Advanced Studies in Contemporary Mathematics
28 (2018), No. 4, pp. 669 – 679

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.4.669

THE DIMENSIONS OF GORENSTEIN $\mathcal{X}_{\mathcal{Y}}$-FLAT MODULES WITH RESPECT TO A SEMIDUALIZING MODULES

V. BIJU, R. UDHAYAKUMAR*, A. UMAMAHESWARAN, AND M. PARIMALA

Abstract. In this paper, we introduce and investigate the notion of Gorenstein $\mathcal{X}_{\mathcal{Y}}$-flat modules with respect to a semidualizing module and also study the relationship between the G_C-$\mathcal{X}_{\mathcal{Y}}$-flat resolution and the $\mathcal{X}_{\mathcal{Y}}$-flat resolution of a module over $G\mathcal{X}_{\mathcal{Y}}f$-closed ring where $\mathcal{X}$ is a class of left R-modules and $\mathcal{Y}$ is a subclass of $\mathcal{X}$.

2000 Mathematics Subject Classification. 13D05, 13D07, 18G20, 18G25.

Keywords and phrases. Semidualizing module; $G\mathcal{X}_{\mathcal{Y}}f$-closed ring; G_C-$\mathcal{X}_{\mathcal{Y}}$-flat module; G_C-$\mathcal{X}_{\mathcal{Y}}$-flat dimension.

Throughout this paper, R and S are associative rings with identity and all modules are unitary modules. R-Mod (resp., Mod-R) denotes the category of left (resp., right) R-modules. Umamaheswaran et al. in [16] introduced $\mathcal{X}$-injective and $\mathcal{X}$-flat R-modules. Let $\mathcal{X}$ be a class of left R-modules. A left R-module M is called $\mathcal{X}$-injective if $Ext^1_R(X, M) = 0$ for all left R-modules $X \in \mathcal{X}$. A right R-module N is said to be $\mathcal{X}$-flat if $Tor^R_1(N, X) = 0$ for all left R-modules $X \in \mathcal{X}$.

The study of semidualizing modules over commutative Noetherian rings was initiated by Foxby [5], Golod [8], and Vasconcelos [19]. Over a commutative Noetherian ring, Holm and Jørgensen in [10] introduced the C-Gorenstein projective, C-Gorenstein injective and C-Gorenstein flat modules using semidualizing modules and their associated projective, injective and flat classes which are also called G_C-projective, G_C-injective and G_C-flat module respectively. White introduced in [20] the G_C-projective modules and gave a functorial description of the G_C-projective dimension of modules with respect to a semidualizing module C over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [20]. Selvaraj et al. introduced Gorenstein $\mathcal{X}$-flat modules [17] and they investigated about its resolutions and dimensions. Also Udhayakumar et al. [18] studied Gorenstein n-flat modules with respect to a semidualizing module. These works motivate us to introduce Gorenstein $\mathcal{X}_{\mathcal{Y}}$-flat modules with respect to a semidualizing module (See Definition 2.4) and also we study the relation between the G_C-$\mathcal{X}_{\mathcal{Y}}$-flat resolution and the $\mathcal{X}_{\mathcal{Y}}$-flat resolution of a module.

This paper is organized as follows: In Section 2, we recall some definitions that are necessary for our proofs of the main results.

The third author was supported by Harish-Chandra Research Institute (HRI), Allahabad, Institute Postdoctoral Fellowship, Code No. M170563.

In Section 3, we introduce C-$\mathcal{X}$-flat, C-$\mathcal{X}$-injective and G_C-$\mathcal{X}_\mathcal{Y}$-flat modules and also establish the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-$\mathcal{X}_\mathcal{Y}$-flat resolution and the $\mathcal{X}$-flat resolution of a module over a $G\mathcal{X}_\mathcal{Y}f$-closed ring.

In Section 4, we discuss some properties of G_C-$\mathcal{X}_\mathcal{Y}$-flat dimension of modules. In particular, as an application of the results obtained in Section 3, we obtain a criterion for computing such a dimension.

