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A NOTE ON GRADED RING WITH PRIME SPECTRUM

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Abstract. In this paper, we study some algebraic and topological properties of the complement Zariski topology of a graded ideal in a commutative ring. We also determine both algebraic and topological characterizations.

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1. Introduction

Throughout this paper, all rings considered are commutative with identity. Let R be a commutative ring and consider $Spec(R)$ as the spectrum of all prime ideals of R. The Zariski topology on $Spec(R)$ plays an important role in algebraic geometry. For each ideal I of R, the variety of I is the set $V(I) = \{P \in Spec(R) : I \subseteq P\}$. Then the set $\{V(I) : I \text{ is an ideal of } R\}$ satisfies the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology on $Spec(R)$ ([1]). The development of topologies on the spectrum of prime ideals has led to the establishment of many useful connections between the Zariski topology on the spectrum of prime ideals of the ring and algebraic properties of the ring. After that, the Zariski topology on rings and modules defined by prime ideals and prime submodules, respectively, has attracted considerable attention of some authors ([6],[7],[8],[12]). It was proved that there were close relationships between algebraic properties of a ring or a module and topological properties such as Noetherian, irreducible in [6] and [12].

The Zariski topology on $Spec_G(R)$ was defined in a similar way to that of $Spec(R)$ and some useful results on the Zariski topology of graded prime ideals and graded prime submodules were proved in [5]. In this paper, we continue this investigation for graded prime ideals of a graded ring. We define a subspace $\mathcal{X}_G^I$ of Zariski topology on $Spec_G(R)$ for a graded ideal I of a graded ring R and study the connections between open subspaces of Zariski topology and graded ideals. The reason we focus on this topological space is to obtain some characterizations for graded rings and the radical of graded ideals. Moreover, we also study some algebraic tools that enable us to find some conditions for the subspace of the Zariski topological space to become quasi-compact, irreducible subset or Noetherian spectrum.

We firstly observe that $\Gamma_I = \left\{\tilde{V}_G(J) : J \text{ is a graded ideal of } R\right\}$ satisfies the axioms for closed sets of a topological space on $\mathcal{X}_G^I = Spec_G(R) \backslash V_G(I)$.

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Then we study some connections between algebraic properties of R and a topological property of $\mathcal{X}_G^I$. Hence we prove that $\sqrt{I} = \sqrt{Rr_1 + ... + Rr_n}$, where $r_i \in h(I)$ if $\mathcal{X}_G^I$ is quasi-compact. In addition, we find a necessary and sufficient algebraic condition for $\mathcal{X}_G^I$ to be quasi-compact in Theorem 3.3. We also define the generalization $N_G^I(0)$ of graded radical ideal (Definition 3.4), which is the intersection of all graded prime ideals not containing the graded ideal I of R. Then we study its algebraic structures in order to find a characterization of irreducibility. It is proved in Theorem 3.6 that the complement Zariski topology of a graded ideal I in R is irreducible if and only if $N_G^I(0)$ is a graded prime ideal of a graded ring R. Finally, we close the paper by finding some necessary and sufficient algebraic conditions for $\mathcal{X}_G^I$ to be Noetherian (Theorem 3.9).

2. Preliminary

Let G be a group with identity e. A commutative ring R with identity is a graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g. The homogeneous elements of the ring R are denoted by $h(R)$, i.e. $h(R) = \bigcup_{g\in G} R_g$. If $a \in R$, then the element a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is called the g-component of a in R_g .

Let I be an ideal of R. For $g \in G$, let $I_g = I \cap R_g$. Then I is a graded ideal of R if $I = \bigoplus_{g\in G} I_g$. In this case, I_g is called the g-component of I for $g \in G$. Clearly, the intersection of graded ideals in a graded ring is also a graded ideal. A graded ideal P of a graded ring R is called a graded prime ideal of R if $P \neq R$ and whenever $ab \in P$, either $a \in P$ or $b \in P$ holds, where $a, b \in h(R)$. The graded radical of a graded ideal I is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that if r is a homogeneous element of R, i.e. $r \in h(R)$, then r is an element of the graded radical of I if and only if $r^n \in I$ for some $n \in \mathbb{N}$. The graded radical of I is denoted by $\sqrt{I}$.

