Advanced Studies in Contemporary Mathematics
28 (2018), No. 4, pp. 685 - 698
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.4.685

On one nonlinear optimal control problem with fixed ends

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ABSTRACT. We consider nonlinear general problem of optimal control with fixed ends and apply two approaches for its solving: Pontryagin maximum principle and Galerkin method. We developed the new method of obtaining approximate solution for this problem and explored its basic properties.

2010 MATHEMATICS SABJECT CLASSIFICATION. 34K35, 47N70, 93-XX.

KEYWORDS AND PHRASES. Optimal control problem, Pontryagin maximum principle, Galerkin method, residual function, trial function.

1. Introduction

Optimal control theory began to take shape as a mathematical discipline in the 1950s. The motivation for its development were the actual problems of automatic control, satellite navigation, aircraft control, chemical engineering and a number of other engineering problems.

Optimal control is regarded as a modern branch of the classical calculus of variations, which is the branch of mathematics that emerged about three centuries ago at the junction of mechanics, mathematical analysis and the theory of differential equations. The calculus of variations studies problems of extreme in which it is necessary to find the maximum or the minimum of some numerical characteristic (functional) defined on the set of curves, surfaces, or other mathematical objects of a complex nature.

The development of the calculus of variations is associated with the names of some famous scientists: Bernoulli, Euler, Newton, Lagrange, Weierstrass, Hamilton and others. Optimal control problems differ from variation problems by the additional requirements imposed on sought solution, and these requirements are sometimes difficult and even impossible to fit applying for solving the methods of the calculus of variations. The need for practical methods resulted in further development of variation calculus, which ultimately led to the formation of the modern theory of optimal control. This theory, absorbed all previous achievements in the calculus of variations, and it was enriched with new results and new content. The central results of the theory – the Pontryagin Maximum Principle and the dynamic programming method of Bellman – became widely known in the scientific and engineering community, and these are now widely used in various academic fields.

Optimal control problems are classified on several types: the simplest problem, the two point minimum time problem, the general problem, the problem with intermediate states, the common problem, etc. [1,2,4,7] Our interest is related with the general problem with fixed ends and special form of objective function. We consider two approaches for solving this problem: Pontryagin maximum principle and Galerkin method and compare obtained results.

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2. Statement of the problem

We consider the problem of optimal control in the form

$$J=\int_{t_0}^{t_1}(c_0+c_1x_1+c_2x_2+\dots+c_nx_n)u^2dt\to\min,$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dots\\ \dot{x}_n\end{pmatrix}=\begin{pmatrix}A_{11}&A_{12}&\dots&A_{1n}\\ A_{21}&A_{22}&\dots&A_{2n}\\ \dots&\dots&\dots&\dots\\ A_{n1}&A_{n2}&\dots&A_{nn}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \dots\\ x_n\end{pmatrix}+\begin{pmatrix}B_1\\ B_2\\ \dots\\ B_n\end{pmatrix}u, \qquad (1)$$

$$\begin{pmatrix}x_1(t_0)\\ x_2(t_0)\\ \dots\\ x_n(t_0)\end{pmatrix}=\begin{pmatrix}x_1^0\\ x_2^0\\ \dots\\ x_n^0\end{pmatrix},\ \begin{pmatrix}x_1(t_1)\\ x_2(t_1)\\ \dots\\ x_n(t_1)\end{pmatrix}=\begin{pmatrix}x_1^1\\ x_2^1\\ \dots\\ x_n^1\end{pmatrix},\ u\in U\subseteq R,$$

where $u(t)$ is control variable, $x(t)\in R^n$ is state variable, $x^0=\begin{pmatrix}x_1^0\\ x_2^0\\ \dots\\ x_n^0\end{pmatrix}$, $x^1=\begin{pmatrix}x_1^1\\ x_2^1\\ \dots\\ x_n^1\end{pmatrix}$ are fixed ends of trajectory, c_0, c_1, …, c_n are constants, t_0, t_1 are fixed moments of time.

Or in a brief form

$$J=\int_{t_0}^{t_1}(c_0+c^Tx)u^2dt\to\min,\ \dot{x}=Ax+Bu,\ x(t_0)=x^0,\ x(t_1)=x^1,\ u\in U\subseteq R, \qquad (2)$$

where $c^T=\begin{pmatrix}c_1 & c_2 & \dots & c_n\end{pmatrix}$, A and B are $n\times n$ and $n\times 1$ matrices respectively.

Note that problem (1) (or (2)) is important generalization of the minimum energy problem

$$J=\int_{t_0}^{t_1}u^2dt\to\min,\ \dot{x}=Ax+Bu,\ x(t_0)=x^0,\ x(t_1)=x^1,\ u\in U\subseteq R$$

that arises in many applications in mechanics, economy or other areas. Objective function in problem 1 (or 2) contains additional conditions imposed on state variables extending the variety of solving real problems. One of the most interesting applications of the problem (2) is the investigation of efficiency in wireless power transfer system.