1. Preliminaries

In this section, we recall some known definitions and terminologies from [11, 20] that will be used in the rest of the paper.

Let R and S be rings. An (S, R)-bimodule C is called semidualizing [11] if it satisfies the following properties:

(i) ${}_SC$ admits a degree wise finite S-projective resolution;
(ii) C_R admits a degree wise finite R^{op}-projective resolution;
(iii) The homothety map ${}_SS_S \rightarrow Hom_{R^{op}}(C, C)$ is an isomorphism;
(iv) The homothety map ${}_RR_R \rightarrow Hom_S(C, C)$ is an isomorphism;
(v) $Ext^i_S(C, C) = 0$ for any $i \geq 1$;
(vi) $Ext^i_{R^{op}}(C, C) = 0$ for any $i \geq 1$.

Let C be a semidualizing module for a ring R. An R-module is C-projective [11] if it has the form $C \otimes_R P$ for some projective module P. An R-module is called C-injective if it has the form $Hom_R(C, I)$ for some injective module I. Set

$$\mathcal{P}_C(R) = \{C \otimes_R P \mid P\, is\ R\text{-projective}\},$$

and

$$\mathcal{I}_C(R) = \{Hom_R(C, I) \mid I\, is\ R\text{-injective}\}.$$

An R-module is called C-flat [11] if it has the form $C \otimes_R F$ for some flat module F. Set $\mathcal{F}_C(R) = \{C \otimes_R F \mid F \text{ is R-flat}\}$.

If $C = R$ in the above definitions, we see that $\mathcal{P}_C(R)$, $\mathcal{I}_C(R)$ and $\mathcal{F}_C(R)$ are the classes of ordinary projective, injective and flat R-modules, which we usually denote $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ respectively.

A left R-module M is said to be Gorenstein $\mathcal{X}$-flat [17] if there is a $\mathscr{A} \otimes -$ exact exact sequence $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ of $\mathcal{X}$-flat left R-modules with $M = \ker(F^0 \longrightarrow F^1)$ where $\mathscr{A}$ denotes class of all $\mathcal{X}$-injective right R-modules.

2. The resolutions of Gorenstein $\mathcal{X}_\mathcal{Y}$-flat modules with respect to semidualizing modules

In this section, we give the relation between the Gorenstein $\mathcal{X}_\mathcal{Y}$-flat resolution with respect to semidualizing module and the $\mathcal{X}$-flat resolution of a module.

Now, we introduce C-$\mathcal{X}$-flat and C-$\mathcal{X}$-injective modules as follows.

Definition 2.1. *An R-module is called C-$\mathcal{X}$-flat if it has the form $C \otimes_R F$ for some $\mathcal{X}$-flat module F. We denoted by $\mathcal{X}f_C(R) = \{C \otimes_R F \mid F \text{ is } \mathcal{X}\text{-flat}\}$.*

Definition 2.2. *An R-module is called C-$\mathcal{X}$-injective if it has the form $Hom_R(C, E)$ for some $\mathcal{X}$-injective R-module E. We denoted by $\mathcal{FI}i_C(R) = \{Hom_R(C, E) \mid E \text{ is } \mathcal{X}\text{-injective}\}$.*

When $C = R$, the above definitions imply the classes of $\mathcal{X}$-flat and $\mathcal{X}$-injective R-modules. Let $M \in R\text{-}Mod$. Write M^I (resp., $M^{(I)}$) is the direct product (resp., sum) of copies of a module M indexed by a set I. We denote $Add_R M$ (resp., $Prod_R M$) the subclass of $R\text{-}Mod$ consisting of all modules isomorphic to direct summands of direct sums (resp., direct products) of copies of M. We start with the following

Proposition 2.3. $\mathcal{FI}f_C(R) = Add_R C$.

