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ON CENTRAL FUBINI POLYNOMIALS ASSOCIATED WITH CENTRAL FACTORIAL NUMBERS OF THE SECOND KIND

DAE SAN KIM, JONGKYUM KWON, DMITRY V. DOLGY, AND TAEKYUN KIM

Abstract. In this paper, we introduce central Fubini polynomials in association with the central factorial numbers of the second kind and derive some properties of them by means of generating functions and p-adic fermionic integrals on $\mathbb{Z}_p$. In addition, we obtain some results relating central factorial numbers of the second kind and Euler numbers of the second kind.

1. Introduction and preliminaries

Let p be a fixed prime number with $p \equiv 1 (mod\ 2)$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. It is well known that Fubini polynomials are defined by the generating function as

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}, \quad \text{(see [5, 6, 8, 10, 14])}. \tag{1.1}$$

When $x = 1$, $F_n(1)$ are called ordered Bell numbers.

The Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [9, 11, 12, 15, 16, 18])}. \tag{1.2}$$

where $k \in \mathbb{N} \cup \{0\}$.

From (1.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!} &= \sum_{k=0}^{\infty} x^k k! \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} k! S_2(n,k)x^k\Big)\frac{t^n}{n!}.\end{aligned} \tag{1.3}$$

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Thus, by (1.3), we get

$$F_n(x) = \sum_{k=0}^{n} k! S_2(n,k) x^k, (n \geq 0), \text{ (see [6, 8, 10, 14, 20]).} \tag{1.4}$$

For $n \geq 0$, the central factorial numbers of the second kind are defined by

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}, (n \geq 0), \text{ (see [1-4, 7, 17, 19-21]),} \tag{1.5}$$

where $x^{[k]} = x(x + \frac{k}{2} - 1)(x + \frac{k}{2} - 2) \cdots (x - \frac{k}{2} + 1)$.
From (1.5), we note that the generating function of the central factorial numbers of the second kind is given by

$$\frac{1}{k!}(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T(n,k) \frac{t^n}{n!}, (k \geq 0). \tag{1.6}$$

By (1.6), we easily get

$$T(n,k) = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i (\frac{k}{2} - i)^n, (n, k \geq 0). \tag{1.7}$$

As is well known, the Euler polynomials are given by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ (see [12]).} \tag{1.8}$$

When $x = 0$, $E_n = E_n(0)$ are called Euler numbers.
Whereas the Euler numbers of the second kind are defined by

$$\text{sech } t = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}, \text{ (see [12]).} \tag{1.9}$$

Thus, by (1.8) and (1.9), we easily get $E_n^* = 2^n E_n(\frac{1}{2}), (n \geq 0)$.
Let f be a continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic integral on $\mathbb{Z}_p$ of f is defined by Kim as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x)(-1)^x, \text{ (see [12]).} \tag{1.10}$$

From (1.10), we have the following integral equation.

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \text{ (see [12]),} \tag{1.11}$$

where $f_1(x) = f(x+1)$ is the translation of f by $+1$.

In [14], the Fubini polynomial is represented by the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} (x(1-e^t))^y d\mu_{-1}(y) = \frac{2}{1-x(e^t-1)} = 2\sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}. \tag{1.12}$$

In this paper, we introduce central Fubini polynomials in association with the central factorial numbers of the second kind and derive some properties of them by means of generating functions and p-adic fermionic integrals on $\mathbb{Z}_p$. In addition, we obtain some results relating central factorial numbers of the second kind and Euler numbers of the second kind.

2. Central Fubini polynomials

The equation (1.4) suggests us to define the central Fubini polynomials as

$$F_n^{(C)}(x) = \sum_{k=0}^{n} k!T(n,k)x^k, (n \geq 0). \tag{2.1}$$

From (2.1), we can derive the following generating function of $F_n^{(C)}(x), (n \geq 0)$

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(C)}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} x^k k! T(n,k)\Big)\frac{t^n}{n!} = \sum_{k=0}^{\infty} x^k k! \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} x^k (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \frac{1}{1-x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}.\end{aligned} \tag{2.2}$$

By (2.2), we get

$$\frac{1}{1-x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} F_n^{(C)}(x)\frac{t^n}{n!}. \tag{2.3}$$