Let $\mathcal{X}_G = Spec_G(R)$ be the set of all graded prime ideals of R and $V_G(I) = \{P \in Spec_G(R) : I \subseteq P\}$ for a graded ideal I of R.

$\Gamma = \{V_G(I) : I$ is a graded ideal of $R\}$ satisfies the axioms of closed subsets of a topological space, which is called the Zariski topology on $Spec_G(R)$ as follows:

i) $V_G(0) = Spec_G(R)$ and $V_G(R) = \emptyset$,

ii) $\bigcap_{i\in\Lambda} V_G(I_i) = V_G\left(\sum_{i\in\Lambda} I_i\right) = V_G\left(\bigcup_{i\in\Lambda} I_i\right)$,

iii) $V_G(I) \cup V_G(J) = V_G(I \cap J) = V_G(IJ)$, where I, J and $\{I_i\}_{i\in\Lambda}$ are graded ideals of the graded ring R. ([11])

If the graded ideal I is generated by A, then it is clear that $V_G(A) = V_G(I)$ and also $V_G(I) = V_G(\sqrt{I})$ for any graded ideal I of R. Therefore it can be easily seen that $V_G(rR) = V_G(r)$ for any $r \in h(R)$.

We recall some topological definitions from [4] and [7] as follows:

Let X be a topological space and let D be a subset of X.

i) If every open cover of X has a finite subcover, X is said to be quasi-compact.

ii) If $X \neq \emptyset$ and for every decomposition $X = X_1 \cup X_2$ with closed subsets $X_1, X_2 \subseteq X$, either $X = X_1$ or $X = X_2$ holds, then X is said to be irreducible.

iii) If for every nonempty open set $U \subseteq X$, $U \cap D \neq \emptyset$, then D is said to be dense in X.

iv) If the closed subsets of X satisfy the descending chain condition, X is said to be Noetherian.

v) If there is an open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ for any two points $x, y \in X$, X is said to be a T_0-space.

3. Spectrum of Graded Prime Ideals

Firstly, let us give some properties of the graded variety.

Let $\mathcal{X}_G^I = Spec_G(R) \backslash V_G(I)$ and $\Gamma_I = \left\{ \tilde{V}_G(J) : J \text{ is a graded ideal of } R \right\}$, where $\tilde{V}_G(J) = V_G(J) \backslash V_G(I)$ and I is a graded ideal of a graded ring R.

Proposition 3.1. *Let A, B, I and $\{I_i\}_{i\in\Lambda}$ be graded ideals of the graded ring R. Then the following hold for variety of ideals:*

i) $\tilde{V}_G(0) = \mathcal{X}_G^I$ *and* $\tilde{V}_G(R) = \emptyset$,

ii) $\bigcap\limits_{i\in\Lambda} \tilde{V}_G(I_i) = \tilde{V}_G\left(\sum\limits_{i\in\Lambda} I_i\right) = \tilde{V}_G\left(\bigcup\limits_{i\in\Lambda} I_i\right)$,

iii) $\tilde{V}_G(A) \cup \tilde{V}_G(B) = \tilde{V}_G(A \cap B) = \tilde{V}_G(AB)$.

Proof. *i*) Let $U = \tilde{V}(K)$, where K is a graded ideal of R. If $K = 0$, $V(0) = Spec_G(R)$. Thus

$$\tilde{V}_G(0) = V_G(0) \backslash V_G(I) = Spec_G(R) \backslash V_G(I) = \mathcal{X}_G^I$$

and $\mathcal{X}_G^I \in \Gamma_I$.

Since $(Spec_G(R), \Gamma)$ is a topological space, $V(R) = \emptyset$ and hence $\tilde{V}_G(R) = V_G(R) \backslash V_G(I) = \emptyset \backslash V_G(I) = \emptyset \in \Gamma_I$.

ii) For $i \in \Lambda$, let $\left\{ \tilde{V}(B_i) \right\}$ be any family of Γ_I, where B_i is a graded ideal of R. Since

$$\bigcap_{i\in\Lambda} V_G(B_i) = V_G\left(\bigcup_{i\in\Lambda} B_i\right),$$

$$\begin{aligned} \bigcap_{i\in\Lambda} \tilde{V}_G(B_i) &= \bigcap_{i\in\Lambda} (V_G(B_i) \backslash V_G(I)) = \left(\bigcap_{i\in\Lambda} (V_G(B_i)\right) \backslash V_G(I) \\ &= V_G\left(\bigcup_{i\in\Lambda} B_i\right) \backslash V_G(I) \\ &= \tilde{V}_G\left(\bigcup_{i\in\Lambda} B_i\right) \in \Gamma_I \end{aligned}$$