Proof. Let $F \in R\text{-}Mod$ be $\mathcal{X}$-flat. Then F is isomorphic to a direct summand of $K^{(J)}$ for some cardinal J, where K is $\mathcal{X}$-flat generator. So $C \otimes_R F$ is isomorphic to a direct summand of $C \otimes_R K^{(J)}$ ($\cong C^{(J)}$), and hence $C \otimes_R F \in Add_R C$. Thus we have $\mathcal{FI}f_C(R) \subseteq Add_R C$. Conversely, for any $M \in Add_R C$, there exists $N \in R\text{-}Mod$ such that $M \oplus N \cong C^{(J)}$ for some cardinal J. Note that $\mathcal{B}_C(R)$ is closed under direct sums and direct summands by [11, Proposition 4.2]. Since $C \cong C \otimes_R R \in \mathcal{B}_C(R)$ by [11, Lemma 5.1], both $C^{(J)}$ and M are in $\mathcal{B}_C(R)$. Since $Hom_R(C, M) \oplus Hom_R(C, N) \cong Hom_R(C, C^{(J)}) \cong R^{(J)}$, $Hom_R(C, M) \in R\text{-}Mod$ is $\mathcal{X}$-flat. Thus $M \in \mathcal{FI}f_C(R)$ by [11, Lemma 5.1]. Therefore $Add_R C \subseteq \mathcal{FI}f_C(R)$. □

Now we introduce G$\mathcal{X}_\mathcal{Y}$-flat module as follows

Definition 2.4. *Let $\mathcal{Y}$ be a subclass of $\mathcal{X}$ and it is containing all injective modules. A left R-module M is said to be Gorenstein $\mathcal{X}_\mathcal{Y}$-flat (G$\mathcal{X}_\mathcal{Y}$-flat for short) if there is an exact sequence*

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of $\mathcal{X}$-flat right R-modules such that $M \cong im(F_0 \to F^0)$ and such that $- \otimes Y$ leaves the above sequence exact whenever $Y \in \mathcal{Y}$.

Definition 2.5. *A ring R is said to be left $G\mathcal{X}_\mathcal{Y}f$-closed if $G\mathcal{X}_\mathcal{Y}f$ is closed under extensions, i.e., for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of left R-modules if M_1 and M_3 are in $G\mathcal{X}_\mathcal{Y}f$, then M_2 is in $G\mathcal{X}_\mathcal{Y}f(R)$, where $G\mathcal{X}_\mathcal{Y}f(R)$ is the class of all Gorenstein $\mathcal{X}_\mathcal{Y}$-flat R-modules.*

Example 2.6. *Every Gorenstein flat module is Gorenstein $\mathcal{X}_\mathcal{Y}$-flat; It follows that every GF-closed ring is $G\mathcal{X}_\mathcal{Y}f$-closed ring.*

Definition 2.7. *A complete $\mathcal{X}_\mathcal{Y}\mathcal{F}_C$-resolution is a $\mathcal{Y}_C \otimes_R -$ that leaves the sequence:*

$$(1) \qquad \cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

exact, with all F_i and F^i are $\mathcal{X}$-flat and $Y \in \mathcal{Y}$. A module $M \in R\text{-}Mod$ is called G_C-$\mathcal{X}_\mathcal{Y}$-flat if there exists a complete $\mathcal{X}_\mathcal{Y}\mathcal{F}_C$-resolution as in (1) with $M = Coker(F_1 \to F_0)$. Let $\mathcal{GX}_\mathcal{Y}f_C(R)$ be the class of G_C-$\mathcal{X}_\mathcal{Y}$-flat modules in $R\text{-}Mod$.

It is trivial that in case ${}_RC_R = {}_RR_R$, the G_C-$\mathcal{X}_\mathcal{Y}$-flat modules are just the Gorenstein $\mathcal{X}_\mathcal{Y}$-flat modules.

Using the definition, we immediately get the following results.

