the symmetry of the polygon, the distance would keep the same ration after transformation,that $L_{c'd'} = L_{a'b'}$. So, the remaining two points on $\mathbb{C}2$ plane could be set as:

$$-D = A = k, \quad k(0 < k < 1). \tag{7}$$

After the above steps, the problem turns out to be the solution of parameter k. And two methods on the solution of the problem were discussed on the below.

Method 1: Solved by Complex Analytic Method Directly

According to the parameters given in Fig.2, the Schwarz-Christoffel mapping, i.e,Form.2 could be shifted as:

$$f(x) = Kg(x) + c = K\int_0^{\varepsilon}\prod_{i=1}^{n-1}(x-a_i)^{\frac{\theta_i}{\pi}-1}dx + c; \tag{8}$$

$$= K\int_0^{\varepsilon}(x^2-1)^{-\frac{1}{2}}(x^2-k^2)^{-\frac{1}{2}}dx + c. \quad (n=5)$$

Each vertex of polygon P is expressed as:

$$Z_i = K\int_0^{a_i}(x^2-1)^{-\frac{1}{2}}(x^2-k^2)^{-\frac{1}{2}}dx + c. \tag{9}$$

The length of polygon's edge were expressed as:

$$|Z_{i+1}Z_i| = |Z_{i+1}-Z_i| = K\left|\int_{a_i}^{a_{i+1}}(x^2-1)^{-\frac{1}{2}}(x^2-k^2)^{-\frac{1}{2}}dx\right|. \tag{10}$$

Then, following equations could be set up as Eq.11.

$$\frac{\overline{ab}}{\overline{oa}} = 2\times\frac{\overline{ab}}{\overline{da}} = \frac{\left|\int_1^{\frac{1}{k}}(k^2x^2-1)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}}dx\right|}{\left|\int_0^{1}(k^2x^2-1)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}}dx\right|} = \frac{5}{7}. \tag{11}$$

Then set: $k' = \sqrt{1-k^2}, \quad x = 1/\sqrt{1-k'^2y^2}$,
and equ.11 could be transformed as:

(12)
$$\frac{\left| i \times \int_0^1 (k'^2y^2-1)^{-\frac{1}{2}}(y^2-1)^{-\frac{1}{2}}dy \right|}{\left| \int_0^1 (k^2x^2-1)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}}dx \right|} = \frac{F(k', \pi/2)}{F(k, \pi/2)} = \frac{5}{7}.$$

Here, complete elliptic integral of the first kind in the canonical form is:

(13) $$F(k, \pi/2) = \int_0^1 \frac{dz}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

In particular,the Schwarz-Christoffel mapping yields as a special case complete elliptic integral. And a ratio derived from Eq.12 must be put forward as:

(14) $$m = \frac{7 \times F(k', \pi/2)}{5 \times F(k, \pi/2)}.$$

The ratio indicates the accuracy of the calculation, and variable step iterative method must be adopted in the numerical solution of this equation.

Since

(15) $$\frac{\overline{oa}}{\overline{oa} + \overline{ab}} = \frac{3.5}{2.5 + 3.5} \approx 0.583,$$

so the initial value of k was set at 0.6.

The step-by-step results of iterative could be acquired in MATLAB, which are described as Table 1. After 30 iterations, the numerical solution with the accuracy of 1E-15 is :

$$k = 0.9062749012906616.$$

Method 2: Solve by The Numerical Method based on Sc-toolkit

The numerical calculation method must adopt iteration method in the solution of linear equations. In Sc-toolkit [6], Gauss-Jacobi integral method was adopted in the solution.

TABLE 1. Results of the complex analytic method

No.	k	m
1	0.6	1.59555494557737
2	0.5	1.79096619963941
3	0.8	1.22841272588751
4	0.9	1.01574804747843
5	0.91	0.990416584722543
6	0.905	1.00323841366840
7	0.906	1.00070002535597
8	0.9065	0.999426075140706
9	0.9062	1.00019082881466
10	0.90625	1.00006344990818
11	0.90628	0.999987007219460
12	0.90627	1.00001248939504
13	0.906275	0.999999748467193
14	0.9062749	1.00000000328888
15	0.90627495	0.999999875878055
16	0.90627491	0.999999977806721
17	0.906274905	0.999999990547803
...	...	...
30	0.90627490129066l6	1

Assume point$(0, 2.5000001)$was attached to the polygon as an auxiliary point, the polygon may be reconstructed in MATLAB as:

$$\texttt{p} = \texttt{polygon}([-3.5 + 2.5\texttt{i}, -3.5 + 0\texttt{i}, 3.5 + 0\texttt{i}, 3.5 + 2.5\texttt{i},$$

$$0 + 2.5000001i]);$$

When Sc-toolkit was invoked to find the solution of Schwarz-Christoffel mapping, its angles $'beta'$ was counted as:

$$[\texttt{alpha}, \texttt{isccw}, \texttt{index}] = \texttt{angle(p)}; \texttt{beta} = \texttt{alpha} - \texttt{1};$$

Then,after called function $\texttt{hpparam}()$ of sc-toolkit in MATLAB, pre-vertices z and constant K were obtained.

$$[\texttt{z}, \texttt{K}, \texttt{qdat}] = \texttt{hpparam}(\texttt{p}, \texttt{theta}).$$

According to this rule,for the example represented in Fig.2, auxiliary points e could be presetting variously for comparison. Each time,the new polygon p' constructed by vertices $abcde$ is similar to polygon p formally, and parameter k could be solved through a method based on sc-toolkit, the results are listed in Table 2.

