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A NOTE ON CENTRAL FACTORIAL NUMBERS

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Abstract. In this paper, we study the central factorial numbers of the second kind. In particular, we derive some identities and recurrence relations arising from the generating function. In addition, we will give some applications related to special polynomials and numbers.

1. Introduction

For $n \in \mathbb{N} \cup \{0\}$, it is well known that the stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (see[4,5,6,13,16]). \tag{1.1}$$

Note that

$$S_1(n+1,k) = S_1(n,k-1) - nS_1(n,k), \quad (1 \le k \le n). \tag{1.2}$$

From (1.1), we can derive the following generating function to be

$$\frac{1}{k!}\Big(\log(1+t)\Big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (see[7,8,9,10,11]). \tag{1.3}$$

The stirling number of the second kind is defined as

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \in \mathbb{N} \cup \{0\}). \tag{1.4}$$

By (1.4), we easily get

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (k \in \mathbb{N} \cup \{0\}), \quad (see[4,6,7,8,9]). \tag{1.5}$$

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The central factorial $x^{[n]}$ is defined by

$$\begin{aligned} x^{[n]} &= x(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdots(x-\frac{n}{2}+1) \\ &= x\cdot(x+\frac{n}{2}-1)_{n-1}, \quad (n\geq 1), \quad (see[1,2,3,4,15]). \end{aligned} \tag{1.6}$$

It is known that, for all nonnegative integer n and k $(k \leq n)$, the central factorial number $t(n,k)$ of the first kind is defined by

$$x^{[n]} = \sum_{k=0}^{n} t(n,k)x^k, \quad (see[4,5,12,14,15]). \tag{1.7}$$

Thus, we have

$$\begin{aligned} \sum_{k=0}^{n} t(n,k)x^k &= x^{[n]} = x\Big(x+\frac{n}{2}-1\Big)\Big(x+\frac{n}{2}-2\Big)\cdots\Big(x-\frac{n}{2}+2\Big)\Big(x-\frac{n}{2}+1\Big) \\ &= x\Big(x+\frac{n-2}{2}\Big)\Big(x+\frac{n-2}{2}-1\Big)\cdots\Big(x-\frac{n-2}{2}+1\Big)\Big(x-\frac{n-2}{2}\Big) \\ &= x\Big(x+\frac{n-2}{2}-1\Big)\cdots\Big(x-\frac{n-2}{2}+1\Big)\Big(x^2-(\frac{n-2}{2})^2\Big) \\ &= \sum_{k=0}^{n-2} t(n-2,k)x^k\Big(x^2-\frac{(n-2)^2}{4}\Big) \\ &= \sum_{k=0}^{n}\Big\{t(n-2,k-2)-\frac{1}{4}(n-2)^2t(n-2,k)\Big\}x^k, \end{aligned} \tag{1.8}$$

where $n \in \mathbb{N}$ with $n \geq 2$.

By (1.8), we easily get

$$t(n,k) = t(n-2,k-2) - \frac{1}{4}(n-2)^2 t(n-2,k), \quad (n\geq 2), \quad (see[3,4]). \tag{1.9}$$

It is not difficult to show that the generating function of the central number of the first kind is given by

$$\frac{1}{k!}\Big(2\log\Big(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\Big)\Big)^k = \sum_{n=k}^{\infty} t(n,k)\frac{t^n}{n!}, \quad (see[17,18]), \tag{1.10}$$

where $t \in \mathbb{C}$ with $|t| < 1$.