The purpose of this research is in obtaining the efficient methods of solution of the problem (2). For general case of a mathematical model the problem of obtaining the exact solutions is very complicated. The most numerical methods using for solution of the problem (2): Newton's method, Gradient method and so on [10,11] can give an approximate solution with some acceptable or not acceptable accuracy. And the problem of convergence of applied methods has very important role. Numerical solution of the problem (2) was developed by a number of researchers, for instance, L.T. Aschepkov [2,6], F.P. Vasiliev [5,11], R.P. Fedorenko [8]. In general case, when we solve optimal control problem, there isn't a formula for getting unknown initial values for conjugate variables. It's worth to mention that there is a bad convergence of initial approximation for the values of conjugate variables to the values that put zeros for residual functions because of permanent getting by them their local minimum [9,10,11]. The latter means that neither Newton's method nor Gradient method don't give a good result.

In this paper, applying Pontryagin maximum principle we obtain the form of optimal control and utilizing Galerkin method with the proper choice of trial functions we get approximate solution of the problem (2). We show that under some conditions Galerkin method allows to obtain an exact solution of the problem (2). This advantage of Galerkin method can be generalized on the other types of optimal control problems.

3. Main results

By classification, the problem (2) is the general problem of optimal control [1,2,6,7], that is, the problem that has mobile ends of an integral curve. It has the form

$$\begin{aligned} &J_0 = \Phi_0(x(t_0), x(t_1), t_0, t_1) \to \min \\ &J_i = \Phi_i(x(t_0), x(t_1), t_0, t_1)\begin{cases} \le 0, i = 1,2,\dots, m_0, \\ = 0, i = m_0+1,\dots, m, \end{cases} \\ &\dot{x} = f(x,u,t),\ u \in U \subseteq R^r,\ t_1 \ge t_0. \end{aligned} \tag{3}$$

Here $\Phi_0, \Phi_1, \dots, \Phi_m$ are the given functions of the class $C_1(R^n \times R^n \times R \times R \to R)$, m_0 is an integer nonnegative number, and m is a natural number. If $m_0 = 0$ or $m_0 = m$, then the general problem only has constraints-equalities $J_i = 0, i = 0,1,\dots, m$, or only constraints-inequalities $J_i \le 0, i = 0,1,\dots, m$, respectively. The *process* is said to be a quaternion $x(t), u(t), t_0, t_1$ that satisfies all conditions of the general problem except, possibly, the first condition. A process $x(t), u(t), t_0, t_1$ is regarded to be *optimal* if for any other process $\tilde{x}(t), \tilde{u}(t), \tilde{t}_0, \tilde{t}_1$, the following inequality is true

$$\Phi_0(x(t_0), x(t_1), t_0, t_1) \le \Phi_0(\tilde{x}(t_0), \tilde{x}(t_1), \tilde{t}_0, \tilde{t}_1).$$

Function $u(t)$ is called optimal control and $x(t)$ - optimal trajectory. The general problem consists in determining the optimal process. The necessary conditions of optimality are given by the Pontryagin maximum principle [1].

Case 1. Unbounded range of control.
We seek optimal control in the class of continuous on segment $[t_0, t_1]$ functions. Infinite controls we define as unusable.

Transforming the problem (2) to standard form (3), yields

$$\begin{aligned} &x_{n+1}(t_1) \to \min, \\ &\dot{x} = Ax + Bu,\ \dot{x}_{n+1} = (c_0 + c^T x)u^2, \\ &x(t_0) = x^0,\ x_{n+1}(t_0) = 0, \\ &x(t_1) = x^1,\ u \in R,\ t \in [t_0, t_1]. \end{aligned} \tag{4}$$

Theorem 1. Optimal control of the problem (4) is

$$u(t) = \frac{\psi^T B}{2\mu(c_0 + c^T x)} \quad \text{if } \mu(c_0 + c^T x) < 0 \text{ for all } t \in [t_0, t_1] \text{ and}$$

$$u(t) = \pm\infty \text{ if } \mu(c_0 + c^T x) \ge 0 \text{ for at least one } t \in [t_0, t_1].$$

We denote optimal control by $u^{opt}(t)$.

Proof. According to the maximum principle, if $x(t),u(t),t_0,t_1$ is an optimal process of the general problem (3) then there exist vector $\lambda=(\lambda_0,\lambda_1,...,\lambda_m)$ and continuous solution $\psi(t)$ of a conjugate system of differential equations $\dot{\psi}=-H(\psi,x(t),u(t),t)$, satisfying conditions:

1) non-triviality, non-negativity, and complementary slackness

$$\lambda\neq 0,\ \lambda_i\geq 0, i=0,1,...,m_0,\ \lambda_i\Phi_i(x(t_0),x(t_1),t_0,t_1)=0, i=1,2,...,m_0;$$

2) transversality

$$\psi(t_0)=L_{x(t_0)}(\lambda,x(t_0),x(t_1),t_0,t_1),\ \psi(t_1)=-L_{x(t_1)}(\lambda,x(t_0),x(t_1),t_0,t_1),$$
$$\frac{d}{dt_0}L(\lambda,x(t_0),x(t_1),t_0,t_1)=0,\ \frac{d}{dt_1}L(\lambda,x(t_0),x(t_1),t_0,t_1)=0;$$