One can prove the other equality with standard arguments.

iii) Let $\tilde{V}(U_1)$ and $\tilde{V}(U_2)$ be in Γ_I, where U_1 and U_2 are graded ideals of R. Since $(Spec_G(R), \Gamma)$ is a topological space,

$$V_G(U_1) \cup V_G(U_2) = V_G(U_1 \cap U_2)$$

and thus

$$\begin{aligned} \tilde{V}_G(U_1) \cup \tilde{V}_G(U_2) &= [(V_G(U_1) \cup V_G(U_2))] \backslash V_G(I) \\ &= [(V_G(U_1 \cap U_2))] \backslash V_G(I) \\ &= \tilde{V}_G(U_1 \cap U_2) \in \Gamma_I \end{aligned}$$

□

Then $\Gamma_I = \left\{ \tilde{V}_G(J) : J \text{ is a graded ideal of } R \right\}$ satisfies the axioms for closed sets of a topological space on $\mathcal{X}_G^I$, called the complement Zariski topology of I in R on $\mathcal{X}_G^I$. Clearly, the open subspace $\mathcal{X}_G^I$ of Zariski topology is equal to $Spec_G(R)$ when $I = R$ and $\mathcal{X}_G^I = \emptyset$ when $I = 0$.

The following lemma reveals some connections between $\mathcal{X}_G^I$ and a graded ideal I.

Lemma 3.2. *Let I be a graded ideal of a graded ring R. The following hold for every element $r, s \in h(R)$:*

i) $(\mathcal{X}_G^I)^r = \mathcal{X}_G^I \backslash \tilde{V}_G(r)$ *forms a base for the complement Zariski topology of I in R on $\mathcal{X}_G^I$.*

ii) $(\mathcal{X}_G^I)^r \cap (\mathcal{X}_G^I)^s = (\mathcal{X}_G^I)^{rs}$.

iii) $(\mathcal{X}_G^I)^r = \emptyset$ *if and only if* $rI \subseteq \sqrt{0}$.

iv) $(\mathcal{X}_G^I)^r = (\mathcal{X}_G^I)^s$ *if and only if* $\sqrt{rI} = \sqrt{sI}$.

Proof. *i)* If $\mathcal{X}_G^I$ is empty, then $(\mathcal{X}_G^I)^r = \emptyset$, which is the trivial case. Therefore we can assume that $\mathcal{X}_G^I \neq \emptyset$.

Let $U \subset \mathcal{X}_G^I$ be an open set. Then $U = \mathcal{X}_G^I \backslash \tilde{V}_G(J) = Spec_G(R) \backslash (V_G(J) \cup V_G(I))$, where J is a graded ideal of R. Notice that $J = \bigcup_{g \in G} J_g =< h(J) >$. Then $V_G(J) = V_G(h(J)) = \bigcap_{r \in h(J)} V_G(r)$ and

$$\begin{aligned} U &= \mathcal{X}_G^I \backslash \tilde{V}_G \left(\bigcup_{g \in G} J_g \right) = \mathcal{X}_G^I \backslash \bigcap_{r_i \in h(J)} \tilde{V}_G(r_i) \\ &= \bigcup_{r_i \in h(J)} (\mathcal{X}_G^I \backslash \tilde{V}(r_i)) = \bigcup_{r_i \in h(J)} (\mathcal{X}_G^I)^{r_i}. \end{aligned}$$

Thus it is proved that $(\mathcal{X}_G^I)^r$ is a base for the complement Zariski topology of I of R on $\mathcal{X}_G^I$.

ii) Let $P \in (\mathcal{X}_G^I)^r \cap (\mathcal{X}_G^I)^s$. Then $P \in (\mathcal{X}_G^I)^r$ and $P \in (\mathcal{X}_G^I)^s$, which means $rI \nsubseteq P$ and $sI \nsubseteq P$. Since $P \in \mathcal{X}_G$, $rsI \nsubseteq P$ and so $P \in (\mathcal{X}_G^I)^{rs}$, implying $(\mathcal{X}_I)^r \cap (\mathcal{X}_I)^s \subseteq (\mathcal{X}_I)^{rs}$.