From (1.9), we note that

$$t(2n,1) = 0 \ \ and \ \ t(2n+1,1) = (-1)^n \frac{(2n)!)^2}{2^{4n}(n!)^2}. \tag{1.11}$$

On the other hand,

$$x^{[2n]} = x(x+n-1)(x+n-2)\cdots x(x+1)\cdots(x-1)\cdots(x-n+1)$$
$$= x^2\big(x^2-(n-1)^2\big)\big(x^2-(n-2)^2\big)\cdots\big(x^2-2^2\big)\big(x^2-1^2\big). \qquad (1.12)$$

Thus, by (1.12), we get

$$\big(x^2-1^2\big)\big(x^2-2^2\big)\cdots\big(x^2-(n-1)^2\big) = \sum_{k=1}^{2n} t(2n,2k)x^{2k-2}, \quad (n\in\mathbb{N}). \qquad (1.13)$$

For all nonnegative integers n, k $(k \le n)$, central factorial numbers of the second kind are defined by the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} T(n,k)x^{[k]}, \quad (see[4,14]). \qquad (1.14)$$

Note that

$$\begin{aligned} x^n &= x^2x^{n-2} = x^2\sum_{k=0}^{n-2} T(n-2,k)x^{[k]} \\ &= \sum_{k=0}^{n-2} T(n-2,k)x^{[k]}\big(x^2-\frac{k^2}{4}+\frac{k^2}{4}\big) \\ &= \sum_{k=0}^{n-2} T(n-2,k)x^{[k]}\Big(x^2-\frac{k^2}{4}\Big)+\frac{1}{4}\sum_{k=0}^{n-2}k^2T(n-2,k)x^{[k]} \\ &= \sum_{k=0}^{n-2} T(n-2,k)x^{[k+2]}+\frac{1}{4}\sum_{k=0}^{n-2}k^2T(n-2,k)x^{[k]} \\ &= \sum_{k=2}^{n} T(n-2,k-2)x^{[k]}+\sum_{k=0}^{n-2}\frac{k^2}{4}T(n-2,k)x^{[k]} \\ &= \sum_{k=0}^{n}\Big\{T(n-2,k-2)+\frac{k^2}{4}T(n-2,k)\Big\}x^{[k]}. \end{aligned} \qquad (1.15)$$

By (1.14) and (1.15), we easily get

$$T(n,k) = T(n-2,k-2)+\frac{k^2}{4}T(n-2,k), \quad (n,k\ge 2). \qquad (1.16)$$

From (1.6), we note that the central factorial $x^{[n]}$ is associated sheffer sequence which is given by

$$x^{[n]} \sim (1, 2\sinh 2t), \quad (n\ge 0), \quad (see[15]). \qquad (1.17)$$

Thus, by (1.10),(1.17), we get

$$\sum_{n=0}^{\infty} x^{[n]}\frac{t^n}{n!} = e^{x\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right)} = \left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)^{2x}. \tag{1.18}$$

It is well known that

$$\frac{x^{2k}}{\left(1-x^2\right)\left(1-(2x)^2\right)\cdots\left(1-(kx)^2\right)} = \sum_{n=0}^{\infty} T(2n,2k)x^{2n}, \quad (see[3,4]). \tag{1.19}$$

In this paper, we study some properties of central factorial numbers of the second kind. In particular, we derive some identities and recurrence relations arising from the generating function. In addition, we give some relations between the central numbers of the second kind and special numbers.

2. Central factorial numbers of the second kind

Let

$$\frac{x^{2k}}{\left(1-x^2\right)\left(1-(2x)^2\right)\cdots\left(1-(kx)^2\right)} = \sum_{l=0}^{k} \frac{A_l}{1-(lx)^2}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} A_l &= \frac{1}{l^2(l^2-1^2)(l^2-2^2)\cdots(l^2-(l-1)^2)(l^2-(l+1)^2)\cdots(l^2-k^2)} \\ &= \prod_{i=1,i\neq l}^{k}\left(\frac{1}{l^2-i^2}\right)\frac{1}{l^2}, \quad (k\in\mathbb{N}). \end{aligned}$$

From (1.19) and (2.1), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} T(2n,2k)x^{2n} &= \sum_{l=0}^{k}\frac{1}{1-(lx)^2}\prod_{i=1,i\neq l}^{k}\left(\frac{1}{l^2-i^2}\right)\frac{1}{l^2} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{k} l^{2n-2}\Big(\prod_{i=1,i\neq l}^{k}\left(\frac{1}{l^2-i^2}\right)\Big)\right)x^{2n}. \end{aligned} \tag{2.2}$$