3) maximum of Hamiltonian

$$H(\psi(t),x(t),u(t),t)=\max_{v\in U}H(\psi(t),x(t),v,t),\quad t_0\leq t\leq t_1$$

with Lagrange and Hamilton functions

$$L(\lambda,x(t_0),x(t_1),t_0,t_1)=\sum_{i=0}^{m}\lambda_i\Phi_i(x(t_0),x(t_1),t_0,t_1),\ H(\psi,x,u,t)=\sum_{j=1}^{n}\psi_j f_j(x,u,t).$$

We form Lagrange and Hamilton functions for the problem (4)

$$H(\psi,x,u,t)=\psi^T(Ax+Bu)+\psi_{n+1}(c_0+c^Tx)u^2,$$
$$L(\lambda,x(t_0),x(t_1),t_0,t_1)=\lambda_0x_{n+1}(t_1)+\sum_{i=1}^{n}\lambda_i(x_i(t_0)-x_i^0)+\sum_{i=n+1}^{2n}\lambda_i(x_i(t_1)-x_i^1)+\lambda_{2n+1}x_{n+1}(t_0).$$

Since the left end of a trajectory and times moments t_0 and t_1 are fixed, corresponding transversality conditions are satisfied automatically (see [1,2]) and we can simplify the second, in above, function omitting the terms related with initial conditions $x(t_0)=x^0$ and $x_{n+1}(t_0)=0$. That is, the Lagrange function can be rewritten

$$L(\lambda,x(t_0),x(t_1),t_0,t_1)=\lambda_0x_{n+1}(t_1)+\sum_{i=1}^{n}\lambda_i(x_i(t_1)-x_i^1).$$

We get the conjugate system

$$\begin{cases}\dot{\psi}=-A^T\psi-c\psi_{n+1}u^2\\ \dot{\psi}_{n+1}=0\end{cases}.$$

Its solution is $\psi(t,\tilde{C}_1,\tilde{C}_2,...,\tilde{C}_n,\mu)$, $\psi_{n+1}=\mu$, where $\tilde{C}_1,\tilde{C}_2,...,\tilde{C}_n,\mu$ are constants of integration. Maximum of the Hamiltonian condition is reduced to solution of the extreme problem

$$H(\psi,x,u,t)=...+\psi^TBu+\mu(c_0+c^Tx)u^2\to\max_{u\in R},\quad t_0\leq t\leq t_1,$$

where we denote the terms that don't depend on control variables by three dot. Analysis of the latter problem arrive us at the following optimal control function:

$$u^{opt}(t)=\frac{\psi^TB}{2\mu(c_0+c^Tx)},\ \text{for}\ \mu(c_0+c^Tx)<0,$$

or in coordinate form

$$u^{opt}(t)=\frac{1}{2\mu(c_0+c_1x_1+c_2x_2+...+c_nx_n)}\sum_{j=1}^{n}B_j\psi_j(t).\tag{5}$$

If $\mu(c_0+c^Tx)\geq 0$ we obtain unusable control $u(t)=\pm\infty$. Theorem is proven.

To obtain solution of the problem (4), we solve the system 2n differential equations corresponding to optimal control (5)

$$\dot{x}_i = \frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij} x_j + \frac{1}{2\mu(c_0 + c_1 x_1 + c_2 x_2 + ... + c_n x_n)} B_i \sum_{j=1}^{n} B_j \psi_j$$
$$\dot{\psi}_i = \frac{d\psi_i}{dt} = -\sum_{j=1}^{n} A_{ji} \psi_j - c_i \frac{1}{4\mu^2 (c_0 + c_1 x_1 + c_2 x_2 + ... + c_n x_n)^2} \left(\sum_{j=1}^{n} B_j \psi_j \right)^2, \quad i = \overline{1,n} \quad (6)$$

with 2n boundary conditions

$$x(t_0) = x^0, \; x(t_1) = x^1.$$

Differential equation $\dot{x}_{n+1} = (c_0 + c^T x)u^2$ and initial condition $x_{n+1}(t_0) = 0$ can be omitted here since they define an objective function of the original problem.

Nonlinearity and complexity of the obtained system (6) unable us to get its solution analytically. One of the effective methods for its solving is Galerkin one. The advantage of this method is the possibility due to proper choice of the trial functions to obtain an exact solution.

According to Galerkin method [3], we construct approximate solution in the form

$$x_j \approx \hat{x}_j = x_j^0 + \sum_{k=1}^{M} a_{j,k} \varphi_{j,k}(t),$$
$$\psi_j \approx \hat{\psi}_j = \sum_{k=1}^{M} a_{n+j,k} \varphi_{n+j,k}(t), \quad j = \overline{1,n}. \quad (7)$$

Here $\varphi_{j,k}(t), \quad j = \overline{1,2n} \;, k = \overline{1,M}$ are trial functions satisfying the following conditions

$$\varphi_{j,k}(t_0) = 0, \quad j = \overline{1,n} \;, k = \overline{1,M}, \quad (8)$$

and

$$\forall j = \overline{1,n}, \; \exists k,l,m : \varphi_{j,k}(t_1) \neq 0, \; \varphi_{n+j,l}(t_0) \neq 0, \; \varphi_{n+j,m}(t_1) \neq 0.$$

First condition guarantees that $x(t_0) = x^0$ and second one ensures non-triviality of a conjugate function $\psi(t)$.

