

DOUBLE IDEAL LACUNARY STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

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ABSTRACT. In this paper, we introduce new notion, namely, \mathcal{I} -lacunary double statistical convergence in topological groups. We also investigate some inclusion relations between \mathcal{I} -double statistical and \mathcal{I} -lacunary double statistical convergence.

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1. INTRODUCTION

The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [6] and also independently by Schoenberg [32] for real and complex sequences, but rapid developments were started after the papers of Šalát [19] and Fridy [8]. Nowadays it has become one of the most active area of research in the field of summability. Di Maio and Kočinac [11] introduced the concept of statistical convergence in topological spaces and statistical Cauchy condition in uniform spaces and established the topological nature of this convergence. Quite recently Savas [31] studied \mathcal{I}_θ -statistical convergence for sequences in topological groups where more references on this important summability method can be found. In many branches of science and engineering we often come across double sequences, i.e. sequences of matrices and certainly there are situations where either the idea of ordinary convergence does not work or the underlying space does not serve our purpose. Therefore to deal with such situations we have to introduce some new type of measures which can provide a better tool and suitable frame work. Cakalli and Savas [3] studied the statistical convergence of double sequences to topological groups. Also lacunary statistical convergence of double sequences in topological groups was studied in [30]. Recently Savas [29] introduced new notion, namely, \mathcal{I}_λ -double statistical convergence in topological groups and also some inclusion relations between \mathcal{I} -double statistical and \mathcal{I}_λ -double statistical convergence were investigated. Mursaleen et al. [13] studied generalized statistical convergence and statistical core of double sequences.

Recall that a subset E of the set \mathbb{N} of natural numbers is said to have "natural density" $\delta(E)$ if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in E\}|$$

where the vertical bars denote the cardinality of the enclosed set. The number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0,$$

(see, Fridy [8]). Before continuing with this paper we present some definitions and preliminaries:

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $(k_r)(k_{r-1})^{-1}$ will be abbreviated by q_r .

In another direction, in [7], a new type of convergence called lacunary statistical convergence was introduced as follows: A sequence (x_k) of real numbers is said to be lacunary statistically convergent to L (or, S_θ -convergent to L) if for any $\epsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [7] the relation between lacunary statistical convergence and statistical convergence was established among other things. Some works on lacunary statistical convergence can be found in ([10],[20],[21],[23]). The (relatively more general) concept of \mathcal{I} -convergence was introduced by Kostyrko et al.[9] in a metric space. Later on it was further studied by Dems [5] and Das et al.[4]. More investigations in this direction and more applications of ideals can be found in [24, 25, 26, 27, 28].

2. DEFINITIONS AND NOTATIONS

The following definitions and notions will be needed.

Definition 2.1. A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of \mathbb{N} if the following conditions hold:

- (a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
- (b) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$,

Definition 2.2. A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be an ideal of \mathbb{N} if the following conditions hold:

- (a) $\emptyset \notin F$,
- (b) $A, B \in F$ implies $A \cap B \in F$,
- (c) $A \in F$, $A \subset B$ implies $B \in F$,

If \mathcal{I} is a proper ideal of \mathbb{N} (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of \mathbb{N} . It is called the filter associated with the ideal.

Definition 2.3. A proper ideal \mathcal{I} is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 2.4. (See [9]) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in \mathbb{N} . The sequence $\{x_k\}$ of elements of \mathbb{R} is said to be \mathcal{I} -convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}$.

By X , we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset A of X , $s(A)$ will denote the set of all sequences (x_n) such that x_n is in A for $n = 1, 2, \dots$, $c(X)$ will denote the set of all convergent sequences. In [1], a sequence (x_k) in X is called to be statistically convergent to an element L of X if for each neighbourhood U of 0,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k - L \notin U\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in X is denoted by $st(X)$.

Also, Cakalli [2] defined lacunary statistical convergence in topological groups as follows: A sequence (x_k) is said to be S_θ -convergent to L (or lacunary statistically convergent to L) if for each neighborhood U of 0, $\lim_{r \rightarrow \infty} (h_r)^{-1} |k \in I_r : x_k - L \notin U| = 0$. In this case, we write

$$S_\theta - \lim_{k \rightarrow \infty} x_k = L \quad \text{or} \quad x_k \rightarrow L(S_\theta).$$

3. DEFINITIONS AND NOTATIONS

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural case density can be defined as follows: The lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\{\frac{K_{m,n}}{mn}\}_{m,n=1,1}^{\infty,\infty}$ has a limit then we say that K has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N} \times \mathbb{N}\}$. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero), while the set $\{(i, 2j) : i, j \in \mathbb{N} \times \mathbb{N}\}$ has double natural density $1/2$.