Therefore, by comparing the coefficients on the both sides of (2.2), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$ and $k \geq 0$, we have.*

$$T(2n,2k) = \sum_{l=0}^{k} l^{2n-2}\Big(\prod_{i=1,i\neq l}^{k}\left(\frac{1}{l^2-i^2}\right)\Big).$$

Let

$$f(t) = 2\log\Big(\frac{t}{2} + \sqrt{1+\frac{t^2}{4}}\Big).$$

Then the inverse function of $f(t)$ is given by

$$f^{-1}(t) = e^{\frac{t}{2}} - e^{-\frac{t}{2}}.$$

We note that the generating function of the central factorial number of the second kind is the inverse function of the generating function of the central factorial number of the first kind which is given by

$$\frac{1}{k!}(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}. \tag{2.3}$$

From (2.3), we have

$$\begin{aligned}\sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!} &= \frac{1}{k!}(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \frac{1}{k!}e^{-\frac{k}{2}t}(e^t-1)^k \\ &= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}e^{(i-\frac{k}{2})t} \\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\big(i-\frac{k}{2}\big)^n\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Thus, by (2.4), we have

$$T(n,k) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\big(i-\frac{k}{2}\big)^n, \quad (n,k\geq 0). \tag{2.5}$$

Now, we observe that

$$\begin{aligned}\frac{1}{k!}e^{-\frac{k}{2}t}(e^t-1)^k &= e^{-\frac{k}{2}t}\frac{1}{k!}(e^t-1)^k \\ &= \Big(\sum_{l=0}^{\infty}\big(-\frac{k}{2}\big)^l\frac{t^l}{l!}\Big)\Big(\sum_{m=k}^{\infty}S_2(m,k)\frac{t^m}{m!}\Big) \\ &= \sum_{n=k}^{\infty}\Big(\sum_{m=k}^{n}S_2(m,k)(-1)^{n-m}\big(\frac{k}{2}\big)^{n-m}\binom{n}{m}\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

From (2.3) and (2.6), we have

$$\begin{aligned} T(n,k) &= \sum_{m=k}^{n} \binom{n}{m} \Big(-\frac{1}{2}\Big)^{n-m} S_2(m,k) k^{n-m} \\ &= \sum_{m=0}^{n-k} \binom{n}{m} \Big(-\frac{1}{2}\Big)^{m} S_2(n-m,k) k^{m}. \end{aligned} \tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *For $n, k \geq 0$, we have*

$$\begin{aligned} T(n,k) &= \sum_{m=0}^{n-k} \binom{n}{m} \Big(-\frac{1}{2}\Big)^{m} S_2(n-m,k) k^{m} \\ &= \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \Big(i - \frac{k}{2}\Big)^{n}. \end{aligned}$$

It is well known that the following generating function of the stirling number of the second kind is given by

$$\begin{aligned} \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} &= \frac{1}{k!} \sum_{l=0}^{k} \frac{1}{1-lt} \binom{k}{l} (-1)^{k-l} \\ &= \sum_{n=0}^{\infty} \Big(\frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} l^n \Big) t^n \\ &= \sum_{n=0}^{\infty} \Big(\frac{1}{k!} \Delta^k 0^n \Big) t^n = \sum_{n=k}^{\infty} S_2(n,k) t^n, \end{aligned} \tag{2.8}$$

where Δ is difference operator with $\Delta f(x) = f(x+1) - f(x)$.