The choice of a set of trial functions is critical for realization of Galerkin method. The basic requirement is: the functions $\varphi_{j,k}(t), \quad j = \overline{1,2n} \;, k = \overline{1,M}$ must be linearly independent on given interval $[t_0, t_1]$.

Assuming that trial functions are continuous and differentiable, we have

$$\frac{d}{dt} x_j \approx \frac{d}{dt} \hat{x}_j = \sum_{k=1}^{M} a_{j,k} \frac{d}{dt} \varphi_{j,k}(t), \quad (9)$$
$$\frac{d}{dt} \psi_j \approx \frac{d}{dt} \hat{\psi}_j = \sum_{k=1}^{M} a_{n+j,k} \frac{d}{dt} \varphi_{n+j,k}(t), \quad j = \overline{1,n}.$$

Substituting (7) into (6) and equating corresponding derivatives in (6) and (9) yields

$$\sum_{k=1}^{M} a_{i,k} \frac{d}{dt} \varphi_{i,k}(t) \approx$$
$$\sum_{j=1}^{n} A_{ij} (x_j^0 + \sum_{k=1}^{M} a_{j,k} \phi_{j.k}(t)) +$$
$$\frac{1}{2\mu(c_0 + c_1(x_1^0 + \sum_{k=1}^{M} a_{1,k}\phi_{1,k}(t)) + c_2(x_2^0 + \sum_{k=1}^{M} a_{2,k}\phi_{2,k}(t)) + \ldots + c_n(x_n^0 + \sum_{k=1}^{M} a_{n,k}\phi_{n,k}(t)))} B_i \sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t),$$

$$\sum_{k=1}^{M} a_{n+i,k} \frac{d}{dt} \varphi_{n+i,k}(t) \approx$$
$$-\sum_{j=1}^{n} A_{ji} \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t) -$$
$$c_j \frac{1}{4\mu^2(c_0 + c_1(x_1^0 + \sum_{k=1}^{M} a_{1,k}\phi_{1,k}(t)) + c_2(x_2^0 + \sum_{k=1}^{M} a_{2,k}\phi_{2,k}(t)) + \ldots + c_n(x_n^0 + \sum_{k=1}^{M} a_{n,k}\phi_{n,k}(t)))^2} \times$$
$$\left(\sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t) \right)^2, \quad i = \overline{1,n}.$$

Residual functions for variables x_i and ψ_i are accordingly

$$R_{[t_0,t_1]}^{x_i} = \sum_{k=1}^{M} a_{i,k} \frac{d}{dt} \varphi_{i,k}(t) -$$
$$\sum_{j=1}^{n} A_{ij} (x_j^0 + \sum_{k=1}^{M} a_{j,k} \phi_{j.k}(t)) -$$
$$\frac{1}{2\mu(c_0 + c_1(x_1^0 + \sum_{k=1}^{M} a_{1,k}\phi_{1,k}(t)) + c_2(x_2^0 + \sum_{k=1}^{M} a_{2,k}\phi_{2,k}(t)) + \ldots + c_n(x_n^0 + \sum_{k=1}^{M} a_{n,k}\phi_{n,k}(t)))} B_i \sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t),$$

$$R_{[t_0,t_1]}^{\psi_i} = \sum_{k=1}^{M} a_{n+i,k} \frac{d}{dt} \varphi_{n+i,k}(t) +$$

$$\sum_{j=1}^{n} A_{ji} \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t) +$$
$$c_j \frac{1}{4\mu^2(c_0 + c_1(x_1^0 + \sum_{k=1}^{M} a_{1,k}\phi_{1,k}(t)) + c_2(x_2^0 + \sum_{k=1}^{M} a_{2,k}\phi_{2,k}(t)) + \ldots + c_n(x_n^0 + \sum_{k=1}^{M} a_{n,k}\phi_{n,k}(t)))^2} \times$$
$$\left(\sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t) \right)^2, \quad i = \overline{1,n}.$$

By Galerkin method [3], we have

$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{x_s} W_{s,k} dt = 0\,,\ s = \overline{1,n};\ \ k = \overline{1,M}\ ,$$

$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{\psi_{s-n}} W_{s,k} dt = 0\,,\ s = \overline{n+1,2n};\ \ k = \overline{1,M}\ ,$$

where $W_{s,k}$ are weight functions. For simplicity, we choose weight functions $W_{s,k}$ as

$$W_{s,k} = \varphi_{s,k}\,,\ s = \overline{1,2n};\ \ k = \overline{1,M}\ .$$

To satisfy the condition $x(t_1) = x^1$, we add the term

$$(\hat{x}_{s-n} - x_{s-n}^1)\widetilde{W}_{s,k}\Big|_{t=t_1}\,,\ s = \overline{n+1,2n};\ \ k = \overline{1,M}$$

into the second equation. It is reasonable, for convenience, to take

$$\widetilde{W}_{s,k} = -\varphi_{s,k}\,,\ s = \overline{n+1,2n};\ \ k = \overline{1,M}\ .$$