In (2.3), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(C)}(x)\frac{t^n}{n!} &= \frac{1}{1-xe^{-\frac{t}{2}}(e^t-1)} = \sum_{k=0}^{\infty} x^k e^{-\frac{kt}{2}}(e^t-1)^k \\ &= \sum_{k=0}^{\infty} x^k k! \sum_{n=k}^{\infty}\left(\sum_{i=k}^{n}\binom{n}{i}(-\frac{k}{2})^{n-i}S_2(i,k)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{i=k}^{n} x^k k!\binom{n}{i}(-\frac{k}{2})^{n-i}S_2(i,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By comparing the coefficients on both sides of (2.4), we get

$$F_n^{(C)}(x) = \sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}k!(-\frac{k}{2})^{n-i}S_2(i,k)x^k, (n \geq 0). \tag{2.5}$$

From (1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!} &= \frac{1}{1-x(e^t-1)} = \frac{1}{1-xe^{\frac{t}{2}}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\ &= \sum_{k=0}^{\infty}x^k k!\sum_{n=k}^{\infty}\left(\sum_{i=k}^{n}\binom{n}{i}T(i,k)(\frac{k}{2})^{n-i}\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}x^k k!T(i,k)(\frac{k}{2})^{n-i}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Comparing the coefficients on both sides of (2.6), we obtain

$$F_n(x) = \sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}k!(\frac{k}{2})^{n-i}T(i,k)x^k, (n \geq 0). \tag{2.7}$$

By (1.9), we get

$$\begin{aligned}\sum_{n=0}^{\infty}E_n^*\frac{t^n}{n!} &= \text{sech } t = \frac{2}{e^t+e^{-t}} = \frac{2}{2+(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^2}\\ &= \frac{1}{1+\frac{1}{2}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^2} = \sum_{k=0}^{\infty}(-\frac{1}{2})^k(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^{2k}\\ &= \sum_{k=0}^{\infty}(-\frac{1}{2})^k(2k)!\sum_{n=2k}^{\infty}T(n,2k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-\frac{1}{2})^k(2k)!T(n,2k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

By comparing the coefficients on both sides of (2.8), we get

$$E_n^* = \sum_{k=0}^{n}(-\frac{1}{2})^k(2k)!T(n,2k) = \sum_{k=0}^{[\frac{n}{2}]}(-\frac{1}{2})^k(2k)!T(n,2k), (n \geq 0). \tag{2.9}$$

From (1.11), we can derive the following equation (2.10):

$$\begin{aligned}\int_{\mathbb{Z}_p}(x(e^{-\frac{t}{2}}-e^{\frac{t}{2}}))^y d\mu_{-1}(y) &= \frac{2}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\ &= 2\sum_{n=0}^{\infty}F_n^{(C)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{Z}_p}(x(e^{-\frac{t}{2}}-e^{\frac{t}{2}}))^y d\mu_{-1}(y) &= \int_{\mathbb{Z}_p}(-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}}))^y d\mu_{-1}(y)\\ &= \int_{\mathbb{Z}_p}(-x(e^{-(-\frac{t}{2})}-e^{-\frac{t}{2}}))^y d\mu_{-1}(y) \qquad (2.11)\\ &= 2\sum_{n=0}^{\infty}(-1)^n F_n^{(C)}(-x)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by (2.10) and (2.11), we get

$$F_n^{(C)}(x)=(-1)^n F_n^{(C)}(-x),(n\geq 0).$$

From (1.11), we can derive the following equation (2.12).

$$\int_{\mathbb{Z}_p}(x(e^{-\frac{t}{2}}-e^{\frac{t}{2}}))^{y+1}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(x(e^{-\frac{t}{2}}-e^{\frac{t}{2}}))^y d\mu_{-1}(y)=2. \qquad (2.12)$$

By (2.12), we easily get

$$x(e^{-\frac{t}{2}}-e^{\frac{t}{2}})\frac{2}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}+\frac{2}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}=2. \qquad (2.13)$$

Thus, from (2.13), we have

$$\begin{aligned}1 &= x\sum_{k=0}^{\infty}\Big((-\frac{1}{2})^k-(\frac{1}{2})^k\Big)\frac{t^k}{k!}\sum_{m=0}^{\infty}F_m^{(C)}(x)\frac{t^m}{m!}+\sum_{n=0}^{\infty}F_n^{(C)}(x)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\Big\{x\sum_{m=0}^{n}\binom{n}{m}F_m^{(C)}(x)\Big((-\frac{1}{2})^{n-m}-(\frac{1}{2})^{n-m}\Big)+F_n^{(C)}(x)\Big\}\frac{t^n}{n!} \qquad (2.14)\\ &= \sum_{n=0}^{\infty}\Big\{F_n^{(C)}(x)-x\sum_{m=0}^{n}\binom{n}{m}F_m^{(C)}(x)\Big((\frac{1}{2})^{n-m}-(-\frac{1}{2})^{n-m}\Big)\Big\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides of (2.14), we have