Proposition 2.8. *If* $(F_i)_{i\in I}$ *is a family of* G_C-$\mathcal{X}_\mathcal{Y}$*-flat modules, then* $\bigoplus F_i$ *is* G_C-$\mathcal{X}_\mathcal{Y}$*-flat.*

Proposition 2.9. *An R-module M is* G_C-$\mathcal{X}_\mathcal{Y}$*-flat if and only if M admits a* $\mathcal{X}_\mathcal{Y} f_C(R)$*-resolution T with* $Hom_R(C,E)\otimes_R T$ *exact for any* $\mathcal{X}$*-injective E. and* $Tor^R_{\geq 1}(Hom_R(C,E),M)=0$.

The following result is due to [15].

Proposition 2.10. *Let C be a semidualizing R-module. Then the class* $\mathcal{G}\mathcal{X}_\mathcal{Y} f_C(R)$ *is closed under kernels of epimorphisms and extensions.*

Proposition 2.11. *If F is* $\mathcal{X}$*-flat R-module, then F and* $C\otimes_R F$ *are* G_C-$\mathcal{X}_\mathcal{Y}$*-flat. Thus, every R-module admits a* G_C-$\mathcal{X}_\mathcal{Y}$*-flat resolution.*

Proof. Follows from [10, Example 2.8(a), Propositions 2.1, 2.13(1) and 2.15] and since the class of G_C-$\mathcal{X}_\mathcal{Y}$-flat modules contains the class of $\mathcal{X}$-flat modules, every R-module admits a G_C-$\mathcal{X}_\mathcal{Y}$-flat resolution. □

Theorem 2.12. *Let R be a* $GX_\mathcal{Y} f$*-closed ring and C is semidualizing module, then the class* $\mathcal{G}\mathcal{X}\mathcal{Y} f_C(R)$ *of* G_C-$\mathcal{X}_\mathcal{Y}$*-flat R-modules is projectively resolving and closed under direct summands.*

Proof. Using the dual of Theorem 2.8 in [20] and together with the [15, Lemma 5.2], we see that $\mathcal{G}\mathcal{X}_\mathcal{Y} f_C(R)$ is projectively resolving. Now, comparing Proposition 2.5 with Proposition 1.4 in [9], we get $\mathcal{G}\mathcal{X}_\mathcal{Y} f_C(R)$ is closed under direct summands. □

Proposition 2.13. *Let R be a* $G\mathcal{X}_\mathcal{Y} f$*-closed ring. Then every cokernel in a complete* $\mathcal{X}_\mathcal{Y}\mathcal{F}_C$*–resolution is* G_C-$\mathcal{X}_\mathcal{Y}$*-flat.*

Proof. Follows from Proposition 2.9, Theorem 2.12 and [15, Lemma 5.4]. □

Lemma 2.14. *Let R be a* $G\mathcal{X}_\mathcal{Y} f$*-closed ring and let* $M\in R$*-Mod be* G_C-$\mathcal{X}_\mathcal{Y}$*-flat. Then there exists* $\mathcal{Y}_C\otimes -$ *exact sequences:*

$$0\to M\to G\to N\to 0$$

and

$$0\to K\to F\to M\to 0$$

in R-Mod where N,K *are* G_C-$\mathcal{X}_\mathcal{Y}$*-flat,* $G\in Add_R C$*, and F is* $\mathcal{X}$*-flat.*

Proof. It follows from the definition of G_C-$\mathcal{X}$-flat modules and Proposition 2.13 . □

The following result plays a crucial role in this section.

Lemma 2.15. *Let R be a* $G\mathcal{X}_\mathcal{Y} f$*-closed ring and suppose that*

$$0\to A\to G_1\xrightarrow{f} G_0\to M\to 0 \tag{2}$$

is an exact sequence in R-Mod where G_0,G_1 *are* G_C-$\mathcal{X}_\mathcal{Y}$*-flat. Then we have the following exact sequences:*

$$0\to A\to C_1\to G\to M\to 0, \tag{3}$$

and

$$0\to A\to H\to F\to M\to 0 \tag{4}$$

where $C_1\in Add_R C$*, F is* $\mathcal{X}$*-flat, and* G,H *are* G_C-$\mathcal{X}_\mathcal{Y}$*-flat.*