Finally, we get the following linear system of 2nM equations in 2nM variables $a_{j,k}$:

$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{x_s} \varphi_{s,k} dt = 0\,,\ s = \overline{1,n};\ \ k = \overline{1,M}\ , \tag{10}$$
$$\int_{t_0}^{t_1} R_{[t_0,t_1]}^{\psi_{s-n}} \varphi_{s,k} dt + (\hat{x}_{s-n}^0 + \sum_{k=1}^{M} a_{s-n,k}\varphi_{s-n,k}(t_1) - x_{s-n}^1)\widetilde{W}_{s,k}(t_1) = 0\,,\ s = \overline{n+1,2n};\ \ k = \overline{1,M}\ .$$

Solution of the linear system (10) gives an approximate solution (7) of the problem (4). The accuracy of approximation depends on the choice of trial functions and the exact solution of optimal control problem. Convergence of the method is related with the solvability of the system (10). Proper choice of weight and trial functions allows us to get an exact solution.

Case 2. Bounded range of control.
Here we assume that the range of control U is given by interval $u_{\min} \le u \le u_{\max}$. We consider piecewise continuous on segment $[t_0,t_1]$ functions as the class of controls. Transforming the problem (2) to standard form (3), gives

$$\begin{aligned} & x_{n+1}(t_1) \to \min, \\ & \dot{x} = Ax + Bu\,,\ \dot{x}_{n+1} = (c_0 + c^T x)u^2\,, \\ & x(t_0) = x^0\,,\ x_{n+1}(t_0) = 0\,, \\ & x(t_1) = x^1\,,\ u_{\min} \le u \le u_{\max}\,,\ t \in [t_0,t_1]\,. \end{aligned} \tag{11}$$

Theorem 2. Extreme control of the problem (11) is

$$u^{ext}(t) = \left\{ u_{\min},\ \ or\ \ \frac{\psi^T B}{2\mu(c_0 + c^T x)},\ \ or\ \ u_{\max} \right\},\ \ t_0 \le t \le t_1\,.$$

Note that extreme control is one satisfying all conditions of maximum principle. Optimal control is always extreme but not conversely.

Proof. We repeat in general the steps of the Theorem 1. Applying the maximum principle to the problem (11) arrive us at the extreme problem

$$\psi^T Bu + \mu(c_0 + c^T x)u^2 \to \max_{u_{\min} \le u \le u_{\max}}, \quad t_0 \le t \le t_1,$$

where $\psi(t)$ is a solution of conjugate system

$$\begin{cases} \dot{\psi} = -A^T \psi - c\psi_{n+1} u^2 \\ \dot{\psi}_{n+1} = 0 \end{cases}.$$

Its solution is $\psi(t, \tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n, \mu)$, $\psi_{n+1} = \mu$, where $\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n, \mu$ are constants of integration. Solution of the latter extreme problem, depending on coefficient $\mu(c_0 + c^T x)$, gives the extreme control

$$u^{ext}(t) = \left\{ u_{\min}, \ or \ \frac{\psi^T B}{2\mu(c_0 + c^T x)}, \ or \ u_{\max} \right\}, \quad t_0 \le t \le t_1.$$ Theorem is proved.

From the Theorem 2 we conclude that extreme control can have several break points $\tau_1, \tau_2, ...$ and combination of function $\frac{\psi^T B}{2\mu(c_0 + c^T x)}$ and boundary values $u_{\min}$, $u_{\max}$ on interval $t_0 \le t \le t_1$.

Observe that if $c_0 + c_1 x_1 + c_2 x_2 + ... + c_n x_n > 0$ integral

$$\int_{t_0}^{t_1} (c_0 + c_1 x_1 + c_2 x_2 + ... + c_n x_n) u^2 dt \le \int_{t_0}^{t_1} (c_0 + c_1 x_1 + c_2 x_2 + ... + c_n x_n) u^2_{\min(\max)} dt$$

for any $u_{\min} \le u \le u_{\max}$. Then, in case $\mu(c_0 + c^T x) < 0$, $\mu < 0$, optimal control must have the least number of boundary values $u_{\min}$ and $u_{\max}$ on segment $t_0 \le t \le t_1$. Thus, we get the following possibilities for the form of $u^{ext}(t)$:

1. $u^{ext}(t) = \frac{\psi^T B}{2\mu(c_0 + c^T x)}, \ t_0 \le t \le t_1$ or

2. $u^{ext}(t) = \begin{cases} u_{\min}, t_0 \le t \le \tau, \\ \frac{\psi^T B}{2\mu(c_0 + c^T x)}, \tau < t \le t_1 \end{cases}$ or

3. $u^{ext}(t) = \begin{cases} u_{\max}, t_0 \le t \le \tau, \\ \frac{\psi^T B}{2\mu(c_0 + c^T x)}, \tau < t \le t_1 \end{cases}$ or