$$F_n^{(C)}(x)-x\sum_{m=0}^{n}\binom{n}{m}F_m^{(C)}(x)\Big((\frac{1}{2})^{n-m}-(-\frac{1}{2})^{n-m}\Big)=\begin{cases}1, & \text{if } n=0,\\ 0, & \text{if } n\geq 1,\end{cases} \qquad (2.15)$$

For $n\in\mathbb{N}$, by (2.40), we get

$$F_n^{(C)}(x)=x\sum_{m=0}^{n}\binom{n}{m}F_m^{(C)}(x)\Big((\frac{1}{2})^{n-m}-(-\frac{1}{2})^{n-m}\Big). \qquad (2.16)$$

Thus, for $n \in \mathbb{N}$, we have

$$\begin{aligned} F_n^{(C)}(x) &= x\sum_{m=0}^{n-1}\binom{n}{m}F_m^{(C)}(x)\Big((\frac{1}{2})^{n-m}-(-\frac{1}{2})^{n-m}\Big) \\ &= x\sum_{m=0}^{n-2}\binom{n}{m}F_m^{(C)}(x)\Big((\frac{1}{2})^{n-m}-(-\frac{1}{2})^{n-m}\Big)+xnF_{n-1}^{(C)}(x) \end{aligned} \tag{2.17}$$

By (2.17), we get

$$F_n^{(C)}(x)-xnF_{n-1}^{(C)}(x) = x\sum_{m=0}^{n-2}\binom{n}{m}F_m^{(C)}(x)\Big((\frac{1}{2})^{n-m}-(-\frac{1}{2})^{n-m}\Big), \tag{2.18}$$

where $n \in \mathbb{N}$ with $n \geq 2$.

Making use of (2.34) in [13], we observe that

$$\begin{aligned} \int_{\mathbb{Z}_p}\sinh xt\; d\mu_{-1}(x) &= -\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} = -\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2\cosh\frac{t}{2}} \\ &= -\frac{1}{2}\sum_{l=0}^{\infty}T(l,1)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{1}{2^m}E_m^*\frac{t^m}{m!} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}E_m^*\frac{1}{2^m}T(n-m,1)\binom{n}{m}\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

On the other hand,

$$\int_{\mathbb{Z}_p}\sinh xt\; d\mu_{-1}(x) = \sum_{n=1}^{\infty}\frac{E_{2n-1}}{(2n-1)!}t^{2n-1}. \tag{2.20}$$

Therefore, by (2.19) and (2.20), we get

$$\sum_{m=0}^{2n-1}(\frac{1}{2})^m\binom{2n-1}{m}E_m^*T(2n-1-m,1) = -2E_{2n-1}, (n \geq 1), \tag{2.21}$$

and

$$\sum_{m=0}^{2n}(\frac{1}{2})^m\binom{2n}{m}E_m^*T(2n-m,1) = 0, (n \geq 0)). \tag{2.22}$$

Now, we observe that

$$\int_{\mathbb{Z}_p}(1-e^t)^x d\mu_{-1}(x) = \frac{2}{1-(e^t-1)} = 2\sum_{n=0}^{\infty}F_n(1)\frac{t^n}{n!}, \tag{2.23}$$

where $F_n(1)$ are the ordered Bell numbers.

It is not difficult to show that

$$\frac{2}{1-(e^t-1)} = \frac{2}{1-e^{\frac{t}{2}}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = 2\sum_{k=0}^{\infty} e^{\frac{k}{2}t}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k$$

$$= 2\sum_{k=0}^{\infty} k!\left(\sum_{n=k}^{\infty}\sum_{i=k}^{n}\binom{n}{i}T(i,k)(\frac{k}{2})^{n-i}\right)\frac{t^n}{n!} \tag{2.24}$$

$$= 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}k!T(i,k)(\frac{k}{2})^{n-i}\right)\frac{t^n}{n!}.$$

On the other hand,

$$\frac{2}{1-(e^t-1)} = 2\sum_{k=0}^{\infty} k!\frac{1}{k!}(e^t-1)^k = 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} S_2(n,k)k!\right)\frac{t^n}{n!}. \tag{2.25}$$