Proof. Since G_1 is G_C-$\mathcal{X}_\mathcal{Y}$-flat, there exists a short exact sequence $0 \to G_1 \to C_1 \to G' \to 0$ where $C_1 \in Add_R C$ and G' is G_C-$\mathcal{X}_\mathcal{Y}$-flat by Lemma 2.14. Then we have the following pushout diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & Im(f) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & C_1 & \longrightarrow & B & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & G' & = & G' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

Consider the following pushout diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Im(f) & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & G' & = & G' & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

Since G_0 and G' are G_C-$\mathcal{X}_\mathcal{Y}$-flat, G is also G_C-$\mathcal{X}_\mathcal{Y}$-flat by Theorem 2.12. Connecting the middle rows in the above two diagrams, we get the first desired exact sequence (3).

Since G_0 is G_C-$\mathcal{X}_\mathcal{Y}$-flat, there exists an exact sequence $0 \to G'' \to F \to G_0 \to 0$ where F is $\mathcal{X}$-flat and G'' is G_C-$\mathcal{X}_\mathcal{Y}$-flat by Lemma 2.14. Then we have the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G'' & = & G'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Im(f) & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G'' & = & G'' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & Im(f) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since G_1 and G'' are G_C-$\mathcal{X}_\mathcal{Y}$-flat, H is also G_C-$\mathcal{X}_\mathcal{Y}$-flat by Theorem 2.12. Connecting the middle rows in the above two diagrams, we get the second desired exact sequence (4). □

Gao et al. [6] investigated yoke and Gorenstein yoke modules, i.e., yoke of Gorenstein flat resolutions of modules over a right coherent ring. In a similar manner we introduce C-yoke and G_C-yoke module as follows.

Definition 2.16. *Let n be a positive integer. An R-module A is called an C-yoke module (of M) if there exists an exact sequence*

$$0 \to A \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in R-Mod where all F_i are C-$\mathcal{X}$-flat.

Definition 2.17. *Let n be a positive integer. An R-module A is called G_C-yoke R-module (of M) if there exists an exact sequence*

$$0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

in R-Mod where all G_i are G_C-$\mathcal{X}_\mathcal{Y}$-flat.

The following result establishes the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-$\mathcal{X}_\mathcal{Y}$-flat resolution and the $\mathcal{X}$-flat resolution of a module.

Lemma 2.18. *Let R be a $G\mathcal{X}_{\mathcal{Y}}f$-closed ring and let $n \geq 1$ and*

$$0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

be an exact sequence in R-Mod where all G_i are G_C-$\mathcal{X}_{\mathcal{Y}}$-flat. Then we have the following:

(i) *There exist exact sequences:*

$$0 \to A \to C_{n-1} \to \cdots \to C_1 \to C_0 \to N \to 0$$

and

$$0 \to M \to N \to G \to 0$$

in R-Mod where all $C_i \in Add_R C$ and G is G_C-$\mathcal{X}_{\mathcal{Y}}$-flat.

(ii) *There exist exact sequences*

$$0 \to B \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

and

$$0 \to H \to B \to A \to 0$$

in R-Mod where all F_i are $\mathcal{X}$-flat and H is G_C-$\mathcal{X}_{\mathcal{Y}}$-flat.

Proof. We proceed by induction on n.

(i) When $n = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in R-Mod. Since we have a $\mathcal{Y}_C \otimes_R -$ exact exact sequence $0 \to G_0 \to C_0 \to G \to 0$ in R-Mod where $C_0 \in Add_R C$ and G is G_C-$\mathcal{X}_{\mathcal{Y}}$-flat by Lemma 2.14, we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

The middle row and the last column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod where all G_i are G_C-$\mathcal{X}_{\mathcal{Y}}$-flat. Put $K = Coker(G_{n-1} \to G_{n-2})$. By Lemma 2.15, we have an exact sequence

$$0 \to A \to C_{n-1} \to G'_{n-2} \to K \to 0$$

in R-Mod where $C_{n-1} \in Add_R C$ and G'_{n-2} is G_C-$\mathcal{X}_{\mathcal{Y}}$-flat. Put $A' = Im(C_{n-1} \to G'_{n-2})$. Then we get an exact sequence $0 \to A' \to G'_{n-2} \to G_{n-3} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod. Hence, by the induction hypothesis, we get the desired result.