Let $P \in (\mathcal{X}_G^I)^{rs} = \mathcal{X}_G \backslash V_G(rsI)$. Then $rsI \nsubseteq P$. Since $P \in Spec_G(R)$, $r \notin P$, $s \notin P$ and $I \nsubseteq P$, which means that $P \in \mathcal{X}_G \backslash V_G(rI)$ and $P \in \mathcal{X}_G \backslash V_G(sI)$. Thus $P \in (\mathcal{X}_G^I)^r \cap (\mathcal{X}_G^I)^s$, implies $(\mathcal{X}_I)^{rs} \subseteq (\mathcal{X}_G^I)^r \cap (\mathcal{X}_G^I)^s$.

iii) Let $(\mathcal{X}_I)^r = \emptyset$. Then $\mathcal{X}_G = V_G(rI)$. Thus $rI \subseteq P$ for every graded prime ideal P of R and so $rI \subseteq \sqrt{0}$.

Let $rI \subseteq \sqrt{0}$. Then $rI \subseteq P$ for every graded prime ideal P of R. Thus $\mathcal{X}_G = V_G(rI)$ and so $(\mathcal{X}_I^G)^r = \emptyset$.

v) Let $(\mathcal{X}_I^G)^r = (\mathcal{X}_I^G)^s$. Then $V_G(rI) = V_G(sI)$, implying $\sqrt{rI} = \sqrt{sI}$.

Let $\sqrt{rI} = \sqrt{sI}$. Then $V_G(rI) = V_G(sI)$, implying $(\mathcal{X}_I)^r = (\mathcal{X}_I)^s$. □

A proper graded ideal I of a graded ring R is said to satisfy the condition $(*)$ if there is a finite subset Δ of Λ such that $\sqrt{\langle\{r_i \in h(R) : i \in \Lambda\}\rangle} = \sqrt{\langle\{r_j \in h(R) : j \in \Delta\}\rangle}$, whenever $\sqrt{I} \subseteq \sqrt{\langle\{r_i \in h(R) : i \in \Lambda\}\rangle}$. For example, I satisfies the condition $(*)$ if $R/\sqrt{I}$ is a Noetherian ring.

In this paper, let us denote the finite set $\Delta = \{1, 2, ..., n\}$ for a positive integer n.

Some relationships between algebraic properties and a topological property are given as follows:

Theorem 3.3. *Let I be a proper graded ideal of a graded ring R. Let $(\mathcal{X}_G^I)^r = \mathcal{X}_G^I \backslash \tilde{V}_G(r) = Spec_G(R)\backslash(V_G(rI))$ for $r \in h(R)$. Then the following hold:*

i) $(\mathcal{X}_G^I)^r$ is quasi-compact for every homogenous element $r \in h(R)$.

ii) If $\mathcal{X}_G^I$ is quasi-compact, then $\sqrt{I} = \sqrt{Rr_1 + ... + Rr_n}$, where $r_i \in h(I)$.

iii) If I satisfies the condition $(*)$, then $\mathcal{X}_G^I$ is quasi-compact.

iv) The space $\mathcal{X}_G^I$ is a T_0-space for Zariski topology.

Proof. *i*) Let $\left\{(\mathcal{X}_G^I)^{a_i} : i \in \Lambda\right\}$ be an open cover of $(\mathcal{X}_G^I)^r$ for each $i \in \Lambda$, where $r, a_i \in h(R)$. Then

$$\begin{aligned} (\mathcal{X}_G^I)^r &= \mathcal{X}_G^I \backslash \tilde{V}_G(r) \subseteq \bigcup_{i\in\Lambda} (\mathcal{X}_G^I)^{a_i} = \bigcup_{i\in\Lambda} \left(\mathcal{X}_G^I \backslash \tilde{V}_G(a_i)\right) \\ &= \mathcal{X}_G^I \backslash \tilde{V}_G\left(\sum_{i\in\Lambda} Ra_i\right) \end{aligned}$$