Recently the studies of double sequences has a rapid growth. Mursaleen and Edely [12] extended the idea of statistical convergence of single sequences to double sequences of scalars and established relations between statistical convergence and strongly Cesàro summable double sequences. R. Patterson studied the analogues of some fundamental theorems of summability theory. Also the double lacunary statistical convergence was introduced

by Patterson and Savas [17].

Mursaleen and Edely has given main definition as follows:

Definition 3.1. ([12]). A double sequences $x = (x_{k,l})$ is said to be P-statistically convergent to L provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \{ \text{number of } (k,l) : k < m \text{ and } l < n, |x_{k,l} - L| \geq \epsilon \} = 0,$$

In this case we write $st^2 - \lim_{k,l} x(k,l) = L$ and we denote the set of all statistical convergent double sequences by st^2 .

It is clear that a convergent double sequence is also st^2 -convergent but the inverse is not true, in general. Also note that a st^2 -convergence need not be bounded. For example, the sequence $x = (x_{k,l})$ defined by

$$x(k,l) = \begin{cases} kl, & \text{if } k \text{ and } l \text{ are square} \\ 1, & \text{otherwise} \end{cases}$$

is st^2 -convergent. Nevertheless it neither convergent nor bounded.

The double sequence $\theta = \{(k_r, l_s)\}$ is called **double lacunary** if there exist two increasing of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, θ is determine by $I_r = \{(k) : k_{r-1} < k \leq k_r\}$, $I_s = \{(l) : l_{s-1} < l \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ \& } l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\theta_{r,s}}$.

Let $K \subseteq \mathbf{N} \times \mathbf{N}$ has double lacunary density $\delta_2^\theta(K)$ if

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : (k,l) \in K\}|$$

exists.

In 2005, R. F. Patterson and E. Savas [17] studied double lacunary statistically convergence by giving the definition for complex sequences as follows:

Definition 3.2. Let θ be a double lacunary sequence; the double number sequence x is st_θ^2 -convergent to L provided that for every $\epsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{kl} - L| \geq \epsilon\}| = 0.$$

In this case write $st_\theta^2 - \lim x = L$ or $x_{kl} \rightarrow L(S_\theta^2)$.

More investigation in this direction and more applications of double lacunary and double sequences can be found in [20], [21] and [23].

Quite recently, Cakalli and Savas[3] defined the statistical convergence of double sequences $x = (x_{k,l})$ of points in a topological group as follows.

In a topological group X , double sequence $x = (x_{kl})$ is called statistically convergent to a point L of X if for each neighbourhood U of 0 the set

$$\{(k,l), k \leq n; \text{ and } l \leq m : x_{kl} - L \notin U\}$$

has double natural density zero. In this case we write $S^2\text{-}\lim_{k,l} x_{kl} = L$ and we denote the set of all statistically convergent double sequences by $S^2(X)$.

The definition of double lacunary statistical convergence in topological groups was defined by Savas [30] as follows:

Definition 3.3. A sequence $x = (x_{kl})$ is said to be S_θ^2 -convergent to L (or double lacunary statistically convergent to L) if for each neighborhood U of 0,

$$P\text{-}\lim_{r,s \rightarrow \infty} (h_{rs})^{-1} |(k,l) \in I_{rs} : x_{kl} - L \notin U| = 0$$

In this case, we write

$$S_\theta^2\text{-}\lim_{k,l \rightarrow \infty} x_{kl} = L \quad \text{or} \quad x_{kl} \rightarrow L(S_\theta^2)$$

In this presentation, our goal is to extend a few results known in the literature from ordinary (single) sequences to double sequences in topological groups and to give some important inclusion theorems.

By the convergence of a double sequence we mean the convergence in Pringsheims sense ([18]). A double sequence $x = (x_{kl})$ of real numbers is said to be convergent in the Pringsheim's sense or P -convergent if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$ whenever $k, l \geq N$ and L is called Pringsheim limit (denoted by $P\text{-}\lim x = L$).

In a topological group X , the above definitions become as in the following: a double sequence $x = (x_{kl})$ of points in X is said to be convergent to a point to a point L in X in the Pringsheims sense if for every neighbourhood U of 0 there exists $N \in \mathbb{N}$ such that $x_{kl} - L \in U$ whenever $k, l \geq N$. L is called the Pringsheim limit of x .