Now, we observe that

$$\begin{aligned} &\frac{(-1)^k t^{2k}}{\big(1-t^2\big)\big(1-(2t)^2\big)\cdots\big(1-(kt)^2\big)} \\ &= \frac{t^k \cdot (-t)^k}{(1-t)(1-2t)\cdots(1-kt)(1+t)(1+2t)\cdots(1+kt)} \\ &= \Big(\sum_{l=0}^{\infty} S_2(l,k) t^l \Big) \Big(\sum_{j=0}^{\infty} S_2(j,k) (-1)^j t^j \Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{l=0}^{n} S_2(l,k) S_2(n-l,k) (-1)^{n-l} \Big) t^n. \end{aligned} \tag{2.9}$$

Thus, by (2.9), we get

$$\frac{t^{2k}}{\big(1-t^2\big)\big(1-(2t)^2\big)\cdots\big(1-(kt)^2\big)}=\sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}S_2(l,k)S_2(n-l,k)(-1)^{k-l}\Big)t^n. \tag{2.10}$$

From (1.19) and (2.10), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}T(2n,2k)t^{2n}&=\frac{t^{2k}}{\big(1-t^2\big)\big(1-(2t)^2\big)\cdots\big(1-(kt)^2\big)}\\&=\sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}S_2(l,k)S_2(n-l,k)(-1)^{k-l}\Big)t^n\\&=\sum_{n=0}^{\infty}\Big(\sum_{l=0}^{2n}S_2(l,k)S_2(2n-l,k)(-1)^{k-l}\Big)t^{2n}\\&\quad+\sum_{n=0}^{\infty}\Big(\sum_{l=0}^{2n+1}S_2(l,k)S_2(2n+1-l,k)(-1)^{k-l}\Big)t^{2n+1}.\end{aligned} \tag{2.11}$$

Therefore, by comparing the coefficients on the both sides of (2.11), we obtain the following theorem.

Theorem 2.3. *For $n,k\geq 0$, we have*

$$\sum_{l=0}^{2n}S_2(2n-l,k)S_2(l,k)(-1)^{k-l}=T(2n,2k),$$

and

$$\sum_{l=0}^{2n+1}S_2(2n+1-l,k)S_2(l,k)(-1)^{k-l}=0.$$

Now, we observe that

$$\begin{aligned}\frac{1}{k!}(e^t-1)^k&=\frac{1}{k!}e^{\frac{k}{2}t}\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)^k=\Big(\sum_{m=0}^{\infty}\frac{1}{m!}\big(\frac{k}{2}\big)^m t^m\Big)\Big(\sum_{l=k}^{\infty}T(l,k)\frac{t^l}{l!}\Big)\\&=\sum_{n=k}^{\infty}\Big(\sum_{l=k}^{n}\binom{n}{l}T(l,k)\big(\frac{k}{2}\big)^{n-l}\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

Therefore, by (1.5) and (2.12), we obtain the following theorem.

Theorem 2.4. *For $n,k\geq 0$, we have*

$$S_2(n,k)=\sum_{l=k}^{n}\binom{n}{l}\big(\frac{k}{2}\big)^{n-l}T(l,k).$$

From (2.3), we note that

$$\begin{aligned}\sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\Big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\Big)^k = \frac{1}{k!}e^{-\frac{k}{2}t}\Big(e^t - 1\Big)^k \\ &= \Big(\sum_{l=0}^{\infty}\big(-\frac{k}{2}\big)^l\frac{t^l}{l!}\Big)\Big(\sum_{i=k}^{\infty}S_2(i,k)\frac{t^i}{i!}\Big) \\ &= \sum_{n=k}^{\infty}\Big(\sum_{i=k}^{n}\binom{n}{i}\big(-\frac{1}{2}\big)^{n-i}S_2(i,k)k^{n-i}\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

Therefore, by comparing the coefficients on the both sides of (2.13), we obtain the following theorem.

Theorem 2.5. *For $n, k \geq 0$, we have*

$$\begin{aligned}T(n,k) &= \sum_{i=k}^{n}\binom{n}{i}\big(-\frac{1}{2}\big)^{n-i}S_2(i,k)k^{n-i} \\ &= \sum_{i=0}^{n-k}\binom{n}{i}(-1)^i\big(\frac{1}{2}\big)^i S_2(n-i,k)k^i.\end{aligned}$$