and $\tilde{V}_G\left(\sum_{i\in\Lambda} Ra_i\right) \subseteq \tilde{V}_G(r)$, implying $\sqrt{rR} \subseteq \sqrt{\sum_{i\in\Lambda} Ra_i}$. Then there is a positive integer n such that $r^n \in \sum_{i\in\Lambda} Ra_i$. Thus there exists $i_1, i_2, ..., i_t \in \Lambda$ and $r_{i_1}, r_{i_2}, ..., r_{i_t} \in h(R)$ such that $r^n = a_{i_1}r_{i_1} + a_{i_2}r_{i_2} + ... + a_{i_t}r_{i_t}$. Let $\Delta = \{i_1, i_2, ..., i_t\} \subseteq \Lambda$. Thus $r^n \in \sum_{j\in\Delta} Ra_j$ implies $\tilde{V}_G\left(\sum_{j\in\Delta} Ra_j\right) \subseteq \tilde{V}_G(r^n) = \tilde{V}_G(r)$. Therefore $\bigcap_{j\in\Delta} \tilde{V}_G(a_j) \subseteq \tilde{V}_G(r)$ and so $\bigcup_{j\in\Delta} (\mathcal{X}_G^I \backslash \tilde{V}_G(a_j)) \supseteq \mathcal{X}_G^I \backslash \tilde{V}_G(r)$ and

$$(\mathcal{X}_G^I)^r \subseteq \bigcup_{j\in\Delta} (\mathcal{X}_G^I)^{a_j}$$

Since Δ is a finite set, $(\mathcal{X}_G^I)^r$ is quasi-compact.

ii) Let $\mathcal{X}_G^I$ be quasi-compact.

Let $I=\bigcup\limits_{g\in G}I_g=<h(I)>$. Then $V_G\left(\bigcup\limits_{g\in G}I_g\right)=V_G(I)$ and so $\tilde{V}_G\left(\bigcup\limits_{g\in G}I_g\right)=\emptyset$. Thus

$$\begin{aligned}
\mathcal{X}_G^I &= \mathcal{X}_G^I\backslash\emptyset=\mathcal{X}_G^I\backslash\tilde{V}_G\left(\bigcup_{g\in G}I_g\right)\\
&= \mathcal{X}_G^I\backslash\tilde{V}_G\left(<h(I)>\right)\\
&= \mathcal{X}_G^I\backslash\tilde{V}_G\left(\sum_{r_i\in h(I)}Rr_i\right)=\mathcal{X}_G^I\backslash\tilde{V}_G\left(\bigcup_{r_i\in h(I)}Rr_i\right)\\
&= \mathcal{X}_G^I\backslash\left(\bigcap_{r_i\in h(I)}\tilde{V}_G(r_i)\right)\\
&= \bigcup_{r_i\in h(I)}\left(\mathcal{X}_G^I\backslash\tilde{V}_G(r_i)\right)=\bigcup_{r_i\in h(I)}(\mathcal{X}_G^I)^{r_i}.
\end{aligned}$$

Since $\mathcal{X}_G^I$ is quasi-compact, there is a finite set $\Delta=\{1,2,...n\}\subseteq\Lambda$ such that $\mathcal{X}_G^I=\bigcup\limits_{\substack{i\in\Delta\\ r_i\in h(I)}}(\mathcal{X}_G^I)^{r_i}=\mathcal{X}_G^I\backslash\tilde{V}_G\left(\sum\limits_{i\in\Delta,r_i\in h(I)}Rr_i\right)$. Thus $\tilde{V}_G\left(\sum\limits_{i\in\Delta,r_i\in h(I)}Rr_i\right)=\emptyset$, which means that $V_G\left(\sum\limits_{i\in\Delta,r_i\in h(I)}Rr_i\right)\subseteq V_G(I)$. Thus $\sqrt{I}\subseteq\sqrt{\sum\limits_{i\in\Delta,r_i\in h(I)}Rr_i}$.

On the other hand, we have $\sqrt{\sum\limits_{i\in\Delta,r_i\in h(I)}Rr_i}\subseteq\sqrt{I}$ and thus $\sqrt{I}=\sqrt{Rr_1+Rr_2+...Rr_n}$.

iii) Let I satisfy the condition $(*)$.