Throughout \mathcal{I}_2 will stand for a proper strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{kl})$ of real number is said to be convergent to the number L with respect to the ideal \mathcal{I} , if for each $\varepsilon > 0$

$$A(\varepsilon) = \{(k, l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we write $I\text{-}\lim_{k,l} x_{kl} = L$

Now we are ready to give the main definitions of \mathcal{I} - double statistical convergence and \mathcal{I} - double lacunary statistical convergence in topological groups as follows:

Definition 3.4. A double sequence $x = (x_{kl})$ of points in a topological group X , is said to be \mathcal{I} - double statistically convergent to L or $S(\mathcal{I}_2)$ -convergent to L , if for each neighbourhood U of 0 and $\delta > 0$

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{k \leq m \text{ and } l \leq n : x_{kl} - L \notin U\}| \geq \delta\} \in \mathcal{I}_2.$$

In this case we write $x_{kl} \rightarrow L(S(\mathcal{I}_2))$. The set of all \mathcal{I} - double statistically convergent sequences will be denoted by simply $S(\mathcal{I}_2)(X)$.

Remark 3.5. For $\mathcal{I}_2 = \mathcal{I}_{2fin} = \{A \subset \mathbb{N} \times \mathbb{N}, A \text{ is finite}\}$, $S(\mathcal{I}_2)$ -convergence coincides with double statistical convergence in a topological group X which was studied by Cakalli and Savas [3].

Definition 3.6. A sequence $x = (x_{kl})$ of points in a topological group X , is said to be \mathcal{I} -lacunary double statistically convergent to L or $S(\mathcal{I}_2)$ -lacunary convergent to L if for each neighbourhood U and any $\delta > 0$

$$\left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(k, l) \in I_{rs} : x_{kl} - L \notin U\}| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write

$$S^\theta(\mathcal{I}_2) - \lim_{k, l \rightarrow \infty} x_{kl} = L \text{ or } x_{kl} \rightarrow L(S^\theta(\mathcal{I}_2))$$

and define

$$S^\lambda(\mathcal{I}_2)(X) = \{x = (x_{kl}) : \text{for some } L, S^\theta(\mathcal{I}_2) - \lim_{k, l \rightarrow \infty} x_{kl} = L\}$$

Remark 3.7. For $\mathcal{I}_2 = \mathcal{I}_{2fin} = \{A \subset \mathbb{N} \times \mathbb{N}, A \text{ is finite}\}$, \mathcal{I} -lacunary double statistical convergence becomes lacunary double statistical convergence in topological groups, (see, [30]).

It is obvious that every \mathcal{I} -lacunary double statistically convergent sequence has only one limit, that is, if a double sequence is \mathcal{I}_2^θ -statistically convergent to L_1 and L_2 then $L_1 = L_2$

We now prove the following theorems.

4. INCLUSION THEOREMS

In this section, we prove some analogues for double sequences. For single sequences such results have been proved by Savas [31]. We now have

Theorem 4.1. For any double lacunary sequence $\theta = \{(k_r, l_s)\}$, $S(\mathcal{I}_2)(X) \subseteq S_\theta(\mathcal{I}_2)(X)$ if $\liminf q_r > 1$ and $\liminf \bar{q}_s > 1$.

Proof. Suppose that $\liminf q_r > 1$, and $\liminf \bar{q}_s > 1$, $\liminf q_r = \alpha_1$ and $\liminf \bar{q}_s = \alpha_2$, say. Write $\beta_1 = (\alpha_1 - 1)/2$ and $\beta_2 = (\alpha_2 - 1)/2$. Then there exist two positive integers r_0 and s_0 such that $q_r \geq 1 + \beta_1$ for $r \geq r_0$ and $\bar{q}_s \geq 1 + \beta_2$ for $s \geq s_0$. Thus for $r \geq r_0$, and $s \geq s_0$,

$$\begin{aligned} & h_{rs}(k_r)^{-1}(l_s)^{-1} \\ &= (1 - (k_{r-1})(k_r)^{-1})(1 - (l_{s-1})(l_s)^{-1}) \\ &= (1 - (q_r)^{-1})(1 - (q_s)^{-1}) \\ &\geq (1 - (1 + \beta_1)^{-1})(1 - (1 + \beta_2)^{-1}) \\ &= (\beta_1(1 + \beta_1)^{-1})(\beta_2(1 + \beta_2)^{-1}). \end{aligned}$$