The central difference operator δ is defined as

$$\delta f(x) = f(x+\frac{1}{2}) - f(x-\frac{1}{2}). \tag{2.14}$$

From (2.14), we note that

$$\begin{aligned}\delta^2 f(x) &= \delta\big(\delta f(x)\big) = \delta\big(f(x+\frac{1}{2}) - f(x-\frac{1}{2})\big) = f(x+1) - f(x) \\ &\quad - f(x) + f(x-1) = \binom{2}{2}f(x+1) - \binom{2}{1}f(x) + \binom{2}{0}f(x-1),\end{aligned}$$

and

$$\begin{aligned}\delta^3 f(x) &= f(x+\frac{3}{2}) - f(x+\frac{1}{2}) - 2f(x+\frac{1}{2}) + 2f(x-\frac{1}{2}) + f(x-\frac{1}{2}) - f(x-\frac{3}{2}) \\ &= \binom{3}{3}f(x+\frac{3}{2}) - \binom{3}{2}f(x+\frac{1}{2}) + \binom{3}{1}f(x-\frac{1}{2}) - \binom{3}{0}f(x-\frac{3}{2}).\end{aligned}$$

Continuing this process, we obtain

$$\delta^k f(x) = \sum_{l=0}^{k}\binom{k}{l}f(x+l-\frac{k}{2})(-1)^{k-l}. \tag{2.15}$$

From (2.3), we note that

$$\begin{aligned}\sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\Big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\Big)^k = \frac{1}{k!}e^{-\frac{k}{2}t}(e^t-1)^k \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{\left(l-\frac{k}{2}\right)t} \\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(l-\frac{k}{2}\big)^n\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.16}$$

Let us take $f(x)=x^n$, $(n\ge 0)$. Then, by (2.15), we get

$$\delta^k 0^n = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(l-\frac{k}{2}\big)^n, \quad (n,k\ge 0). \tag{2.17}$$

From (2.16) and (2.17), we have

$$\sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\Big(\frac{1}{k!}\delta^k 0^n\Big)\frac{t^n}{n!}. \tag{2.18}$$

Thus, by comparing the coefficients on the both sides of (2.18), we obtain the following theorem.

Theorem 2.6. *For $n,k\ge 0$, we have*

$$\frac{1}{k!}\delta^k 0^n = \begin{cases} T(n,k), & \text{if}\, n\ge k, \\ 0, & \text{if}\, k>n. \end{cases}$$

Now, we define the central polynomials of the second kind which are given by the generating function to be

$$\frac{1}{k!}e^{xt}\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)^k = \sum_{n=k}^{\infty} T(n,k|x)\frac{t^n}{n!}. \tag{2.19}$$

Note that

$$\begin{aligned}\frac{1}{k!}e^{xt}\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)^k &= \frac{1}{k!}\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)^k e^{xt} = \Big(\sum_{l=k}^{\infty} T(l,k)\frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty} x^m\frac{t^m}{m!}\Big) \\ &= \sum_{n=k}^{\infty}\Big\{\sum_{l=k}^{n}\binom{n}{l}T(l,k)x^{n-l}\Big\}\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Thus, by (2.19) and (2.20), we get

$$T(n,k|x) = \sum_{l=k}^{n}\binom{n}{l}T(l,k)x^{n-l} \quad (n,k\ge 0). \tag{2.21}$$

When $x=0$, we get $T(n,k|0)=T(n,k)$.

From (2.19), we have

$$
\begin{aligned}
&\frac{1}{k!}e^{xt}\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)^k=\frac{1}{k!}e^{\left(x-\frac{k}{2}\right)t}(e^t-1)^k\\
&=\frac{1}{k!}e^{\left(x-\frac{k}{2}\right)t}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt}=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{\left(x-\frac{k}{2}+l\right)t}\\
&=\sum_{n=0}^{\infty}\Big(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(x-\frac{k}{2}+l\big)^n\Big)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\Big(\frac{1}{k!}\delta^k x^n\Big)\frac{t^n}{n!}.
\end{aligned}
\tag{2.22}
$$

Therefore, by (2.19) and (2.22), we obtain the following theorem.

