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SOME IDENTITIES FOR q-BERNOULLI NUMBERS AND POLYNOMIALS ARISING FROM q-BERNSTEIN POLYNOMIALS

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Abstract. In this paper, we study q-Bernoulli numbers and polynomials which are different from Carlitz q-Bernoulli numbers and polynomials and arising from p-adic invariant integrals on $\mathbb{Z}_p$, and investigate some properties of these numbers and polynomials. Then we will consider p-adic invariant integrals on $\mathbb{Z}_p$ of the q-Bernstein polynomials and show that they can be expressed in terms of the q-Bernoulli numbers. In addition, from such p-adic invariant integrals we will derive some identities for the q-Bernoulli numbers.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$.

Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$ and let $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Assume that $f(x)$ is uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \ \ \text{(see [6 − 8])}, \tag{1.1}$$

and the p-adic invariant integral on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{q\to 1}\int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \ \ \text{(see [6 − 8])}. \tag{1.2}$$

From (1.2), we note that

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_1(x) - \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = f'(0), \ \ \text{(see [6 − 8])}. \tag{1.3}$$

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By (1.3), we easily get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.4}$$

where $B_n(x)$ are the ordinary Bernoulli polynomials.
When $x = 0$, $B_n = B_n(0)$, $(n \geq 0)$, are called the Bernoulli numbers.
From (1.4), we can easily derive the following equation.

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = B_n(x), \quad (n \geq 0), \quad \text{(see [6, 7, 13, 14, 16, 17])}. \tag{1.5}$$

From (1.3) and (1.5), we note that

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing B^n by B_n.
In [3, 4], L. Carlitz Considered q-Bernoulli numbers which are recursively given by

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.
He also defined q-Bernoulli polynomials given by

$$\beta_{n,q}(x) = \left(q^x \beta_q + [x]_q\right)^n = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}, \quad (n \geq 0), \quad \text{(see [3])}.$$

It is easy to show that $\lim_{q\to 1} \beta_{n,q} = B_n$, $(n \geq 0)$.
In view of (1.4), we may consider the q-Bernoulli polynomials, different from Carlitz's q-Bernoulli polynomials, as follows:

$$\sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_1(y), \quad \text{(see [5])}. \tag{1.6}$$

When $x = 0$, $B_{n,q} = B_{n,q}(0)$, $(n \geq 0)$, are called the q-Bernoulli numbers.
From (1.6), we note that the Witt's formula for q-Bernoulli polynomials is given by

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_1(y), \quad (n \geq 0),$$

and

$$B_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_1(x), \quad (n \geq 0), \quad \text{([see [5, 11])}. \tag{1.7}$$

By (1.7), we easily get

$$B_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} B_{l,q} = \left(q^x B_q + [x]_q\right)^n, \quad (n \geq 0), \tag{1.8}$$

with the usual convention about replacing B_q^n by $B_{n,q}$.

From (1.2) and (1.7), we note that

$$B_{0,q}=1,\quad (qB_q+1)^n-B_{n,q}=\begin{cases}\frac{\log q}{q-1}, & \text{if } n=1,\\ 0, & \text{if } n>1.\end{cases}$$

For $x\in\mathbb{Z}_p$ and $n\in\mathbb{N}\cup\{0\}$, the p-adic q-Bernstein operators are given by

$$B_{n,q}(f|x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k}=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_{k,n}(x|q), \tag{1.9}$$

where $f(x)$ is continuous on $\mathbb{Z}_p$.

The p-adic q-Bernstein polynomials of degree n are defined by

$$B_{k,n}(x|q)=\binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k},\quad \text{(see [2, 9])}, \tag{1.10}$$

where $n,k\geq 0$, and $x\in\mathbb{Z}_p$.

Note that $\lim_{q\to 1}B_{k,n}(x|q)=B_{k,n}(x)$, where $B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ are the ordinary Bernstein polynomials of degree n (see $[1,10,12-15,17,18,19,20]$).

In this paper, we study q-Bernoulli numbers and polynomials which are different from Carlitz q-Bernoulli numbers and polynomials and arising from p-adic invariant integrals on $\mathbb{Z}_p$, and investigate some properties of these numbers and polynomials. Then we will consider p-adic invariant integrals on $\mathbb{Z}_p$ of the q-Bernstein polynomials and show that they can be expressed in terms of the q-Bernoulli numbers. In addition, from such p-adic invariant integrals we will derive some identities for the q-Bernoulli numbers.