Let $\{A_i:i\in\Lambda\}$ be an open cover of $\mathcal{X}_I^G$. Since A_i can be expressed as a union of the sets $(\mathcal{X}_G^I)^r$, we may assume that $A_i=(\mathcal{X}_G^I)^{r_i}$ for every $i\in\Lambda$, where $r_i\in h(R)$. Then

$$\begin{aligned}
\mathcal{X}_G^I &= \bigcup_{i\in\Lambda}(\mathcal{X}_G^I)^{r_i}=\bigcup_{i\in\Lambda}\left(\mathcal{X}_G^I\backslash\tilde{V}_G(r_i)\right)\\
&= \mathcal{X}_G^I\backslash\bigcap_{i\in\Lambda}\tilde{V}_G(r_i)\\
&= \mathcal{X}_G^I\backslash\tilde{V}_G\left(\sum_{i\in\Lambda}Rr_i\right).
\end{aligned}$$

Thus $\tilde{V}_G\left(\sum\limits_{i\in\Lambda}Rr_i\right)=\emptyset$ and so $V_G\left(\sum\limits_{i\in\Lambda}Rr_i\right)\subseteq V_G(I)$.

In this case, $\sqrt{I}\subseteq\sqrt{\sum\limits_{i\in\Lambda}Rr_i}$. By the condition $(*)$, there is a finite subset $\Delta\subseteq\Lambda$ such that $\sqrt{\sum\limits_{i\in\Lambda}Rr_i}=\sqrt{\sum\limits_{j\in\Delta}Rr_j}$. Then $V_G\left(\sum\limits_{j\in\Delta}Rr_j\right)\subseteq V_G(I)$ and so $\tilde{V}_G\left(\sum_{j\in\Delta} Rr_j\right)=\emptyset$. Then

$$\begin{aligned}\mathcal{X}_G^I &= \mathcal{X}_G^I\backslash\tilde{V}_G\left(\sum_{j\in\Delta} Rr_j\right)=\mathcal{X}_G^I\backslash\bigcap_{j\in\Delta}\tilde{V}_G(r_j)\\ &= \bigcup_{j\in\Delta}\left(\mathcal{X}_G^I\backslash\tilde{V}_G(r_j)\right)=\bigcup_{j\in\Delta}(\mathcal{X}_G^I)^{r_j}.\end{aligned}$$

Since $\mathcal{X}_G^I$ is covered by a finite number of $(\mathcal{X}_G^I)^{r_j}$, $\mathcal{X}_G^I$ is quasi-compact.

iv) Let $P,Q\in\mathcal{X}_G^I$ and $P\neq Q$. Then $P\backslash Q\neq\emptyset$ or $Q\backslash P\neq\emptyset$. Assume that $P\backslash Q\neq\emptyset$. Then there exists a homogeneous element $r\in P\backslash Q$ for $r\in h(R)$. Then $P\notin\mathcal{X}_G^I\backslash\tilde{V}_G(rR)$ and since $rR\nsubseteq Q$, it follows that $Q\notin\tilde{V}_G(rR)$. Thus $Q\in\mathcal{X}_G^I\backslash\tilde{V}_G(rR)$ and $\mathcal{X}_G^I$ is a T_0-space since $\mathcal{X}_G^I\backslash\tilde{V}_G(rR)$ is an open set. □

The following definition is also a generalization of a graded radical ideal of a graded ring.

Definition 3.4. *Let I be a graded ideal of a graded ring R. The set $N_G^I(T)$ is defined as the intersection of all graded prime ideals containing T which does not contain I.*

It can be easily observed that $N_G^I(T)$ is equivalent to the graded radical of a graded ideal T when $I=R$.

Algebraic properties of the graded ideal $N_G^I(T)$ are given as follows:

Proposition 3.5. *Let I be a proper graded ideal of a graded ring R. Then we have*

i) $N_G^I(T)$ is a graded ideal of R.

ii) $N_G^{I/K}(T/K)=N_G^I(T)/K$, where $K\subseteq T$ is a graded ideal of R.

iii) $N_G^I(0)=N_G^{\sqrt{I}}(0)$.

Proof. *i*) Since the intersection of graded ideals is also a graded ideal, it is clear.

ii) $$N_G^{I/K}(T/K)=\bigcap_{\substack{T/K\subseteq P_i/K\\ I/K\nsubseteq P_i/K}}(P_i/K)=\left(\bigcap_{\substack{T\subseteq P_i\\ I\nsubseteq P_i}}P_i\right)/K=N_G^I(T)/K.$$

iii) Since $\sqrt{I}$ is a minimal graded ideal containing I, $N_G^I(0)=N_G^{\sqrt{I}}(0)$. □

A connection between a topological property of $\mathcal{X}_G^I$ and an algebraic property of $N_G^I(0)$ is given by the following theorem.