Theorem 2.7. *For $n, k \geq 0$, we have*

$$
\frac{1}{k!}\delta^k x^n=\begin{cases} T(n,k|x), & \text{if}\, n\geq k,\\ 0, & \text{if}\, k>n.\end{cases}
$$

Now, we observe that

$$
\begin{aligned}
\delta^k x^{n+1}&=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(x-\frac{k}{2}+l\big)^{n+1}\\
&=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(x-\frac{k}{2}+l\big)^{n}\big(x-\frac{k}{2}+l\big)\\
&=\big(x-\frac{k}{2}\big)\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(x-\frac{k}{2}+l\big)^{n}\\
&+\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l\big(x-\frac{k}{2}+l\big)^{n}\\
&=\big(x-\frac{k}{2}\big)\delta^k x^n+k\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{k-l}\big(x-\frac{k}{2}+l\big)^{n}\\
&=\big(x-\frac{k}{2}\big)\delta^k x^n+k\sum_{l=0}^{k}\Big\{\binom{k}{l}-\binom{k-1}{l}\Big\}(-1)^{k-l}\big(x-\frac{k}{2}+l\big)^{n}\\
&=\big(x-\frac{k}{2}\big)\delta^k x^n+k(\delta^k x^n+\delta^{k-1}x^n).
\end{aligned}
\tag{2.23}
$$

From (2.23), we note that

$$
\delta^k x^{n+1}=\big(x-\frac{k}{2}\big)\delta^k x^n+k(\delta^k x^n+\delta^{k-1}x^n). \tag{2.24}
$$

For $n \geq k$, by **Theorem 2.7**, we get

$$\begin{aligned} T(n+1,k|x) &= \frac{1}{k!}\delta^k x^{n+1} = \frac{1}{k!}\Big\{\big(x-\frac{k}{2}\big)\delta^k x^n + k(\delta^k x^n + \delta^{k-1}x^n)\Big\} \\ &= \big(x-\frac{k}{2}\big)\frac{1}{k!}\delta^k x^n + k\frac{1}{k!}\delta^k x^n + \frac{1}{(k-1)!}\delta^{k-1}x^n \qquad (2.25) \\ &= \big(x-\frac{k}{2}\big)T(n,k|x) + kT(n,k|x) + T(n,k-1|x), \end{aligned}$$

where $n, k \in \mathbb{N}$ with $n \geq k$.

Therefore, by (2.25), we obtain the following theorem.

Theorem 2.8. *For $n, k \in \mathbb{N}$ with $n \geq k$, we have*

$$T(n+1,k|x) = \big(x+\frac{k}{2}\big)T(n,k|x) + T(n,k-1|x)$$

Corollary 2.9. *For $n, k \in \mathbb{N}$ with $n \geq k$, we have*

$$T(n+1,k) = \frac{k}{2}T(n,k) + T(n,k-1).$$

3. Further Remark

As is known, the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \qquad (3.1)$$

Now, we observe that

$$\frac{2}{e^t+1} = \Big(\frac{e^t-1}{2}+1\Big)^{-1} = \sum_{l=0}^{\infty} \frac{e^{\frac{lt}{2}}\big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)^l}{2^l}(-1)^l. \qquad (3.2)$$

By (3.1) and (3.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} &= \sum_{l=0}^{\infty}\big(-\frac{1}{2}\big)^l l! e^{\left(\frac{l}{2}+x\right)t}\frac{1}{l!}\Big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\Big)^l \\ &= \sum_{l=0}^{\infty}\big(-\frac{1}{2}\big)^l l! \sum_{n=l}^{\infty} T\big(n,l|\frac{l}{2}+x\big)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\big(-\frac{1}{2}\big)^l l! T\big(n,l|\frac{l}{2}+x\big)\Big)\frac{t^n}{n!}. \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 3.1. *For $n \geq 0$, we have*

$$E_n(x) = \sum_{l=0}^{n} \big(-\frac{1}{2}\big)^l l! T\big(n, l \big| \frac{l}{2} + x\big).$$