2. Some integrals for q-Bernstein polynomials and their connections with q-Bernoulli numbers

First, we observe that

$$\int_{\mathbb{Z}_p}[1-x+y]_{q^{-1}}^n d\mu_1(y)=(-1)^nq^n\int_{\mathbb{Z}_p}[x+y]_q^n d\mu_1(y),\quad (n\geq 0). \tag{2.1}$$

By (1.7) and (2.1), we get

$$B_{n,q^{-1}}(1-x)=(-1)^nq^nB_{n,q}(x),\quad (n\geq 0). \tag{2.2}$$

From (1.7), we note that

$$B_{0,q}(x)=1,\quad B_{n,q}(x)=\frac{\log q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^lq^{lx}\frac{l}{q^l-1}. \tag{2.3}$$

Thus, from (2.3) we can derive the generating function for $B_{n,q}(x)$ as follows:

$$\begin{aligned}F(t,x)&=e^{\frac{t}{1-q}}+\frac{\log q}{1-q}t\sum_{m=0}^{\infty}q^{x+m}e^{[x+m]_qt}\\&=\sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Now, for $n > 1$, we observe that

$$\begin{aligned} B_{n,q}(2) &= (q^2 B_q + 1 + q)^n = \big(q(qB_q+1)+1\big)^n = \sum_{l=0}^{n} \binom{n}{l} q^l (qB_q+1)^l \\ &= 1 + \binom{n}{1} q(qB_q+1) + \sum_{l=2}^{n} \binom{n}{l} q^l (qB_q+1)^l \\ &= 1 + nq\Big(\frac{\log q}{q-1} + B_{1,q}\Big) + \sum_{l=2}^{n} \binom{n}{l} q^l B_{l,q} \\ &= nq\frac{\log q}{q-1} + \sum_{l=0}^{n} \binom{n}{l} q^l B_{l,q} \\ &= nq\frac{\log q}{q-1} + (qB_q+1)^n = nq\frac{\log q}{q-1} + B_{n,q}. \end{aligned} \tag{2.5}$$

Therefore we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$B_{n,q^{-1}}(1-x) = (-1)^n q^n B_{n,q}(x).$$

In particular, if $n \in \mathbb{N}$ with $n > 1$, then we have

$$B_{n,q}(2) = nq\frac{\log q}{q-1} + B_{n,q}.$$

For $n > 1$, from Theorem 2.1, we note that

$$\begin{aligned} \int_{\mathbb{Z}_p} [x]_q^n d\mu_1(x) &= \int_{\mathbb{Z}_p} [x+2]_q^n d\mu_1(x) - nq\frac{\log q}{q-1} \\ &= (-1)^n q^{-n} \int_{\mathbb{Z}_p} [x-1]_{q^{-1}}^n d\mu_1(x) - nq\frac{\log q}{q-1} \\ &= \int_{\mathbb{Z}_p} [1-x]_q^n d\mu_1(x) - nq\frac{\log q}{q-1}. \end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p} [1-x]_q^n d\mu_1(x) = \int_{\mathbb{Z}_p} [x]_q^n d\mu_1(x) + nq\frac{\log q}{q-1}.$$

By (1.10), we easily get

$$B_{n-k,n}(1-x|q^{-1}) = B_{n,k}(x|q), \quad (n, k \geq 0). \tag{2.7}$$

Thus, for $m, n \in \mathbb{N} \cup \{0\}$ with $n-k>1$ and from Theorem 2.2, we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x|q)d\mu_1(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x|q^{-1})d\mu_1(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n-l}d\mu_1(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\Big(B_{n-l,q^{-1}}+(n-l)\frac{\log q}{q-1}\Big) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}B_{n-l,q^{-1}}+\binom{n}{k}\frac{n\log q}{q-1}\delta_{k,0} \\
&- \frac{k\log q}{q-1}\binom{n}{k}\delta_{k,1} \\
&= \begin{cases} B_{n,q^{-1}}+n\frac{\log q}{q-1}, & \text{if } k=0, \\ n(B_{n-1,q^{-1}}-B_{n,q^{-1}})-n\frac{\log q}{q-1}, & \text{if } k=1, \\ \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}B_{n-l,q^{-1}}, & \text{if } k>1. \end{cases}
\end{aligned} \tag{2.8}$$

Note here that $\delta_{k,n}$ are the Kronecker's delta. On the other hand, noting that

$$[1-x]_{q^{-1}} = 1-[x]_q, \tag{2.9}$$

we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x|q)d\mu_1(x) &= \binom{n}{k}\int_{\mathbb{Z}_p}[x]_q^k[1-x]_{q^{-1}}^{n-k}d\mu_1(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\int_{\mathbb{Z}_p}[x]_q^{l+k}d\mu_1(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l B_{k+l,q}.
\end{aligned} \tag{2.10}$$

Therefore, by (2.8) and (2.10), we obtain the following theorem.

Theorem 2.3. *Let $n, k \in \mathbb{N} \cup \{0\}$ with $n > k+1$.*
For $k > 1$, we have

$$\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}B_{n-l,q^{-1}} = \sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l B_{l+k,q}.$$

In particular,

$$B_{n-1,q^{-1}} - B_{n,q^{-1}} - n\frac{\log q}{q-1} = \sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l B_{l+1,q},$$

and

$$B_{n,q^{-1}} + n\frac{\log q}{q-1} = \sum_{l=0}^{n}\binom{n}{l}(-1)^l B_{l,q}.$$

For $m, n, k \in \mathbb{N} \cup \{0\}$ with $m+n > 2k+1$, we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x|q) B_{k,m}(x|q) d\mu_1(x) &= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} [x]_q^{2k} [1-x]_{q^{-1}}^{n+m-2k} d\mu_1(x) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^{n+m-l} d\mu_1(x) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \Big(B_{n+m-l,q^{-1}} + (n+m-l) \frac{\log q}{q-1} \Big) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} B_{n+m-l,q^{-1}} \\
&\quad + \binom{n}{k}\binom{m}{k} \delta_{k,0} \frac{\log q}{q-1} (n+m) \\
&= \begin{cases} B_{n+m,q^{-1}} + (n+m)\frac{\log q}{q-1}, & \text{if } k=0, \\ \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} B_{n+m-l,q^{-1}}, & \text{if } k \geq 1. \end{cases}
\end{aligned} \tag{2.11}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x|q) B_{k,m}(x|q) d\mu_1(x) &= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} [x]_q^{2k} [1-x]_{q^{-1}}^{n+m-2k} d\mu_1(x) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{2k+l} d\mu_1(x) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l B_{2k+l,q}.
\end{aligned} \tag{2.12}$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.4. *Let $m, n, k \in \mathbb{N} \cup \{0\}$ with $m+n > 2k+1$.*
For $k \geq 1$, we have

$$\sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l B_{2k+l,q} = \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} B_{n+m-l,q^{-1}}.$$

In particular, when $k=0$, we have

$$B_{n+m,q^{-1}} + (n+m)\frac{\log q}{q-1} = \sum_{l=0}^{n+m} \binom{n+m}{l} (-1)^l B_{l,q}.$$

For $s \in \mathbb{N}$, with $s \geq 2$, let $k, n_1, n_2, \cdots n_s \in \mathbb{N} \cup \{0\}$, with $\sum_{i=1}^{s} n_i > sk+1$. Then we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} \prod_{i=1}^{s} B_{k,n_i}(x|q) d\mu_1(x) &= \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} [x]_q^{sk} [1-x]_{q^{-1}}^{n_1+\cdots+n_s-sk} d\mu_1(x) \\
&= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^{n_1+\cdots+n_s-l} d\mu_1(x) \\
&= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \Big(B_{n_1+\cdots+n_s-l,q^{-1}} \\
&\quad + \big(\sum_{i=1}^{s} n_i - l\big) \frac{\log q}{q-1} \Big)
\end{aligned}$$

$$\begin{aligned}
&= \prod_{i=1}^{s} \binom{n_i}{k} \Big\{ \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} B_{n_1+\cdots+n_s-l,q^{-1}} \\
&\quad + \big(\sum_{i=1}^{s} n_i\big) \frac{\log q}{q-1} \delta_{k,0} \Big\} \qquad (2.13) \\
&= \begin{cases} B_{n_1+\cdots+n_s,q^{-1}} + \left(\sum_{i=1}^{s} n_i\right) \frac{\log q}{q-1}, & \text{if } k=0, \\ \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} B_{n_1+\cdots+n_s-l,q^{-1}}, & \text{if } k \geq 1. \end{cases}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{Z}_p} \prod_{i=1}^{s} B_{k,n_i}(x|q) d\mu_1(x) &= \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} [x]_q^{sk} [1-x]_{q^{-1}}^{n_1+\cdots+n_s-sk} d\mu_1(x) \\
&= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{sk+l} d\mu_1(x) \qquad (2.14) \\
&= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l B_{sk+l,q}.
\end{aligned}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.5. *For $s \in \mathbb{N}$, with $s \geq 2$, let $k, n_1, n_2, \cdots, n_s \in \mathbb{N} \cup \{0\}$ with $n_1+n_2+\cdots+n_s > sk+1$. For $k \geq 1$, we have*

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l B_{sk+l,q} = \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} B_{n_1+\cdots+n_s-l,q^{-1}}.$$

In particular, when $k=0$, we have

$$B_{n_1+\cdots+n_s,q^{-1}} + \Big(\sum_{i=1}^{s} n_i\Big) \frac{\log q}{q-1} = \sum_{l=0}^{n_1+\cdots+n_s} \binom{n_1+\cdots+n_s}{l} (-1)^l B_{l,q}.$$

3. Conclusion

Here we studied q-Bernoulli numbers and polynomials which are different from Carlitz q-Bernoulli numbers and polynomials. They arise naturally from some p-adic invariant integrals on $\mathbb{Z}_p$, as indicated by (1.6). After investigating some properties of these numbers and polynomials, we considered p-adic invariant integrals on $\mathbb{Z}_p$ of the q-Bernstein polynomials and showed that they can be expressed in terms of the q-Bernoulli numbers.

Along the same line, we can introduce a q-analogue of Euler numbers and polynomials by considering the p-adic fermionic integrals instead of the p-adic invariant integrals on $\mathbb{Z}_p$. Then we can investigate fermionic p-adic integrals on $\mathbb{Z}_p$ of the q-Bernstein polynomials and reveal their connections with those q-analogues of q-Euler numbers. These will appear as a separate paper.

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