TABLE 2. Results of the numerical calculation method

No	e	C	D	A	B	E	MRE
1	0+2.49i	-1	-0.90705945411314	0.90705945411387	1	∞	8.66E-04
2	0+2.501i	-1	-0.90619641196954	0.90619641196932	1	∞	8.66E-05
3	0+2.499i	-1	-0.90635338450486	0.90635338450487	1	∞	8.66E-05
4	0+2.5001i	-1	-0.90626705260993	0.90626705261238	1	∞	8.66E-06
5	0+2.4999i	-1	-0.90628274986521	0.90628274986685	1	∞	8.66E-06
6	0+2.50001i	-1	-0.90627411643672	0.90627411637501	1	∞	8.66E-07
7	0+2.49999i	-1	-0.90627568612812	0.90627568613458	1	∞	8.66E-07
8	0+2.500001i	-1	-0.90627482273949	0.90627482282579	1	∞	8.66E-08
9	0+2.499999i	-1	-0.90627497975518	0.90627497975518	1	∞	8.66E-08
10	0+2.5000001i	-1	-0.90627489196844	0.90627489487278	1	∞	8.68E-09
11	0+2.4999999i	-1	-0.90627490667819	0.90627491156009	1	∞	8.64E-09
12	0+2.49999999i	-1	-0.90627489510625	0.90627490926213	1	∞	7.81E-09
13	0+2.50000001i	-1	-0.90627490021979	0.90627490074830	1	∞	8.90E-10

And the final result of parameter k is:

$$\kappa \approx 0.906274901.$$

The relative error (MRE) is close to 1E-09. The comparison also demonstrates that: the closer the point e is to the middle point $(0, 2.5)$,the closer the value k is to the final result.

The reason can be summarized as follows, the pre-image of vertices e is set as infinite, and the iterative process wasn't subject to its coordinates,therefore the affect of the results is mainly reflected by the change of polygon's angle θ_1 andθ_{n-1}. If the initial angle error can be controlled, a higher accuracy of the calculation could be acquired.

3. A Comparison with the Solution of Schwarz-Christoffel Toolbox

3.1. Background and an example.

In electromagnetics area, symmetry principle has a wide application. Usually,the boundary condition of the field is complex,which makes the calculation of the field-related problem to be difficult. Schwarz-Christoffel mapping can transform polygon $p1$, constructed by the boundary especially, to be an axial geometry on the upper plane firstly, and then it was mapped within a rectangle $p2$ through elliptic functions integral. Because the magnetic field lines and magnetic potential lines always keep orthogonality,and magnetic field lines flow from the N pole to the S pole, so the magnetic field lines and magnetic equipotential lines in $p2$ could be acquired simply. After a series of inverse operations, the lines in $p1$ could be acquired relevantly. Schwarz-Christoffel mapping provides an optimization algorithms on the solution of a boundary value problem in electromagnetics.

As illustrated in Fig.3,there is a static magnetic field bounded with axial polygon $'abcd'$ on plane Z with the coordinates as :

$$(1.25, 0); (3.75, 2.5); (-3.75, 2.5); (-1.25, 0).$$

Edge $'ab'$ and $'cd'$ are the pole of the field, which is called as air-gap magnetic field. Through Schwarz-Christoffel mapping and using the numerical method elaborated above, the four vertices on plane Z are converted into four points(named as A, B, C, D individually) on the real axis of the half-plane of plane $\mathbb{C}$, with the coordinates as:

$$(-1, 0); (\kappa, 0); (\kappa, 0); (1, 0),$$

Through elliptic functions, the upper half of plane $\mathbb{C}$ could be mapped into a rectangle on plane W with the coordinates as:

$$(K, 0); (K, K'); (-K, K'); (-K, 0).$$

(Here K and K' are the values of elliptic integrals $F(k, \pi/2)$, $F(k', \pi/2)$.)