Theorem 3.6. *Let I be a proper graded ideal of a graded ring R and $\sqrt{I}\neq\sqrt{0}$. Then $\mathcal{X}_G^I$ is irreducible if and only if $N_G^I(0)$ is a graded prime ideal of R.*

Proof. Let $N_G^I(0)$ be a graded prime ideal of R and K be a nonempty open subset of $\mathcal{X}_G^I$. Then $K=\mathcal{X}_G^I\backslash\tilde{V}_G(E)=Spec_G(R)\backslash(V_G(I)\cup V_G(E))$, where E is a graded ideal of R. Take $P\in K$. Then we have $P\notin V_G(I)\cup V_G(E)$, which means that $I\nsubseteq P$ and $E\nsubseteq P$. Thus $N_G^I(0)\subseteq P$, so $E\nsubseteq N_G^I(0)\subseteq P$. This implies that $N_G^I(0)\notin V_G(E)$ and by the definition of $N_G^I(0)$, we get

$N_G^I(0) \notin V_G(I)$. Thus $N_G^I(0) \in K$. Therefore any nonempty open subset of $\mathcal{X}_G^I$ contains $N_G^I(0)$. This means that $\mathcal{X}_G^I$ is irreducible.

Let $\mathcal{X}_G^I$ be irreducible. Assume that $N_G^I(0)$ is not a graded prime ideal of R. Then there exist elements $a, b \in h(R)$ such that $ab \in N_G^I(0)$ and $a, b \in h(R) \backslash N_G^I(0)$.

Since $\sqrt{I} \neq \sqrt{0}$ and $a \in h(R) \backslash N_G^I(0)$, it follows that $\tilde{V}_G(a) \neq \emptyset$ and $\tilde{V}_G(a) \neq \mathcal{X}_G^I$, which implies $(\mathcal{X}_G^I)^a \neq \emptyset$. By the same argument, $(\mathcal{X}_G^I)^b$ is a nonempty open subset. Therefore, we have

$$\begin{aligned} (\mathcal{X}_G^I)^a \cap (\mathcal{X}_G^I)^b &= (\mathcal{X}_G^I)^{ab} = \mathcal{X}_G^I \backslash \tilde{V}_G(ab) \\ &\subseteq \mathcal{X}_G^I \backslash \tilde{V}_G(N_G^I(0)) \\ &= Spec_G(R) \backslash \big(V_G(N_G^I(0)) \cup V_G(I)\big) = \emptyset. \end{aligned}$$

This contradicts with the hypothesis. Thus $N_G^I(0)$ is a graded prime ideal of R . □

A condition on the ideals N_G^I is needed, which helps us out with going further in finding more connections between a topological space and a ring.

A ring R is said to satisfy the TN-condition for a graded ideal I, if for any chain $N_G^I(U_1) \subseteq N_G^I(U_2) \subseteq N_G^I(U_3) \subseteq ...$, there is an integer m such that $N_G^I(U_m) = N_G^I(U_{m+i})$ for all positive integers i.

Theorem 3.7. *Let I be a proper graded ideal of a graded ring R. The following are equivalent:*

i) R satisfies the TN-condition.

ii) $\mathcal{X}_G^I$ is a Noetherian topological space.

Proof. $(i) \Rightarrow (ii)$ Assume that R satisfies the TN-condition. Take the sequence $\tilde{V}_G(U_1) \supseteq \tilde{V}_G(U_2) \supseteq \tilde{V}_G(U_3) \supseteq ...$, where U_i is a graded ideal of R. Then we have the sequence $N_G^I(U_1) \subseteq N_G^I(U_2) \subseteq N_G^I(U_3) \subseteq ...$ and there exists an integer m such that $N_G^I(U_m) = N_G^I(U_{m+i})$ for all positive integers i since R satisfies the TN-condition. Therefore we have $\tilde{V}_G(U_m) = \tilde{V}_G(U_{m+i})$ for all positive integers i. Thus $\mathcal{X}_G^I$ is Noetherian.