Take any $x = (x_{k,l}) \in S(\mathcal{I}_2)(X)$, and $S(\mathcal{I}_2) - \lim_{k, l \rightarrow \infty} x_{k,l} = L$, say. We have that $S_\theta(\mathcal{I}_2) - \lim_{(k, l) \rightarrow \infty} x_{k,l} = L$. Let us take any neighborhood U of 0. Then for $r \geq r_0$, and $s \geq s_0$ we get

$$\begin{aligned} & (k_r)^{-1}(l_s)^{-1} |\{k \leq k_r, l \leq l_s : x_{kl} - L \notin U\}| \\ &\geq (k_r)^{-1}(l_s)^{-1} |\{(k, l) \in I_{rs} : x_{kl} - L \notin U\}| \\ &= h_{r,s}(k_r)^{-1}(l_s)^{-1}(h_{rs})^{-1} |\{(k, l) \in I_{rs} : x_{kl} - L \notin U\}| \\ &\geq \beta_1(1 + \beta_1)^{-1}\beta_2(1 + \beta_2)^{-1}(h_{rs})^{-1} |\{(k, l) \in I_{rs} : x_{kl} - L \notin U\}|. \end{aligned}$$

Then for any $\delta > 0$, we get

$$\begin{aligned} & \{(r, s) \in \mathbb{N} \times \mathbb{N} : (h_{rs})^{-1} |\{(k, l) \in I_{rs} : x_{kl} - L \notin U\}| \geq \delta\} \\ & \subseteq \{(r, s) \in \mathbb{N} \times \mathbb{N} : (k_r)^{-1} (l_s)^{-1} |\{k \leq k_r, l \leq l_s : x_{kl} - L \notin U\}| \geq \delta \beta_1 (1 + \beta_1)^{-1} \beta_2 (1 + \beta_2)^{-1}\} \in . \end{aligned}$$

□

We now present the next theorem

Theorem 4.2. Let $\theta = \{(k_r, l_s)\}$ and $\theta^1 = \{(k_r, l_s)\}$ be two double lacunary sequences such that $I_{rs} \subset J_{rs}$ for all $(r, s) \in \mathbb{N} \times \mathbb{N}$, if

$$(4.1) \quad \lim_{mn \rightarrow \infty} \inf \frac{h_{rs}}{\ell_{rs}} > 0$$

then $S_{\theta^1}(\mathcal{I}_2)(X) \subseteq S^\theta(\mathcal{I}_2)(X)$.

Proof. Suppose that $I_{rs} \subset J_{rs}$ for all $(r, s) \in \mathbb{N} \times \mathbb{N}$ and let (4.1) be satisfied. For neighbourhood U of 0, we have

$$\{(k, l) \in J_{rs} : x_{kl} - L \notin U\} \supseteq \{(k, l) \in I_{rs} : x_{kl} - L \notin U\}.$$

Therefore we can write

$$(\ell_{rs})^{-1} |\{(k, l) \in J_{rs} : x_{kl} - L \notin U\}| \geq h_{rs} (\ell_{rs})^{-1} (h_{rs})^{-1} |\{(k, l) \in I_{rs} : x_{kl} - L \notin U\}|$$

and so for all $(r, s) \in \mathbb{N} \times \mathbb{N}$ we have,

$$\begin{aligned} & \{(r, s) \in \mathbb{N} \times \mathbb{N} : (h_{rs})^{-1} |\{(k, l) \in I_{rs} : x_{kl} - L \notin U\}| \geq \delta\} \\ & \subseteq \{(r, s) \in \mathbb{N} \times \mathbb{N} : (\ell_{rs})^{-1} |\{(k, l) \in J_{rs} : x_{kl} - L \notin U\}| \geq \delta h_{rs} (\ell_{rs})^{-1}\} \in \mathcal{I}_2 \end{aligned}$$

for all $(r, s) \in \mathbb{N} \times \mathbb{N}$ where $I_{r,s} = \{(k, l) : k_{r-1} < k \leq k_r \text{ \& } l_{s-1} < l \leq l_s\}$, and $J_{r,s} = \{(p, q) : p_{r-1} < p \leq p_r \text{ \& } q_{s-1} < q \leq q_s\}$. Hence $S_\theta^1(\mathcal{I}_2)(X) \subseteq S_\theta(\mathcal{I}_2)(X)$. □

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