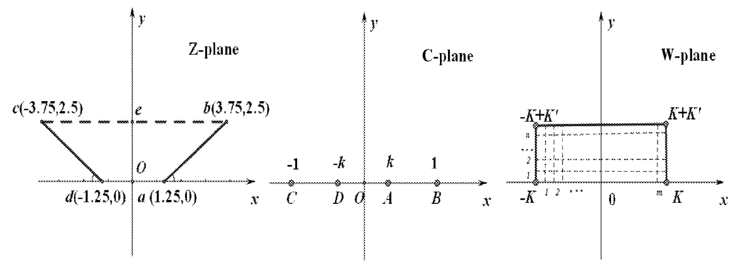

FIGURE 3. an example of axisymmetric geometry for Schwarz-Christoffel mapping

The rectangle on plane W was divided $m * n$ squares (here set $m = 20$, and the value of n is obtained by the ratio of K and K'). Of which, the lateral line represented the magnetic field lines, the longitudinal line represented the equipotential lines. The coordinates of the intersection was set as :

$$\omega = \frac{K}{m} I + i \frac{K'}{n} J,$$

Here,I, J are identification number with corresponding ranks , and corresponding inverse mapping points on plane $\mathbb{C}$ are [16]:

$$\varepsilon = k \times Sn(\omega) \tag{16}$$

Through the Schwarz-Christoffel mapping and formula(2), corresponding mapping points on plane Z could be obtained. When horizontal and vertical points were connected respectively, the schematic of magnetic field lines and equipotential line within polygon *abcd* on plane Z could be plotted.

k is elliptic parameter associated in an one-to-one manner with the geometry of the rectangle. And the Schwarz-Christoffel parameter problem woulb be in principle solved when the value of k was determined.

3.2. **Comparison and Discussion.** For the polygon in example(2), according to the above numerical methods, point $(0, 2.5000001)$ was adopted as the auxiliary point, and polygon p was reshaped in MATLAB as:

$\mathtt{p} = \mathtt{polygon}([-3.75{+}2.5\mathtt{i}, -1.25{+}0\mathtt{i}, 1.25{+}0\mathtt{i}, 3.75{+}2.5\mathtt{i},$

$$0 + 2.5000001i]);$$

Through the method mentioned above, the solution of k (the mapping point on plane $\mathbb{C}$) was:

$$\kappa = -0.49470417839404.$$

In SC Toolkit,it find prevertices on the strip firstly, then transplant to the rectangle from there. Fig.4 shows the process of inverse mapping that shift rectangle mapping to strip in SC Toolkit. Within it, L is the width of the strip(the heigth is fixed as π).

Table 3. Comparison of the results

value	Direct Method	SC Toolkit	RE
K	1.68290772118912	1.68290772316928	1.17E-09
K'	2.16606221124109	2.16606220448869	3.12E-09
$L(-log(k)/\pi)$	0.22402500387113	0.22402500404842	7.91E-10

L is linked algebraically to the conformal modulus, which is hard to seek. The relation of L and k can be described as:

$$L = -log(k)/\pi;$$

Table 3 shows the comparison of K, K' and L that solved by two different methods. The relative error (RE) was controlled 1E-08.

After the above steps of calculation, the magnetic field lines and equipotential line within the polygon p was obtained and described in the right part of Fig.5,the solid lines represented the magnetic field lines, the broken lines the equipotential lines, which also agreed with the real measurements. The left part is the result of the Schwarz-Christoffel toolbox.

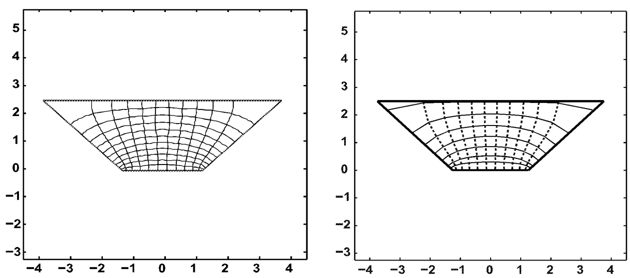

Figure 4. comparison of the solution on the Field

The result from the method promoted in this paper is unanimous agreement with the solution of SC toolbox. But in SC toolbox,the solution includes four steps as:

$$ploygon < - > half{-}palne < - > string < - > rectangle$$

When solved by SC toolbox, the inverse calcilation of the problem and other applications depended on it would be hard to carry out, Which also confined the utilization of the Schwarz- Christoffel mapping.

But,when using the method mentioned aboved to solve the peoblem, it only contains three steps as:

$$ploygon < - > half - palne < - > rectangle$$

Obviously,it has significantly reduced the amount of computation.Besides that,it simplified the processing of further research and also provided a reference for other solutions.

4. Conclusion

In essence, the parameters problem of Schwarz-Christoffel mapping is a type of weak singular integral. The numerical calculation of this problem usually includes three distinct steps. This paper treats the problem by a method of removing(or increase) a little area which including the singularity.

When the Schwarz-Chrisstofel mapping is applied on the solution of an axisymmetric polygon, an auxiliary point could be set right on the symmetry axis, which approaching to the middle point of line connected the first and latest vertex. In numerical calculation, the point was mapped to be ∞. Thereby, the mapped polygon remains the original symmetry. The calculation accuracy is consistent with the distance of the auxiliary point away from the intermediate point, when the distance is controlled within in 1E-08, the accuracy can be controlled within 1E-09. In this way, the numerical process on the boundary value problem could be simplified, and the results meet the practical requirement.

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