Therefore, by (2.24) and (2.25), we get

$$\sum_{k=0}^{n} S_2(n,k)k! = \sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}k!T(i,k)(\frac{k}{2})^{n-i}, (n \geq 0). \tag{2.26}$$

and

$$F_n(1) = \sum_{k=0}^{n} k!S_2(n,k), \quad \text{(see (1.4))}. \tag{2.27}$$

In view of (2.27), we may define the central ordered Bell numbers as

$$F_n^{(C)}(1) = \sum_{k=0}^{n} k!T(n,k), (n \geq 0). \tag{2.28}$$

From (2.28), the generating function of central ordered Bell numbers is easily seen to be

$$\sum_{n=0}^{\infty} F_n^{(C)}(1)\frac{t^n}{n!} = \frac{1}{1-(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}. \tag{2.29}$$

Also, from (2.5) and (2.7), we note that

$$F_n^{(C)}(1) = \sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}k!(-\frac{k}{2})^{n-i}S_2(i,k), \tag{2.30}$$

and

$$F_n(1) = \sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}k!(\frac{k}{2})^{n-i}T(i,k). \tag{2.31}$$

By (1.11), we get

$$\int_{\mathbb{Z}_p}(1-e^t)^{x+1}d\mu_{-1}(x) + \int_{\mathbb{Z}_p}(1-e^t)^{x}d\mu_{-1}(x) = 2. \tag{2.32}$$

Thus, we have

$$(1-e^t)\frac{2}{1-(e^t-1)}+\frac{2}{1-(e^t-1)}=2. \tag{2.33}$$

Clearly, (2.33) is equivalent to

$$\frac{2}{1-(e^t-1)}-\frac{e^t}{1-(e^t-1)}=1. \tag{2.34}$$

Now, (1.12) and (2.34), we obtain

$$2F_n(1)-\sum_{l=0}^{n}\binom{n}{l}F_l(1)=\begin{cases}1, & \text{if } n=0,\\ 0, & \text{if } n\geq 1.\end{cases} \tag{2.35}$$

For $n\in\mathbb{N}$, by (2.35), we get

$$F_n(1)=\sum_{l=0}^{n-1}\binom{n}{l}F_l(1). \tag{2.36}$$

From (1.11), we can derive the following equation.

$$\int_{\mathbb{Z}_p}(e^{-\frac{t}{2}}-e^{\frac{t}{2}})^{x+1}d\mu_{-1}(x)+\int_{\mathbb{Z}_p}(e^{-\frac{t}{2}}-e^{\frac{t}{2}})^{x}d\mu_{-1}(x)=2. \tag{2.37}$$

Thus, we have

$$\int_{\mathbb{Z}_p}(e^{-\frac{t}{2}}-e^{\frac{t}{2}})^{x}du_{-1}(x)=\frac{2}{1-(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}=2\sum_{n=0}^{\infty}F_n^{(C)}(1)\frac{t^n}{n!}. \tag{2.38}$$

By (2.37) and (2.38), we get

$$\sum_{n=0}^{\infty}\left(F_n^{(C)}(1)+\sum_{m=0}^{n}\binom{n}{m}F_m^{(C)}(1)\Big((-\frac{1}{2})^{n-m}-(\frac{1}{2})^{n-m}\Big)\right)\frac{t^n}{n!}=1. \tag{2.39}$$

Comparing the coefficients on both sides of (2.41), we have

$$F_n^{(C)}(1)+\sum_{m=0}^{n}\binom{n}{m}F_m^{(C)}(1)\Big((-\frac{1}{2})^{n-m}-(\frac{1}{2})^{n-m}\Big)=\begin{cases}1, & \text{if } n=0,\\ 0, & \text{if } n\geq 1.\end{cases} \tag{2.40}$$

For $n\in\mathbb{N}$ with $n\geq 2$, we have

$$F_n^{(C)}(1)-nF_{n-1}^{(C)}(1)=\sum_{m=0}^{n-2}\binom{n}{m}F_m^{(C)}(1)\Big((\frac{1}{2})^{n-m}-(-\frac{1}{2})^{n-m}\Big). \tag{2.41}$$

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Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea(corresponding)
E-mail address: mathkjk26@gnu.ac.kr

Hanrimwon, Kwangwoon University, Seoul, 139-701, Republic of Korea.
E-mail address: dvdolgy@gmail.com

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea(corresponding)
E-mail address: tkkim@kw.ac.kr