For $r \in \mathbb{N}$, the higher-order Euler polynomials are given by the generating function to be

$$\Big(\frac{2}{e^t + 1}\Big)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{3.3}$$

On the other hand,

$$\begin{aligned} \Big(\frac{2}{e^t + 1}\Big)^r &= \Big(\frac{e^t - 1}{2} + 1\Big)^{-r} = \sum_{l=0}^{\infty} \binom{r + l - 1}{l} (-1)^l \big(\frac{1}{2}\big)^l (e^t - 1)^l \\ &= \sum_{l=0}^{\infty} \binom{l + r - 1}{l} \big(-\frac{1}{2}\big)^l e^{\frac{l}{2}t} \Big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\Big)^l. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!} &= \Big(\frac{e^t - 1}{2} + 1\Big)^{-r} \\ &= \sum_{l=0}^{\infty} \binom{l + r - 1}{l} \big(-\frac{1}{2}\big)^l e^{\left(\frac{l}{2} + x\right)t} \Big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\Big)^l \\ &= \sum_{l=0}^{\infty} \binom{l + r - 1}{l} \big(-\frac{1}{2}\big)^l \sum_{n=l}^{\infty} l! T\big(n, l \big| \frac{l}{2} + x\big) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{l=0}^{n} \binom{l + r - 1}{l} \big(-\frac{1}{2}\big)^l l! T\big(n, l \big| \frac{l}{2} + x\big)\Big) \frac{t^n}{n!}. \end{aligned} \tag{3.5}$$

Therefore, by comparing the coefficients on the both sides of (3.5), we obtain the following theorem.

Theorem 3.2. *For $r \in \mathbb{N}$ and $n \geq 0$, we have*

$$E_n^{(r)}(x) = \sum_{l=0}^{n} \binom{l + r - 1}{l} \big(-\frac{1}{2}\big)^l l! T\big(n, l \big| \frac{l}{2} + x\big).$$

By (2.3), we get

$$\begin{aligned}\sum_{n=2k}^{\infty} T(n,2k)\frac{t^n}{n!} &= \frac{1}{(2k)!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{2k} = \frac{1}{(2k)!}\left(e^t+e^{-t}-2\right)^k \\ &= \frac{1}{(2k)!}\sum_{l=0}^{k}\binom{k}{l}(e^t-1)^l(e^{-t}-1)^{k-l} \\ &= \frac{k!}{(2k)!}\sum_{l=0}^{k}\frac{1}{l!}(e^t-1)^l\frac{1}{(k-l)!}(e^{-t}-1)^{k-l}.\end{aligned} \tag{3.6}$$

From (1.5), we note that

$$\begin{aligned}\frac{1}{l!}(e^t-1)^l\frac{1}{(k-l)!}(e^{-t}-1)^{k-l} &= \sum_{n=k}^{\infty}\Big(\sum_{i=l}^{n}\binom{n}{i}S_2(i,l) \\ &\quad \times S_2(n-i,k-l)(-1)^{n-i}\Big)\frac{t^n}{n!}.\end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we get

$$\begin{aligned}\sum_{n=2k}^{\infty} T(n,2k)\frac{t^n}{n!} &= \frac{1}{k!\binom{2k}{k}}\sum_{l=0}^{k}\sum_{n=k}^{\infty}\Big(\sum_{i=l}^{n}\binom{n}{i}S_2(i,l) \\ &\quad \times S_2(n-i,k-l)(-1)^{n-i}\Big)\frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty}\Big(\frac{1}{k!\binom{2k}{k}}\sum_{l=0}^{k}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l) \\ &\quad \times S_2(n-i,k-l)(-1)^{n-i}\Big)\frac{t^n}{n!}.\end{aligned} \tag{3.8}$$

By comparing the coefficients on the both sides of (3.8), we get

$$\sum_{l=0}^{k}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)S_2(n-i,k-l)(-1)^{n-i} = \begin{cases} k!\binom{2k}{k}T(n,2k), & \text{if } n \geq 2k, \\ 0, & \text{if } n < 2k.\end{cases}$$

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