Advanced Studies in Contemporary Mathematics
28 (2018), No. 4, pp. 681 - 684
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.4.681

CONTINUITY CONDITIONS FOR LOCALLY BOUNDED FINITE-DIMENSIONAL REPRESENTATIONS OF TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS

A. I. Shtern

Abstract. We obtain a criterion for the continuity of a locally bounded finite-dimensional representation of a totally disconnected locally compact group.

§ 1. Introduction

The continuity conditions for locally bounded finite-dimensional representations of connected locally compact groups are quite nontrivial (see [1–4]).

In the present note, we obtain a simple criterion for the continuity of a locally bounded finite-dimensional representation of a totally disconnected locally compact group.

Below, $\{e\}$ stands for the identity element of a group under consideration.

§ 2. Preliminaries

Lemma 1. *Let G be a totally disconnected locally compact group, and let $\mathcal{N}$ be the family of compact open normal subgroups $N \neq \{e\}$ of open subgroups $O \subset G$ (i.e., $N \in \mathcal{N}$ if and only if there is an open subgroup $O \subset G$ such that N is an open compact normal subgroup of O). Then $\mathcal{N}$ is a nontrivial filter basis convergent to $\{e\}$.*

2010 *Mathematics Subject Classification.* Primary 22A25, Secondary 22A10.

Key words and phrases. Locally compact group, totally disconnected group, finite-dimensional representation, locally bounded representation, continuity conditions.

Proof. Since every neighborhood U of $\{e\}$ contains a compact open subgroup (see Theorem 16 in [5]), which can be taken for O with $N = O$, it follows that $\mathcal{N}$ is nontrivial and

$$\bigcap_{N\in\mathcal{N}} N = \bigcap_{U\ni e} U = \{e\}.$$

If O_1 and O_2 are open subgroups and N_1 and N_2 are compact open normal subgroups of O_1 and O_2, respectively, then $N = N_1 \cap N_2$ is a compact open normal subgroup of $O = O_1 \cap O_2$, which is immediate, and hence $\mathcal{N}$ is a filter basis with respect to inclusion.

Definition. Let π be a locally bounded (not necessarily continuous) finite-dimensional representation of a totally disconnected locally compact group G in a (finite-dimensional) normed linear space E. The set

$$\mathrm{FDG}(\pi) = \bigcap_{N\in\mathcal{N}} \overline{\pi(N)}$$

where $\overline{\pi(N)}$ stands for the closure of the π-image $\pi(N)$ of N, is called the *final discontinuity group* of π.

Lemma 2. *Let π be a locally bounded (not necessarily continuous) finite-dimensional representation of a totally disconnected locally compact group G in a (finite-dimensional) normed linear space E. The set* $\mathrm{FDG}(\pi)$ *is a compact normal subgroup of the closure $\overline{\pi(G)}$ of the π-image $\pi(G)$ of G.*

Proof. Since $\mathcal{N}$ is obviously invariant with respect to the inner automorphisms of G, it follows that this invariance property is inherited by $\mathrm{FDG}(\pi)$.

Moreover, since π is locally bounded and every $N \in \mathcal{N}$ is compact, it follows that $\pi(N)$ is bounded, and therefore its closure is compact, and, since every $\pi(N)$ is a group, it follows that its closure is a group (recall that $\pi(N)$ is bounded), and therefore the intersection of these groups is a group.

This completes the proof of the lemma.

§ 3. Main theorem

Theorem. *Let π be a locally bounded (not necessarily continuous) finite-dimensional representation of a totally disconnected locally compact group G in a (finite-dimensional) normed linear space E. The following conditions are equivalent:*

(1) π *is continuous,*

(2) $\mathrm{FDG}(\pi) = \{1_E\}$ (1_E *stands for the identity operator on* E),

(3) $\mathrm{FDG}(\pi)$ *is contained in a ball in the space* $\mathcal{L}(E)$ *of linear operators on* E (*this space is equipped with the standard operator norm*) *centered at* 1_E *and of radius less than* $\sqrt{3}$.

Proof. (2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (2) follows from the fact that, if (3) holds, then it results from the convergence of the filter $\mathcal{N}$ to $\{e\}$ that there is an $N \in \mathcal{N}$ for which $\pi(N)$ is also contained in a ball centered at 1_E and of radius less than $\sqrt{3}$, and this ball contains no nonidentity subgroups.

(1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1) holds because, as was noted above, if (2) is satisfied, then there is an $N \in \mathcal{N}$ for which $\pi(N)$ is also contained in a ball centered at 1_E and of radius less than $\sqrt{3}$, and, since this ball has no nonidentity subgroups, it follows that $\pi(N) = \{1_E\}$. Since N is open, this implies the continuity of π.

§ 4. Concluding remarks

In [1, 2, 4], for any (not necessarily continuous) finite-dimensional locally bounded representation π of a locally compact topological group G, we have introduced the notion of the so-called discontinuity group $\mathrm{DG}(\pi)$ of the representation π, which is defined as the intersection of closures of the precompact neighborhoods of the identity element of G. The key point of the situation considered above is that $\mathrm{FDG}(\pi) = \mathrm{DG}(\pi)$ for any totally disconnected locally compact group and any representation π with the above properties, and this equation explains the nature of the result.

There is a natural problem related to pseudocharacters on totally disconnected locally compact groups: is it true that every locally bounded pseudocharacter on a totally disconnected locally compact group is automatically continuous?

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Advanced Studies of Contemporary Mathematics.

The research was partially supported by the Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN), theme NIR 0065-2018-0004.

References

1. A. I. Shtern, *Locally bounded finally precontinuous finite-dimensional quasirepresentations of connected locally compact groups*, Mat. Sb. **208** (2017), no. 10, 149–170; English transl., Sb. Math, **208** (2017), no. 10, 1557–1576.
2. A. I. Shtern, *Finite-dimensional quasi-representations of connected Lie groups and Mishchenko's conjecture*, J. Math. Sci. **159** (2009), no. 5, 653–751.
3. A. I. Shtern, *Remarks on finite-dimensional locally bounded finally precontinuous quasirepresentations of locally compact groups*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 4, 469–480.
4. A. I. Shtern, *A version of van der Waerden's theorem and a proof of Mishchenko's conjecture on homomorphisms of locally compact groups*, Izv. Math. **72** (2008), no. 1, 169–205.
5. L. S. Pontryagin, *Topological Groups*, 3rd ed. Izdat. "Nauka", Moscow, 1973; English transl. of the 2nd edition, Gordon and Breach Science Publishers, Inc., New York–London–Paris, 1966.

Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN),
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru