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REMARKS ON RECURRENCE FORMULAS FOR THE APOSTOL-TYPE NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, by differentiating the generating functions for one of the family of the Apostol-type numbers and polynomials with respect to their parameters, we present some partial differential equations including these functions. By making use of these equations, we provide some new formulas, relations and identities including these numbers and polynomials and their derivatives. Furthermore, by using a collection of the generating functions for the aforementioned family and their functional equations, we investigate the numbers and polynomials belonging to this family and their relationships with other well-known special numbers and polynomials including the Apostol-Bernoulli numbers and polynomials of higher order, the Apostol-Euler numbers and polynomials of higher order, the Frobenius-Euler numbers and polynomials of higher order, the λ-array polynomials, the λ-Stirling numbers, and the λ-Bernoulli numbers and polynomials.

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1. INTRODUCTION AND PRELIMINARIES

In recent years, studying on generalization and unification of the well-known special numbers and polynomials has gained importance by virtue of abundance of their potential applications and their connections with very kind of areas such as mathematics, quantum theory, physics and so on (*cf.* [1]-[27]).

Hence, in this paper, our motivation is to study on the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, which are one of the families of Apostol-type numbers and polynomials of higher order defined by the second author [22] in the manner of the following generating functions:

$$F_w(t;\lambda;k) = \frac{1}{(\lambda e^t + \lambda^{-1}e^{-t} + 2)^k} = \sum_{n=0}^{\infty} W_n^{(k)}(\lambda)\frac{t^n}{n!} \tag{1}$$

and

$$G_w(t,x,k;\lambda) = e^{xt}F_w(t;\lambda;k) = \sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!}. \tag{2}$$

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where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, in which $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{C}$ denote the set of non-negative integers and the complex numbers, respectively, $\mathbb{N} := \{1, 2, 3, \cdots\}$ being the set of positive integers.

The following relationship:

$$W_n^{(k)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} W_m^{(k)}(\lambda) \tag{3}$$

holds true between the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, so that, obviously,

$$W_n^{(k)}(\lambda) = W_n^{(k)}(0;\lambda), \tag{4}$$

with

$$W_n(\lambda) = W_n^{(1)}(\lambda) \quad \text{and} \quad W_n(x;\lambda) = W_n^{(1)}(x;\lambda).$$

(*cf.* [22]).

Moreover, for $n \in \mathbb{N}$, the numbers $W_n(\lambda)$ satisfy the following recurrence relation (*cf.* [11], [22]):

$$W_n(\lambda) = \frac{\lambda}{(\lambda+1)^2} \sum_{m=0}^{n-1} \left((-1)^{n-m+1} \lambda^{-1} - \lambda \right) \binom{n}{m} W_m(\lambda). \tag{5}$$

Thus, clearly several values of the numbers $W_n(\lambda)$ are given by (*cf.* [22])

$$\begin{aligned} W_1(\lambda) &= -\frac{\lambda(\lambda-1)}{(\lambda+1)^3}, \\ W_2(\lambda) &= \frac{\lambda\left(\lambda^2-4\lambda+1\right)}{(\lambda+1)^4} \\ W_3(\lambda) &= -\frac{\lambda\left(\lambda^3-11\lambda^2+11\lambda-1\right)}{(\lambda+1)^5}, \end{aligned}$$

and so on.

On the other hand, the numbers $W_n^{(k)}(\lambda)$ satisfy the following computation formula (*cf.* [11], [23]):

$$W_n^{(k)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k-1)}(\lambda) W_{n-m}(\lambda) \tag{6}$$

so that, by (6), a few values of the numbers $W_n^{(k)}(\lambda)$ are given as follows:

$$W_0^{(2)}(\lambda) = \frac{\lambda^2}{(\lambda+1)^4}, W_1^{(2)}(\lambda) = \frac{2\lambda^2(1-\lambda)}{(\lambda+1)^5}, W_2^{(2)}(\lambda) = \frac{4\lambda^2\left(\lambda^2-3\lambda+1\right)}{(\lambda+1)^6},$$

$$W_0^{(3)}(\lambda) = \frac{\lambda^3}{(\lambda+1)^6}, W_1^{(3)}(\lambda) = \frac{3\lambda^3(1-\lambda)}{(\lambda+1)^7}, W_2^{(3)}(\lambda) = \frac{3\lambda^3\left(3\lambda^2-8\lambda+3\right)}{(\lambda+1)^8},$$

and so on.

Remark 1. *The reader may consult the works* [10], [11], [22] and [23] *for further properties, relations and identities including the numbers* $W_n(\lambda)$*, the numbers* $W_n^{(k)}(\lambda)$*, the polynomials* $W_n(x;\lambda)$ *and the polynomials* $W_n^{(k)}(x;\lambda)$*.*

Furthermore, since we derive some new relations between the numbers $W_n^{(k)}(\lambda)$, the polynomials $W_n^{(k)}(x;\lambda)$ and other well-known special numbers and polynomials, we recall them with their generating functions in the following manner:

The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(k)}(x;\lambda)$ of higher order are defined by the following generating function:

$$F_B(t,x;\lambda;k)=\left(\frac{t}{\lambda e^t-1}\right)^k e^{xt}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}(x;\lambda)\frac{t^n}{n!} \tag{7}$$

where $\lambda\in\mathbb{C}$ and $|t+\log\lambda|<2\pi$, so that, in the special case when $x=0$, these polynomials are reduced to the Apostol-Bernoulli numbers $\mathcal{B}_n^{(k)}(\lambda)$ of higher order with

$$\mathcal{B}_n^{(k)}(\lambda)=\mathcal{B}_n^{(k)}(0;\lambda)$$

(*cf.* [15]).

It should be note that

$$\mathcal{B}_n(x;\lambda)=\mathcal{B}_n^{(1)}(x;\lambda),$$
$$\mathcal{B}_n(\lambda)=\mathcal{B}_n^{(1)}(\lambda)$$

and

$$B_n=\mathcal{B}_n(1)$$

where $\mathcal{B}_n(x;\lambda)$, $\mathcal{B}_n(\lambda)$ and B_n denote the Apostol-Bernoulli polynomials, the Apostol-Bernoulli numbers and the classical Bernoulli numbers, respectively (*cf.* [1], [15], [17], [24], [27], [26]; and see also the references cited therein).

In [9], Kim *et al.* considered the modification of the Apostol-Bernoulli polynomials and gave the following generating function for the λ-Bernoulli polynomials $B_n(\lambda;x)$:

$$\frac{\log\lambda+t}{\lambda e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(\lambda;x)\frac{t^n}{n!}, \tag{8}$$

so that,

$$B_n(\lambda)=B_n(\lambda;0)$$

where $B_n(\lambda)$ denotes the λ-Bernoulli numbers and some values of these numbers are given as follows:

$$B_0(\lambda)=\frac{\log\lambda}{\lambda-1},\quad B_1(\lambda)=\frac{\lambda-1-\lambda\log\lambda}{(\lambda-1)^2},\ldots$$

(*cf.* [9], [4], [21]; and see also the references cited therein).

In [9], Kim *et al.* derived a summation formula in connection with λ-Bernoulli numbers and polynomials as follows (*cf.* [9, p. 9]):

$$B_l(\lambda;x)-\lambda^{-k}B_l(\lambda)=\lambda^{-k}l\sum_{n=0}^{k-1}\lambda^n n^{l-1}. \tag{9}$$

The Apostol-Euler polynomials $\mathcal{E}_n^{(k)}(x;\lambda)$ of higher order are defined by means of the following generating function:

$$F_{\mathcal{E}}(t,x;\lambda;k)=\left(\frac{2}{\lambda e^t+1}\right)^k e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x;\lambda)\frac{t^n}{n!} \tag{10}$$

where $\lambda \in \mathbb{C}$ and $|t + \log(\lambda)| < \pi$, so that, in the special case when $x = 0$, these polynomials are reduced to the Apostol-Euler numbers $\mathcal{E}_n^{(k)}(\lambda)$ of higher order with

$$\mathcal{E}_n^{(k)}(\lambda) = \mathcal{E}_n^{(k)}(0; \lambda)$$

(*cf.* [15], [12]; and the references cited therein).

It should be note that

$$\mathcal{E}_n(x; \lambda) = \mathcal{E}_n^{(1)}(x; \lambda),$$
$$\mathcal{E}_n(\lambda) = \mathcal{E}_n^{(1)}(\lambda)$$

and

$$E_n = \mathcal{E}_n(1)$$

where $\mathcal{E}_n(x; \lambda)$, $\mathcal{E}_n(\lambda)$ and E_n denote the Apostol-Euler polynomials, the Apostol-Euler numbers and the classical Euler numbers of the first kind, respectively (*cf.* [1], [12], [15], [17], [24], [26] [27]; and see also the references cited therein).

The Apostol-Genocchi polynomials $\mathcal{G}_n^{(k)}(x; \lambda)$ of higher order are defined by the following generating function:

$$F_G(t, x; \lambda; k) = \left(\frac{2t}{\lambda e^t + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x; \lambda) \frac{t^n}{n!} \tag{11}$$

where $\lambda \in \mathbb{C}$ and $|t| < |\log(-\lambda)|$, so that, in the special case when $x = 0$, these polynomials are reduced to the Apostol-Genocchi numbers $\mathcal{G}_n^{(k)}(\lambda)$ of higher order with

$$\mathcal{G}_n^{(k)}(\lambda) = \mathcal{G}_n^{(k)}(0; \lambda)$$

(*cf.* [16], [13], [14]; and see also the references cited therein).

It should be note that

$$\mathcal{G}_n(x; \lambda) = \mathcal{G}_n^{(1)}(x; \lambda),$$
$$\mathcal{G}_n(\lambda) = \mathcal{G}_n^{(1)}(\lambda),$$

and

$$G_n = \mathcal{G}_n(1)$$

where $\mathcal{G}_n(x; \lambda)$, $\mathcal{G}_n(\lambda)$ and G_n denote the Apostol-Genocchi polynomials, the Apostol-Genocchi numbers and the Genocchi numbers, respectively (*cf.* [16], [13], [14], [17], [26], [27]; and see also the references cited therein).

The Frobenius-Euler polynomials $\mathcal{H}_n^{(k)}(x|\lambda)$ of higher order are defined by means of the following generating function:

$$F_H(t, x; \lambda; k) = \left(\frac{1-\lambda}{e^t - \lambda}\right)^k e^{xt} = \sum_{n=0}^{\infty} \mathcal{H}_n^{(k)}(x|\lambda) \frac{t^n}{n!} \tag{12}$$

where $\lambda (\neq 1) \in \mathbb{C}$, so that, these polynomials are reduced to the Frobenius-Euler numbers $\mathcal{H}_n^{(k)}(\lambda)$ of higher order with

$$\mathcal{H}_n^{(k)}(\lambda) = \mathcal{H}_n^{(k)}(0|\lambda)$$

and a relation between these numbers and polynomials is given by

$$\mathcal{H}_n^{(k)}(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} \mathcal{H}_m^{(k)}(\lambda) \tag{13}$$

(*cf.* [5], [6], [7], [8], [9], [19] [20]; and see also the references cited therein).

The λ-array polynomials $S_k^n(x;\lambda)$ are given by the following generating function:

$$F_A(t,x,k;\lambda) = \frac{\left(\lambda e^t - 1\right)^k}{k!} e^{xt} = \sum_{n=0}^{\infty} S_k^n(x;\lambda) \frac{t^n}{n!} \tag{14}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [2], [20]) which, for $\lambda = 1$, yields the classical array polynomials $S_k^n(x)$, which are defined by the following explicit formula:

$$S_k^n(x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n \tag{15}$$

with $S_0^0(x) = S_n^0(x) = 1$, $S_0^n(x) = x^n$ and $S_0^0(x) = 0$ when $k > n$. Moreover, for $x = 0$, the λ-array polynomials $S_k^n(x;\lambda)$ reduce to the λ-Stirling numbers, that is:

$$S(n,k;\lambda) = S_k^n(0;\lambda), \tag{16}$$

(*cf.* [2], [20], [16], [25]). In the special case of $\lambda = 1$, the λ-Stirling numbers reduce to the Stirling numbers of the second kind, that is

$$S_2(n,k) = S(n,k;1)$$

which are defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{17}$$

where

$$(x)_n = x(x-1)(x-2)\ldots(x-n+1),$$

with $(x)_0 = 1$ (*cf.* [3], [26]; see also the references cited therein).

We give detailed summary covering results of this paper as follows:

In Section 2, by differentiating the generating functions for the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ with respect to their parameters, we present partial differential equations including these functions. By making use of these equations, we provide some new formulas, relations and identities including these numbers and polynomials and their derivatives.

In Section 3, by using a collection of the generating functions for the Apostol-type numbers and polynomials of higher order and their functional equations, we investigate the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ and their relationships with other well-known special numbers and polynomials including the Apostol-Bernoulli numbers and polynomials of higher order, the Apostol-Euler numbers and polynomials of higher order, the Frobenius-Euler numbers and polynomials of higher order, the λ-array polynomials, the λ-Stirling numbers, and the λ-Bernoulli numbers and polynomials.

2. Identities related to the derivative formulas for the functions $F_w(t;\lambda;k)$ and the functions $G_w(t,x,k;\lambda)$

In this section, we provide partial differential equations including the generating functions for the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$.

With the help of these equations, we give some formulas, relations and identities including these numbers and polynomials and their derivatives.

Differentiating both sides of (1) and (2) with respect to t yields respectively the following partial differential equations:

$$\frac{\partial}{\partial t}\left\{F_w(t;\lambda;k)\right\} = -k\left(\lambda e^t - \lambda^{-1}e^{-t}\right)F_w(t;\lambda;k+1), \tag{18}$$

and

$$\frac{\partial}{\partial t}\left\{G_w(t,x;\lambda;k)\right\} = xG_w(t,x;\lambda;k) - k\left(\lambda e^t - \lambda^{-1}e^{-t}\right)G_w(t,x;\lambda;k+1). \tag{19}$$

Combining (2) and (19) yields

$$\begin{aligned}\sum_{n=0}^{\infty} W_{n+1}^{(k)}(x;\lambda)\frac{t^n}{n!} &= x\sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!} - k\left(\sum_{n=0}^{\infty}\left(\lambda-\lambda^{-1}(-1)^n\right)\frac{t^n}{n!}\right) \\ &\quad \times\sum_{n=0}^{\infty} W_n^{(k+1)}(x;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Making use of the Cauchy product rule in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 2.1.

$$W_{n+1}^{(k)}(x;\lambda) = xW_n^{(k)}(x;\lambda) - k\sum_{m=0}^{n}\binom{n}{m}\left(\lambda-\lambda^{-1}(-1)^{n-m}\right)W_m^{(k+1)}(x;\lambda).$$

Combining the special case of Theorem 2.1 when $x = 0$ with (4) yields the following corollary:

Corollary 2.2.

$$W_{n+1}^{(k)}(\lambda) = k\sum_{m=0}^{n}\binom{n}{m}\left(\lambda^{-1}(-1)^{n-m}-\lambda\right)W_m^{(k+1)}(\lambda).$$

By differentiating both sides of (1) with respect to t, we get also another version of (18) as follows:

$$\frac{\partial}{\partial t}\left\{F_w(t;\lambda;k)\right\} = kF_w(t;\lambda;k) - 2k\left(\lambda e^t + 1\right)F_w\left(t;\lambda;k+1\right).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} W_{n+1}^{(k)}(\lambda)\frac{t^n}{n!} = k\sum_{n=0}^{\infty} W_n^{(k)}(\lambda)\frac{t^n}{n!} - 2k\lambda\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{n=0}^{\infty} W_n^{(k+1)}(\lambda)\frac{t^n}{n!} - 2k\sum_{n=0}^{\infty} W_n^{(k+1)}(\lambda)\frac{t^n}{n!}.$$

Thus, by using the Cauchy product rule in the above equation, we get

$$\sum_{n=0}^{\infty} W_{n+1}^{(k)}(\lambda)\frac{t^n}{n!} = k\sum_{n=0}^{\infty}\left(W_n^{(k)}(\lambda) - 2\lambda\sum_{m=0}^{n}\binom{n}{m}W_m^{(k+1)}(\lambda) - 2W_n^{(k+1)}(\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 2.3.

$$W_{n+1}^{(k)}(\lambda) = k\left(W_n^{(k)}(\lambda) - 2\lambda \sum_{m=0}^{n} \binom{n}{m} W_m^{(k+1)}(\lambda) - 2W_n^{(k+1)}(\lambda)\right)$$

or, equivalently,

$$W_{n+1}^{(k)}(\lambda) = kW_n^{(k)}(\lambda) - 2\lambda W_n^{(k+1)}(1;\lambda) - 2W_n^{(k+1)}(\lambda).$$

Differentiating both sides of Eq. (2) with respect to λ yields the following partial differential equation:

$$\frac{\partial}{\partial\lambda}\left\{G_w(t,x;\lambda;k)\right\} = -k\left(e^t - \lambda^{-2}e^{-t}\right)G_w(t,x;\lambda;k+1). \tag{20}$$

From (20), we have

$$\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!} = k\left(\sum_{n=0}^{\infty}\left(\lambda^{-2}(-1)^n - 1\right)\frac{t^n}{n!}\right)\sum_{n=0}^{\infty} W_n^{(k+1)}(x;\lambda)\frac{t^n}{n!}.$$

By making use of the Cauchy product rule in the above equation, we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial\lambda}W_n^{(k)}(x;\lambda)\frac{t^n}{n!} = k\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\left(\lambda^{-2}(-1)^{n-m} - 1\right)W_m^{(k+1)}(x;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 2.4.

$$\frac{\partial}{\partial\lambda}W_n^{(k)}(x;\lambda) = k\sum_{m=0}^{n}\binom{n}{m}\left(\lambda^{-2}(-1)^{n-m} - 1\right)W_m^{(k+1)}(x;\lambda).$$

Combining the special case of Theorem 2.4 when $x = 0$ with (4) yields the following corollary:

Corollary 2.5.

$$\frac{d}{d\lambda}W_n^{(k)}(\lambda) = k\sum_{m=0}^{n}\binom{n}{m}\left(\lambda^{-2}(-1)^{n-m} - 1\right)W_m^{(k+1)}(\lambda).$$

Differentiating both sides of Eq. (2) with respect to x yields the following partial differential equation:

$$\frac{\partial}{\partial x}\left\{G_w(t,x;\lambda;k)\right\} = tG_w(t,x;\lambda;k).$$

From the above equation, we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!} = t\sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!}.$$

Thus, we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}W_n^{(k)}(x;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} nW_{n-1}^{(k)}(x;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 2.6.

$$\frac{\partial}{\partial x} W_n^{(k)}(x;\lambda) = n W_{n-1}^{(k)}(x;\lambda).$$

Remark 2. *According to Roman* [18, Theorem 2.5.8, p. 27]*, a Appell sequence $s_n(x)$ satisfy the following Appell identity:*

$$s_n(x+y) = \sum_{m=0}^{n} \binom{n}{m} s_m(x) y^{n-m}.$$

Hence, by (2.6)*, one can infer that the polynomials $W_n^{(k)}(x;\lambda)$ are Appell sequences and it is clear that the following identity holds for the polynomials $W_n^{(k)}(x;\lambda)$:*

$$W_n^{(k)}(x+y;\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k)}(x) y^{n-m}. \tag{21}$$

Substituting $y = 1$ into (21) yields

$$W_n^{(k)}(x+1;\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k)}(x). \tag{22}$$

Substituting $y = -1$ into (21) also yields

$$W_n^{(k)}(x-1;\lambda) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} W_m^{(k)}(x). \tag{23}$$

By subtracting (23) from (22), we get the following theorem:

Theorem 2.7.

$$W_n^{(k)}(x+1;\lambda) - W_n^{(k)}(x-1;\lambda) = \sum_{m=0}^{n} \left(1-(-1)^{n-m}\right) \binom{n}{m} W_m^{(k)}(x). \tag{24}$$

From (24), we get the following corollary:

Corollary 2.8.

$$\frac{W_n^{(k)}(x+1;\lambda) - W_n^{(k)}(x-1;\lambda)}{2} = \sum_{\substack{m=0\\ n-m \text{ odd}}}^{n} \binom{n}{m} W_m^{(k)}(x). \tag{25}$$

3. Relations between the numbers $W_n^{(k)}(\lambda)$, the polynomials $W_n^{(k)}(x;\lambda)$ and other well-known special numbers and polynomials

In this section, by using a collection of the generating functions for the Apostol-type numbers and polynomials of higher order and their functional equations, we investigate the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ and their relationships with other well-known special numbers and polynomials including the Apostol-Bernoulli numbers and polynomials of higher order, the Apostol-Euler numbers and polynomials of higher order, the Frobenius-Euler numbers and polynomials of higher order, the λ-array polynomials, the λ-Stirling numbers, and the λ-Bernoulli numbers and polynomials.

3.1. **Relationships of the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ with the Apostol-Bernoulli numbers and polynomials of higher order.** Here, by using functional equations of the generating functions for the Apostol-Bernoulli numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, we derive relationship between these numbers and polynomials.

By combining (2) with (7), we get the following functional equation:

$$t^{-2k}F_B(t,x;\lambda;2k) = (-1)^k\,\lambda^{-k}G_w(t,x-k,k;-\lambda). \tag{26}$$

It follows from the above functional equation that

$$\sum_{n=0}^{\infty}\frac{1}{(n+2k)_{2k}}\mathcal{B}_{n+2k}^{(2k)}(x;\lambda)\frac{t^n}{n!} = (-1)^k\,\lambda^{-k}\sum_{n=0}^{\infty}W_n^{(k)}(x-k;-\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields a relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Apostol-Bernoulli polynomials of higher order by the following thereom:

Theorem 3.1.

$$\mathcal{B}_{n+2k}^{(2k)}(x;\lambda) = \frac{(-1)^k\,(n+2k)_{2k}}{\lambda^k}W_n^{(k)}(x-k;-\lambda). \tag{27}$$

Combining (3) with (27) yields a relation between the numbers $W_n^{(k)}(\lambda)$ and the Apostol-Bernoulli polynomials of higher order by the following corollary:

Corollary 3.2.

$$\mathcal{B}_{n+2k}^{(2k)}(x;\lambda) = \frac{(-1)^k\,(n+2k)_{2k}}{\lambda^k}\sum_{m=0}^{n}\binom{n}{m}(x-k)^{n-m}\,W_m^{(k)}(-\lambda). \tag{28}$$

Substituting $k=1$ into (27) yields the following corollary:

Corollary 3.3.

$$\mathcal{B}_{n+2}^{(2)}(x;\lambda) = -\frac{(n+2)\,(n+1)}{\lambda}W_n(x-1;-\lambda). \tag{29}$$

Replacing x by k and λ by $-\lambda$ in (27) and using (4) yields the following corollary:

Corollary 3.4.

$$\mathcal{B}_{n+2}^{(2k)}(k;-\lambda) = \frac{(n+2k)_{2k}}{\lambda^k}W_n^{(k)}(\lambda). \tag{30}$$

Remark 3. *When $k=1$, the equation (30) is reduced to the following relation:*

$$\mathcal{B}_{n+2}^{(2)}(1;-\lambda) = \frac{(n+2)\,(n+1)}{\lambda}W_n(\lambda)$$

which was given by the second author (cf. [23]*).*

3.2. **Relationships of the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ with the Apostol-Euler numbers and polynomials of higher order.** Here, by using functional equations of the generating functions for the Apostol-Euler numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, we derive relationship between these numbers and polynomials.

By combining (2) with (10), we get the following functional equation:

$$2^{-2k} F_{\mathcal{E}}(t,x;\lambda;2k) = \lambda^{-k} G_w(t;x-k,k;\lambda).$$

It follows from the above functional equation that

$$2^{-2k}\sum_{n=0}^{\infty}\mathcal{E}_n^{(2k)}(x;\lambda)\frac{t^n}{n!} = \lambda^{-k}\sum_{n=0}^{\infty}W_n^{(k)}(x-k;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields a relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Apostol-Euler polynomials of higher order by the following theorem:

Theorem 3.5.

$$\mathcal{E}_n^{(2k)}(x;\lambda) = \left(\frac{4}{\lambda}\right)^k W_n^{(k)}(x-k;\lambda). \tag{31}$$

Combining (3) with (31) yields a relation between the numbers $W_n^{(k)}(\lambda)$ and the Apostol-Euler polynomials of higher order by the following corollary:

Corollary 3.6.

$$\mathcal{E}_n^{(2k)}(x;\lambda) = \left(\frac{4}{\lambda}\right)^k \sum_{m=0}^{n}\binom{n}{m}(x-k)^{n-m}W_m^{(k)}(\lambda). \tag{32}$$

Substituting $k=1$ into (31) yields the following corollary:

Corollary 3.7.

$$\mathcal{E}_n^{(2)}(x;\lambda) = \frac{4}{\lambda}W_n(x-1;\lambda). \tag{33}$$

Replacing x by k in (31) and using (4) yields the following corollary:

Corollary 3.8.

$$\mathcal{E}_n^{(2k)}(k;\lambda) = \left(\frac{4}{\lambda}\right)^k W_n^{(k)}(\lambda). \tag{34}$$

Remark 4. *When $k=1$, the equation* (34) *is reduced to the following relation:*

$$\mathcal{E}_n^{(2)}(1;\lambda) = \frac{4}{\lambda}W_n(\lambda)$$

which was given by the second author (cf. [22, p. 2358]*).*

3.3. **Relationships of the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ with the Apostol-Genocchi numbers and polynomials of higher order.** Here, by using functional equations of the generating functions for the Apostol-Genocchi numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, we derive relationship between these numbers and polynomials.

By combining (2) with (11), we get the following functional equation:

$$t^{-2k}F_G(t,x;\lambda;2k) = \left(\frac{4}{\lambda}\right)^k G_w(t,x-k,k;\lambda).$$

It follows from the above functional equation that

$$\sum_{n=0}^{\infty}\frac{1}{(n+2k)_{2k}}\mathcal{G}_n^{(2k)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{4}{\lambda}\right)^k\sum_{n=0}^{\infty}W_n^{(k)}(x-k;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields a following relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Apostol-Genocchi polynomials of higher order by the following theorem:

Theorem 3.9.

$$\mathcal{G}_{n+2k}^{(2k)}(x;\lambda) = (n+2k)_{2k}\left(\frac{4}{\lambda}\right)^k W_n^{(k)}(x-k;\lambda). \tag{35}$$

Combining (3) with (35) yields a relation between the numbers $W_n^{(k)}(\lambda)$ and the Apostol-Genocchi polynomials of higher order by the following corollary:

Corollary 3.10.

$$\mathcal{G}_{n+2k}^{(2k)}(x;\lambda) = (n+2k)_{2k}\left(\frac{4}{\lambda}\right)^k\sum_{m=0}^{n}\binom{n}{m}(x-k)^{n-m}W_m^{(k)}(\lambda). \tag{36}$$

Substituting $k=1$ into (35) yields the following corollary:

Corollary 3.11.

$$\mathcal{G}_{n+2}^{(2)}(x;\lambda) = \frac{4\,(n+2)\,(n+1)}{\lambda}W_n(x-1;\lambda) \tag{37}$$

Replacing x by k in (35) and using (4) yields the following corollary:

Corollary 3.12.

$$\mathcal{G}_{n+2k}^{(2k)}(k;\lambda) = (n+2k)_{2k}\left(\frac{4}{\lambda}\right)^k W_n^{(k)}(\lambda). \tag{38}$$

Remark 5. *When $k=1$, the equation (38) is reduced to the following relation:*

$$\mathcal{G}_{n+2}^{(2)}(1;\lambda) = \frac{4\,(n+2)\,(n+1)}{\lambda}W_n(\lambda)$$

which was given by the second author (cf. [23]*).*

3.4. **Relationships of the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ with the Frobenius-Euler numbers and polynomials of higher order.** Here, by using functional equations of the generating functions for Frobenius-Euler numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, we derive relationship between these numbers and polynomials.

By combining (2) with (12), we get the following functional equation:

$$F_H(t;x;\lambda^{-1};2k) = \frac{(-1)^k(\lambda-1)^{2k}}{\lambda^k} G_w(t,x-k;k;-\lambda).$$

It follows from the above functional equation that

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(2k)}(x|\lambda^{-1})\frac{t^n}{n!} = \frac{(-1)^k(\lambda-1)^{2k}}{\lambda^k}\sum_{n=0}^{\infty} W_n^{(k)}(x-k;-\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields a relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Frobenius-Euler polynomials of higher order by the following theorem:

Theorem 3.13.

$$\mathcal{H}_n^{(2k)}(x|\lambda^{-1}) = \frac{(-1)^k(\lambda-1)^{2k}}{\lambda^k} W_n^{(k)}(x-k;-\lambda). \tag{39}$$

Combining (3) with (39) yields a relation between the numbers $W_n^{(k)}(\lambda)$ and the Frobenius-Euler polynomials of higher order by the following corollary:

Corollary 3.14.

$$\mathcal{H}_n^{(2k)}(x|\lambda^{-1}) = \frac{(-1)^k(\lambda-1)^{2k}}{\lambda^k}\sum_{m=0}^{n}\binom{n}{m}(x-k)^{n-m} W_m^{(k)}(-\lambda). \tag{40}$$

Substituting $k=1$ into (39) yields the following corollary:

Corollary 3.15.

$$\mathcal{H}_n^{(2)}(x|\lambda^{-1}) = -\frac{(\lambda-1)^2}{\lambda} W_n(x-1;-\lambda). \tag{41}$$

Replacing x by k in (39) and using (4) yields the following corollary:

Corollary 3.16.

$$\mathcal{H}_n^{(2k)}(k|\lambda^{-1}) = \frac{(-1)^k(\lambda-1)^{2k}}{\lambda^k} W_n^{(k)}(-\lambda). \tag{42}$$

When $k=1$, the equation (42) is reduced to the following relation:

Corollary 3.17.

$$\mathcal{H}_n^{(2)}(1|\lambda^{-1}) = -\frac{(\lambda-1)^2}{\lambda} W_n(-\lambda). \tag{43}$$

3.5. **Relationships of the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ with the λ-array polynomials, the λ-Stirling numbers, and λ-Bernoulli numbers and polynomials.** Here, by using functional equations of the generating functions for λ-array polynomials and the polynomials $W_n^{(k)}(x;\lambda)$, we derive relationships between the numbers $W_n^{(k)}(\lambda)$, the polynomials $W_n^{(k)}(x;\lambda)$, λ-array polynomials, the λ-Stirling numbers, and the λ-Bernoulli numbers and polynomials.

By combining (2) with (14), we get the following functional equation:

$$F_A(t,-x,2k;\lambda)\,G_w(t,x,k;-\lambda)=\frac{(-1)^k\lambda^k e^{kt}}{(2k)!}.$$

It follows from the above functional equation that

$$\sum_{n=0}^{\infty}S_{2k}^{n}(-x;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}W_n^{(k)}(x;-\lambda)\frac{t^n}{n!}=\frac{(-1)^k\lambda^k}{(2k)!}\sum_{n=0}^{\infty}k^n\frac{t^n}{n!}.$$

By making use of the Cauchy product rule in the above equation, we get

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}(-x;\lambda)\,W_{n-m}^{(k)}(x;-\lambda)\right)\frac{t^n}{n!}=\frac{(-1)^k\lambda^k}{(2k)!}\sum_{n=0}^{\infty}k^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 3.18.

$$\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}(-x;\lambda)\,W_{n-m}^{(k)}(x;-\lambda)=\frac{(-1)^k\lambda^k k^n}{(2k)!}. \tag{44}$$

By (16), substituting $x=0$ into (44) yields a relation between the λ-Stirling numbers and the numbers $W_n^{(k)}(\lambda)$ by the following corollary:

Corollary 3.19.

$$\sum_{m=0}^{n}\binom{n}{m}S(m,2k;\lambda)\,W_{n-m}^{(k)}(-\lambda)=\frac{(-1)^k\lambda^k k^n}{(2k)!}. \tag{45}$$

From (44), we have

$$(-1)^k(2k)!\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}(-x;\lambda)\,W_{n-m}^{(k)}(x;-\lambda)=\lambda^k k^n.$$

Summing the above equation over all $0\le k\le v$ yields the following combinatorial sums:

$$\sum_{k=0}^{v}(-1)^k(2k)!\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}(-x;\lambda)\,W_{n-m}^{(k)}(x;-\lambda)=\sum_{k=0}^{v}\lambda^k k^n.$$

Therefore, by combining the above equation with (9), we get the following theorem:

Theorem 3.20.

$$\sum_{k=0}^{v}\sum_{m=0}^{n}(-1)^k(2k)!\binom{n}{m}S_{2k}^{m}(-x;\lambda)\,W_{n-m}^{(k)}(x;-\lambda)=\frac{\lambda^{v+1}B_{n+1}(\lambda;v+1)-B_{n+1}(\lambda)}{n+1}.$$

On the other hand, combining (2) with (14) yields also the following functional equation:

$$F_A\left(t,x,2k;\lambda\right)G_w(t,x,k;-\lambda)=\frac{(-1)^k\lambda^k e^{(2x+k)t}}{(2k)!}.$$

By using the above functional equation, we have

$$\sum_{n=0}^{\infty}S_{2k}^{n}\left(x;\lambda\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}W_n^{(k)}(x;-\lambda)\frac{t^n}{n!}=\frac{(-1)^k\lambda^k}{(2k)!}\sum_{n=0}^{\infty}(2x+k)^n\frac{t^n}{n!}.$$

By using the Cauchy product rule in the above equation, we have

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}\left(x;\lambda\right)W_{n-m}^{(k)}(x;-\lambda)\right)\frac{t^n}{n!}=\frac{(-1)^k\lambda^k}{(2k)!}\sum_{n=0}^{\infty}(2x+k)^n\frac{t^n}{n!}.$$

Thus, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 3.21.

$$\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}\left(x;\lambda\right)W_{n-m}^{(k)}(x;-\lambda)=\frac{(-1)^k\lambda^k}{(2k)!}\left(2x+k\right)^n.$$

Substituting $\lambda=1$ into the above theorem, we get

$$\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}\left(x\right)W_{n-m}^{(k)}(x;-1)=\frac{(-1)^k}{(2k)!}\left(2x+k\right)^n.$$

Combining the above equation with (15), we arrive at the following corrollary:

Corollary 3.22.

$$\sum_{m=0}^{n}\sum_{j=0}^{2k}(-1)^j\binom{n}{m}\binom{2k}{j}\left(x+j\right)^m W_{n-m}^{(k)}(x;-1)=(-1)^k\left(2x+k\right)^n.$$

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