(ii) When $n = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in R-Mod. Since we have a $\mathcal{Y}_C \otimes_R -$ exact exact sequence $0 \to H \to F_0 \to G_0 \to 0$ in R-Mod where F_0 is $\mathcal{X}$-flat and H is G_C-$\mathcal{X}_\mathcal{Y}$-flat by Lemma 2.14, we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & B & \to & F_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & G_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

The middle row and the first column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod where all G_i are G_C-$\mathcal{X}_\mathcal{Y}$-flat. Put $K = Ker(G_1 \to G_0)$. By Lemma 2.15, we get an exact sequence

$$0 \to K \to G_1' \to F_0 \to M \to 0$$

in R-Mod where F_0 is $\mathcal{X}$-flat and G_1' is G_C-$\mathcal{X}_\mathcal{Y}$-flat. Put $M' = Im(G_1' \to P_0)$. Then we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_2 \to G_1' \to G_0 \to M \to 0$ in R-Mod. So, by the induction hypothesis, we obtain the assertion. □

Here is a version of Schannuel's Lemma for $\mathcal{X}_\mathcal{Y}\mathcal{F}_C$-resolutions.

Proposition 2.19. *Let M be a left R-module, and consider two exact sequences of left R-modules,*

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

and

$$0 \to H_n \to H_{n-1} \to \cdots \to H_0 \to M \to 0,$$

where $G_0, \cdots, G_{n-1}$ and $H_0, \cdots, H_{n-1}$ are G_C-$\mathcal{X}_\mathcal{Y}$-flat. If R is $GX_\mathcal{Y}f$-closed, then G_n is G_C-$\mathcal{X}_\mathcal{Y}$-flat if and only if H_n is G_C-$\mathcal{X}_\mathcal{Y}$-flat.

Proof. It follows from Proposition 2.8 and Proposition 2.12 . □

3. The dimension of Gorenstein $\mathcal{X}_\mathcal{Y}$-flat modules with respect to Semidualizing modules

The class of G_C-$\mathcal{X}_\mathcal{Y}$-flat modules can be used to define the G_C-$\mathcal{X}_\mathcal{Y}$-flat dimension.

Definition 3.1. *For a module $M \in R$-Mod, the G_C-$\mathcal{X}_\mathcal{Y}$-flat dimension of M, denoted by $G_C - X_Y fd_R(M)$, is defined as $inf\{n|$ there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod where all G_i are G_C-$\mathcal{X}_\mathcal{Y}$-flat $\}$.*

We have $G_C - X_Y fd_R(M) \geq 0$, *and* $G_C - X_Y fd_R(M) = \infty$ *if no such integer exists.*

We start with the following standard result.

Lemma 3.2. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in R-Mod.*

(i) $G_C - X_Y fd_R(N) \leq max\, \{G_C - X_Y fd_R(M), G_C - X_Y fd_R(L) + 1\}$, *and the equality holds if* $G_C - X_f d_R(M) \neq G_C - X_Y fd_R(L)$.

(ii) $G_C - X_Y fd_R(L) \leq max\, \{G_C - X_Y fd_R(M), G_C - X_Y fd_R(N) - 1\}$, *and the equality holds if* $G_C - X_Y fd_R(M) \neq G_C - X_Y fd_R(N)$.

(iii) $G_C - X_Y fd_R(M) \leq max\, \{G_C - X_Y fd_R(L), G_C - X_Y fd_R(N)\}$, *and the equality holds if* $G_C - X_y fd_R(N) \neq G_C - X_Y fd_R(L) + 1$.

Proof. It is immediate. □

The proof of the following theorem is similar to [9, Theorem 3.15].