4. $u^{ext}(t) = \begin{cases} \frac{\psi^T B}{2\mu(c_0 + c^T x)}, t_0 \le t \le \tau, \\ u_{\max}, \tau < t \le t_1 \end{cases}$ or

5. $u^{ext}(t) = \begin{cases} \frac{\psi^T B}{2\mu(c_0 + c^T x)}, t_0 \le t \le \tau, \\ u_{\min}, \tau < t \le t_1 \end{cases}$ or

6. $u^{ext}(t) = \begin{cases} u_{\min}, t_0 \le t \le \tau_1, \\ \frac{\psi^T B}{2\mu(c_0 + c^T x)}, \tau_1 < t \le \tau_2, \\ u_{\max}, \tau_2 < t \le t_1 \end{cases}$ or

7. $u^{ext}(t)=\begin{cases} u_{\max}, t_0 \le t \le \tau_1, \\ \dfrac{\psi^T B}{2\mu(c_0+c^T x)}, \tau_1 < t \le \tau_2, \\ u_{\min}, \tau_2 < t \le t_1 \end{cases}$ or

8. $u^{ext}(t)=\begin{cases} u_{\max}, t_0 \le t \le \tau_1, \\ \dfrac{\psi^T B}{2\mu(c_0+c^T x)}, \tau_1 < t \le \tau_2, \\ u_{\min}, \tau_2 < t \le \tau_3, \\ \dfrac{\psi^T B}{2\mu(c_0+c^T x)}, \tau_3 < t \le t_1 \end{cases}$ and so on.

If $\mu(c_0+c^T x)\ge 0$ optimal control consists of different combinations of $u_{\min}$ and/or $u_{\max}$, that is,

9. $u^{ext}(t)=u_{\min},\ t_0 \le t \le t_1$ or

10. $u^{ext}(t)=u_{\max},\ t_0 \le t \le t_1$ or

11. $u^{ext}(t)=\begin{cases} u_{\min}, t_0 \le t \le \tau, \\ u_{\max}, \tau < t \le t_1 \end{cases}$ or

12. $u^{ext}(t)=\begin{cases} u_{\max}, t_0 \le t \le \tau_1, \\ u_{\min}, \tau_1 < t \le \tau_2, \\ u_{\max}, \tau_2 < t \le t_1 \end{cases}$ and so on.

We must check each of these extreme controls on optimality.
Variants 9 and 10 are trivial. After substitution $u_{\max}$ or $u_{\min}$ into (11) we get trajectory

$$x(t)=F(t,t_0)x^0+\int_{t_0}^{t} F(t,\rho)Bu_{\max(\min)}d\rho,$$

where $F(t,\rho)$ is the fundamental matrix for the system $\dot{x}=Ax$. Fulfilling the condition $x(t_1)=x^1$ guarantees the existence of a solution of the problem (11). In the same manner we can obtain solution of the problem (11) for variants 11, 12 and the similar ones.

We discuss variants corresponding to the case $\mu(c_0+c^T x)<0$. Let us consider, for instance, variant 2:

$$u^{ext}(t)=\begin{cases} u_{\min}, t_0 \le t \le \tau, \\ \dfrac{\psi^T B}{2\mu(c_0+c^T x)}, \tau < t \le t_1. \end{cases}$$

On interval $t_0 \le t \le \tau$ trajectory of the problem (11) is defined by

$$x(t)=F(t,t_0)x^0+\int_{t_0}^{t} F(t,\rho)Bu_{\min}d\rho$$

and the terminal point $x^\tau=F(\tau,t_0)x^0+\int_{t_0}^{\tau} F(\tau,t)Bu_{\min}dt$ can be regarded as the initial point for the interval $\tau \le t \le t_1$. Thus, the original problem is reducing to the solution of the system of 2n differential equations

$$\dot{x}_i = \frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij} x_j + \frac{1}{2\mu(c_0 + c_1 x_1 + c_2 x_2 + ... + c_n x_n)} B_i \sum_{j=1}^{n} B_j \psi_j$$

$$\dot{\psi}_i = \frac{d\psi_i}{dt} = -\sum_{j=1}^{n} A_{ji} \psi_j - c_i \frac{1}{4\mu^2 (c_0 + c_1 x_1 + c_2 x_2 + ... + c_n x_n)^2} \left(\sum_{j=1}^{n} B_j \psi_j \right)^2, \quad i = \overline{1,n} \tag{12}$$

with 2n boundary conditions $x(\tau) = x^\tau$, $x(t_1) = x^1$.

By analogy with Case 1, to obtain approximate solution of the problem (11), we apply Galerkin method [3]. According to this method, we construct solution in the form (7), that is,

$$x_j \approx \hat{x}_j = x_j^0 + \sum_{k=1}^{M} a_{j,k} \varphi_{j,k}(t),$$

$$\psi_j \approx \hat{\psi}_j = \sum_{k=1}^{M} a_{n+j,k} \varphi_{n+j,k}(t), \quad j = \overline{1,n},$$

where trial functions $\varphi_{j,k}(t)$, $j = \overline{1,2n}$ $, k = \overline{1,M}$ satisfy conditions (8). Replacing x^0 by x^τ and t_0 by τ we, finally, get the linear system of 2nM equations in 2nM variables $a_{j,k}$:

$$\int_{\tau}^{t_1} R_{[\tau,t_1]}^{x_s} \phi_{s,k} dt = 0, \ s = \overline{1,n}; \ \ k = \overline{1,M}, \tag{13}$$

$$\int_{\tau}^{t_1} R_{[\tau,t_1]}^{\psi_{s-n}} \phi_{s,k} dt + (\hat{x}_{s-n}^\tau + \sum_{k=1}^{M} a_{s-n,k} \phi_{s-n,k}(t_1) - x_{s-n}^1) \tilde{W}_{s,k}(t_1) = 0, \ s = \overline{n+1,2n}; \ \ k = \overline{1,M}.$$

Its solution gives trajectory of the system (11) corresponding to variant 2 of control. In the same manner we obtain solution of the problem (11) associated with variants 3, 4 and others. Comparing the values of objective function yields the optimal process.

5. Illustrating example

Given the general optimal control problem

$$J = \int_0^1 (x_1 + x_2) u^2 dt \to \min,$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \tag{14}$$

$$x_1(0) = 1, \ \ x_2(0) = 0,$$
$$x_1(1) = 1, \ x_2(1) = 1,$$
$$-1 \le u \le 1.$$

Here $u_{\min} = -1$ and $u_{\max} = 1$. We determine solution (optimal process) of the problem (14).

Solution. We transform problem (14) to standard form

$$J = x_3(1) \to \min,$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ \dot{x}_3 = (x_1 + x_2) u^2 \end{cases} \tag{15}$$

$$x_1(0)=1,\;\; x_2(0)=0,\;\; x_3(0)=0\,,$$
$$x_1(1)=1\,,\; x_2(1)=1\,,\;\; -1\le u\le 1\,.$$

According to the Theorem 2 we check all consequent variants of extreme control. We exclude variants 9 and 10 since boundary conditions $x_1(0)=1,\;\; x_2(0)=0\,,\; x_1(1)=1\,,\; x_2(1)=1$ are not satisfied.

Applying Galerkin method for variant 1 of extreme control gives approximate solution of the problem (15):

$$\hat{x}_1(t)=1-0.3949t-0.309t^2+0.704t^3\,,$$
$$\hat{x}_2(t)=-3.584t+8.036t^2-3.456t^3\,,$$

and

$$u^{ext}(t)=\frac{-1.98+13.539t}{2-7.9506t+15.4547t^2-5.504t^3}\,,\;\; 0\le t\le 1\,.$$

Since $u^{ext}(1)>u_{\max}$, we exclude this variant.
Further, we must consider variants, containing one break point τ, that is, variants 2 – 5.
Let us obtain solution for variant 4. According to previous discussion, we have

$$x^{\tau}=F(\tau,t_1)x^1+\int_{t_1}^{\tau}F(\tau,t)Bu_{\max}dt\,,$$

where $F(\tau,t)=\begin{pmatrix}1 & \tau-t\\ 0 & 1\end{pmatrix}$ is corresponding fundamental matrix. Substituting $t_1=1,\; u_{\max}=1$, and $x^1=\begin{pmatrix}1\\1\end{pmatrix}$, yields $x(\tau)=x^{\tau}=\begin{pmatrix}\frac{\tau^2}{2}+\frac{1}{2}\\ \tau\end{pmatrix}$. And solution of the problem (15) is reduced to the case of variant 1 where the point $x(\tau)=x^{\tau}$ plays the role of a terminal point on interval $0\le t\le\tau$. Extreme control on this interval is $u^{ext}(t)=\dfrac{\psi^T B}{2\mu(c_0+c^T x)}$ or in component form

$$u^{ext}(t)=\frac{\psi_2}{2(x_1+x_2)}\;\;,\; 0\le t\le\tau\,.$$

Since the functions ψ_2, x_1 and x_2 are not defined we cannot use directly this control for getting extreme trajectory. Nevertheless, this form of optimal control is the basis for obtaining approximate solution of the problem (15) by Galerkin method.

According to Galerkin method we take trial functions

$$\varphi_{1,1}(t)=\varphi_{2,1}=t\,,\;\; \varphi_{1,2}(t)=\varphi_{2,2}=t^2\,,\;\; \varphi_{1,3}(t)=\varphi_{2,3}=t^3\,;$$
$$\varphi_{3,1}(t)=\varphi_{4,1}=1\,,\;\; \varphi_{3,2}(t)=\varphi_{4,2}=t\,,\;\; \varphi_{3,3}(t)=\varphi_{4,3}=t^2\,,$$

satisfying the conditions (8). We form approximate solution (7) of the system (12) as

$$\begin{aligned}\hat{x}_1&=1+a_{1,1}t+a_{1,2}t^2+a_{1,3}t^3\,,\\ \hat{x}_2&=0+a_{2,1}t+a_{2,2}t^2+a_{2,3}t^3\,,\\ \hat{\psi}_1&=a_{3,1}+a_{3,2}t+a_{3,3}t^2\,,\\ \hat{\psi}_2&=a_{4,1}+a_{4,2}t+a_{4,3}t^2\,.\end{aligned}\qquad(16)$$