Theorem 3.3. *Assume that R is* $G\mathcal{X}_\mathcal{Y}f$*-closed and C is a semidualizing module. If any two of the modules* M, M' *or* M'' *in a short exact sequence* $0 \to M' \to M \to M''$ *have finite* G_C-$\mathcal{X}_\mathcal{Y}$*-flat dimension, then so has the third.*

Proposition 3.4. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in R-Mod. If* $L \neq 0$ *and* N *is* G_C-$\mathcal{X}_\mathcal{Y}$*-flat, then* $G_C - X_Y fd_R(L) = G_C - X_Y fd_R(M)$.

Proof. It follows by Lemma 3.2(3). □

We give a criterion for computing the G_C-$\mathcal{X}_\mathcal{Y}$-flat dimension of modules as follows. It generalizes [9, Theorem 3.14]. We denote $\overline{Add_R C} = Add_R C \cup Add_R R$.

Proposition 3.5. *Let R be a* $G\mathcal{X}_\mathcal{Y}f$*-closed ring. The following statements are equivalent for any* $M \in R$*-Mod and* $n \geq 0$:

(i) $G_C - X_Y fd_R(M) \leq n$;

(ii) *For every non-negative integer t such that* $0 \leq t \leq n$, *there exists an exact sequence* $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ *in R-Mod such that* X_t *is* G_C-$\mathcal{X}_\mathcal{Y}$*-flat and* $X_i \in \overline{Add_R C}$ *for* $i \neq t$.

Proof. $(ii) \Rightarrow (i)$. It is trivial.
$(i) \Rightarrow (ii)$. We proceed by induction on n. Suppose $G_C - X_Y fd_R(M) \leq 1$. Then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ in *R-Mod* where G_0 and G_1 are G_C-$\mathcal{X}_\mathcal{Y}$-flat. By Lemma 2.15 where $A = 0$, we get the exact sequences $0 \to C_1 \to G'_0 \to M \to 0$ and $0 \to G'_1 \to F_0 \to M \to 0$ in *R-Mod* where $C_1 \in Add_R C$, F_0 is $\mathcal{X}$-flat, and G'_0, G'_1 are G_C-$\mathcal{X}_\mathcal{Y}$-flat.

Now suppose $G_C - X_Y fd_R(M) = n \geq 2$. Then there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in *R-Mod* where G_i are G_C-$\mathcal{X}_\mathcal{Y}$-flat for any $0 \leq i \leq n$. Set $A = Coker(G_3 \to G_2)$. By applying Lemma 2.15 to the exact sequence $0 \to A \to G_1 \to G_0 \to M \to 0$, we get an exact sequence $0 \to G_n \to \cdots \to G_2 \to G'_1 \to F_0 \to M \to 0$ in *R-Mod* where G'_1 are G_C-$\mathcal{X}_\mathcal{Y}$-flat and F_0 is $\mathcal{X}$-flat. Set $N = Coker(G_2 \to G'_1)$. Then we have $G_C - X_Y fd_R(N) \leq n-1$. By the induction hypothesis, there exists an exact sequence

$$0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to F_0 \to M \to 0$$

in $R\text{-}Mod$ such that F_0 is $\mathcal{X}$-flat, X_t is G_C-$\mathcal{X}_\mathcal{Y}$-flat, $X_i \in \overline{Add_R C}$ for $i \neq t$ and $1 \leq t \leq n$.

Now we need only to prove (ii) for $t = 0$. Set $B = Coker(G_2 \to G_1)$. By the induction hypothesis, we get an exact sequence $0 \to X_n \to \cdots \to X_3 \to X_2 \to G'_1 \to B \to 0$ in $R\text{-}Mod$ where G'_1 is G_C-$\mathcal{X}_\mathcal{Y}$-flat and $X_i \in \overline{Add_R C}$ for any $2 \leq i \leq n$. Set $D = Coker(X_3 \to X_2)$. Then by applying Lemma 2.15 to the exact sequence $0 \to D \to G'_1 \to G_0 \to M \to 0$, we get the exact sequence $0 \to D \to C_1 \to G'_0 \to M \to 0$ in $R\text{-}Mod$ where $C_1 \in Add_R C$ and G'_0 is G_C-$\mathcal{X}_\mathcal{Y}$-flat. Thus we obtain the desired exact sequence

$$0 \to X_n \to \cdots \to X_2 \to X_1 \to G'_0 \to M \to 0$$

in $R\text{-}Mod$ where all $X_i \in \overline{Add_R C}$ and G'_0 is G_C-$\mathcal{X}_\mathcal{Y}$-flat. □

References

[1] D. Bennis, *Rings over which the class of Gorenstein flat modules is closed under extensions*, Comm. Algebra 37(3) (2009), 855-868.

[2] L. W. Christensen, A. Frankild, and H. Holm, *On Gorenstein projective, injective and flat dimensions - a functorial description with applications*, J. Algebra 302 (1) (2006), 231-279.

[3] E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, De Gruyter Expositions in Mathematics No 30, New York, 2000.

[4] E. Enochs, O. M. G. Jenda and B. Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Shuxue Bannian Kan 10 (1993), 1-9.

[5] H. -B. Foxby, *Gorenstein modules and related modules*, Math. Scand 31 (1972), 267-284.

[6] Z. Gao and F. Wang, *Coherent rings and Gorenstein yoke modules*, J. Algebra Appl. 11 (2) 1250028, 8P (2012).

[7] R. Göbel and J. Trlifaj, *Approximations and Endomorphisms Algebras of Modules*, de Gruyter, Berlin, 2000.

[8] E. S. Golod, *G-dimension and generalized perfect ideals*, Algebraic geometry and its applications, Trudy Mat. Inst. Steklov. 165 (1984), 62-66.

[9] H. Holm, *Gorenstein Homological Dimensions*, J. Pure Appl. Algebra 189 (2004), 167-193.

[10] H. Holm and P. Jørgensen, *Semidualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra 205 (2007), 423-445.

[11] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. 47 (2007), 781-808.

[12] Z. Liu, Z. Huang and A. Xu, *Gorenstein Projective Dimension Relative to a Semidualizing Bimodule*, Comm. Algebra 41 (1) (2013), 1-18.

[13] L. Mao and N. Ding, *FI-injective and FI-flat modules*, J. Algebra 209 (2007), 367-385.

[14] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.

[15] S. Sather-Wagstaff, T. Sharif and D. White, *AB-Contexts and Stability for Gorenstein Flat Modules with Respect to Semidualizing Modules*, Algebr. Represent. Theory 14 (2011), 403-428.

[16] A. Umamaheswaran, R. Udhayakumar, J Cao and C. Selvaraj , *Covers and Envelopes of $\mathcal{X}$-injective and $\mathcal{X}^\perp$-projective modules*, communicated to *Indian J. Pure Appl. Math.* (2018).

[17] C. Selvaraj and A.Umamaheswaran, *Gorenstein $\mathcal{X}$-flat dimension and Tate homology*, Gulf J. Math. 2(4) (2014) 32-42.

[18] R. Udhayakumar and C. Selvaraj, *Gorenstein n-flat dimension with respect to a semidualizing module*, Palestine Journal of Mathematics 5(1) (2016) , 144153.

[19] W. V. Vasconcelos, *Divisor Theory in Module Categories*, North - Holland Publishing Co., Amsterdam, 1974.
[20] D. White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Comm. Algebra 2 (2010), 111-137.

Department of Mathematics, School of Advanced Sciences,, VIT , Vellore - 632014,, Tamil Nadu, India.,
E-mail address: bijuwillwin@gmail.com

Department of Mathematics, School of Advanced Sciences,, VIT, Vellore - 632014,, Tamil Nadu, India.,
E-mail address: udhayaram_v@yahoo.co.in

Department of Mathematics, Harish-Chandra Research Institute (HRI), Chhatnag, Allahabad - 211019, India.
E-mail address: ruthreswaran@gmail.com

Department of Mathematics, Bannari Amman Institute of Technology, Sathyamangalam, Erode - 638401, India.
E-mail address: rishwanthpari@gmail.com