Substituting (16) into (12), forming residual functions and integrating expressions (13), yield the following nonlinear system:

$$\int_0^\tau (a_{1,1}+(2a_{1,2}-a_{2,1})t+(3a_{1,3}-a_{2,2})t^2-a_{2,3}t^3)t\,dt=0,$$

$$\int_0^\tau (a_{1,1}+(2a_{1,2}-a_{2,1})t+(3a_{1,3}-a_{2,2})t^2-a_{2,3}t^3)t^2dt=0,$$

$$\int_0^\tau (a_{1,1}+(2a_{1,2}-a_{2,1})t+(3a_{1,3}-a_{2,2})t^2-a_{2,3}t^3)t^3dt=0,$$

$$\int_0^\tau \left(a_{2,1}+2a_{2,2}t+3a_{2,3}t^2-\frac{a_{4,1}+a_{4,2}t+a_{4,3}t^2}{2(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)}\right)\cdot t\,dt=0,$$

$$\int_0^\tau \left(a_{2,1}+2a_{2,2}t+3a_{2,3}t^2-\frac{a_{4,1}+a_{4,2}t+a_{4,3}t^2}{2(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)}\right)\cdot t^2dt=0,$$

$$\int_0^\tau \left(a_{2,1}+2a_{2,2}t+3a_{2,3}t^2-\frac{a_{4,1}+a_{4,2}t+a_{4,3}t^2}{2(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)}\right)\cdot t^3dt=0,$$

$$\int_0^\tau \left(a_{3,2}+2a_{3,3}t-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)dt-a_{1,1}-a_{1,2}-a_{1,3}=0,$$

$$\int_0^\tau \left(a_{3,2}+2a_{3,3}t-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)\cdot t\,dt-a_{1,1}-a_{1,2}-a_{1,3}=0,$$

$$\int_0^\tau \left(a_{3,2}+2a_{3,3}t-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)\cdot t^2dt-a_{1,1}-a_{1,2}-a_{1,3}=0,$$

$$\int_0^\tau \left((a_{3,1}+a_{4,2})+(a_{3,2}+2a_{4,3})t+a_{3,3}t^2-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)dt-$$

$$a_{2,1}-a_{2,2}-a_{2,3}=0,$$

$$\int_0^\tau \left((a_{3,1}+a_{4,2})+(a_{3,2}+2a_{4,3})t+a_{3,3}t^2-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)\cdot t\,dt-$$

$$a_{2,1}-a_{2,2}-a_{2,3}=0,$$

$$\int_0^{\tau}\left((a_{3,1}+a_{4,2})+(a_{3,2}+2a_{4,3})t+a_{3,3}t^2-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)\cdot t^2dt-$$

$$a_{2,1}-a_{2,2}-a_{2,3}=0.$$

Using Mathematica version 10 software we determine its solution:

$$a_{1,1}=-0.3949,\quad a_{1,2}=-0.309,\quad a_{1,3}=0.704,$$
$$a_{2,1}=-3.584,\quad a_{2,2}=8.036,\quad a_{2,3}=-3.456,$$
$$a_{3,1}=-13.539,\quad a_{3,2}=0,\quad a_{3,3}=0,$$
$$a_{4,1}=-1.98,\quad a_{4,2}=13.539,\quad a_{4,3}=0.$$

Break point $\tau=0.216$ is defined uniquely and, finally, we get approximate solution of the problem (15):

extreme control is $u^{ext}(t)=\begin{cases}\dfrac{-1.98+13.539t}{2-7.9506t+15.4547t^2-5.504t^3}, 0\le t\le 0.216,\\ 1, 0.216<t\le 1\end{cases}$

and corresponding optimal trajectory is

$$\hat{x}_1(t)=1-0.3949t-0.309t^2+0.704t^3,$$
$$\hat{x}_2(t)=-3.584t+8.036t^2-3.456t^3 \qquad \text{on interval } 0\le t\le 0.216$$

and

$$\hat{x}_1(t)=\tfrac{1}{2}(t^2+1),$$
$$\hat{x}_1(t)=t \qquad \text{on interval } 0.216\le t\le 1.$$

Solution of conjugate system is

$$\hat{\psi}_1(t)=-13.539,$$
$$\hat{\psi}_2(t)=-1.98+13.539t.$$

In the same manner we can obtain solution for the variants 2, 3, 5. Comparing the values of objective function for these variants shows that optimal control is variant 4:

$$u^{opt}(t)=\begin{cases}\dfrac{-1.98+13.539t}{2-7.9506t+15.4547t^2-5.504t^3}, 0\le t\le 0.216,\\ 1, 0.216<t\le 1.\end{cases}$$

Acknowledgments

This paper is supported by National Research Foundation of Korea (NRF) grant founded by the Korea government (MEST) (No. 2017R1E1A1A03070882).

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