$(ii) \Rightarrow (i)$ Let $\mathcal{X}_G^I$ be a Noetherian topological space. Take the sequence $N_G^I(U_1) \subseteq N_G^I(U_2) \subseteq N_G^I(U_3)...$, where U_i is a graded ideal of R. Then this yields the sequence $\tilde{V}_G(U_1) \supseteq \tilde{V}_G(U_2) \supseteq \tilde{V}_G(U_3) \supseteq ....$ Since $\mathcal{X}_G^I$ is Noetherian, there exits an integer m such that $\tilde{V}_G(U_m) = \tilde{V}_G(U_{m+i})$ for all positive integers i. This implies $N_G^I(U_m) = N_G^I(U_{m+i})$ for all positive integers i. Therefore R satisfies the TN-condition. □

Corollary 3.8. *Let I be a proper graded ideal of a graded ring R. If R satisfies the TN-condition, then $\sqrt{I} = \sqrt{Rr_1 + ... + Rr_n}$, where $r_i \in h(I)$.*

We close this paper with the following theorem providing us with algebraic and topological properties.

Theorem 3.9. *Let R be a graded ring. Then the following are equivalent:*

i) $\mathcal{X}_G$ is a Noetherian topological space.

ii) $\mathcal{X}_G^I$ is a Noetherian topological space for every graded ideal I of R.

iii) $\mathcal{X}_G^I$ is quasi-compact for every graded ideal I of R.

iv) R satisfies the TN-condition for every graded ideal I of R.

v) R satisfies ascending chain condition on the graded radical ideals of R.

Proof. $(i) \Rightarrow (ii)$ Take the sequence $\tilde{V}_G(U_1) \supseteq \tilde{V}_G(U_2) \supseteq \tilde{V}_G(U_3) \supseteq ...$, where U_i is a graded ideal of R. Then we have the sequence $V_G(U_1) \supseteq V_G(U_2) \supseteq V_G(U_3) \supseteq ....$ Since $\mathcal{X}_G$ is Noetherian, there exists an integer m such that $V_G(U_m) = V_G(U_{m+i})$ for all positive integers i. Thus we have $\tilde{V}_G(U_m) = \tilde{V}_G(U_{m+i})$ for all positive integers i. Thus $\mathcal{X}_G^I$ is Noetherian.

$(ii) \Rightarrow (i)$ Let $V_G(U_1) \supseteq V_G(U_2) \supseteq V_G(U_3) \supseteq ...$, where U_i is a graded ideal of R, be any decreasing chain in $\mathcal{X}_G$. Let $I = \cap U_i$ be a graded ideal of R. Consider the complement Zariski topology $\mathcal{X}_G^I$. Then we have the sequence $\tilde{V}_G(U_1) \supseteq \tilde{V}_G(U_2) \supseteq \tilde{V}_G(U_3) \supseteq ....$ Since $\mathcal{X}_G^I$ is Noetherian, there exists an integer m such that $\tilde{V}_G(U_m) = \tilde{V}_G(U_{m+i})$ for all positive integers i. Thus we have $V_G(U_m) = V_G(U_{m+i})$ for all positive integers i. Thus $\mathcal{X}$ is Noetherian.

$(i) \Rightarrow (v)$ Consider the sequence $\sqrt{U_1} \subseteq \sqrt{U_2} \subseteq \sqrt{U_3} \subseteq ...$, where U_i is a graded ideal of R. Then we have the sequence $V_G(U_1) \supseteq V_G(U_2) \supseteq V_G(U_3) \supseteq ....$ From (i), there exists an integer m such that $V_G(U_m) = V_G(U_{m+i})$ for all positive integers i. Thus we have $\sqrt{U_m} = \sqrt{U_{m+i}}$ for all positive integers i. Then R satisfies ascending chain condition on the graded radical ideals of R.

$(v) \Rightarrow (i)$ Take the sequence $V_G(U_1) \supseteq V_G(U_2) \supseteq V_G(U_3) \supseteq ...$, where U_i is a graded ideal of R. Then we have the sequence $\sqrt{U_1} \subseteq \sqrt{U_2} \subseteq \sqrt{U_3} \subseteq ....$ From (iv), there exists an integer m such that $\sqrt{U_m} = \sqrt{U_{m+i}}$ for all positive integers i. Thus we have $V_G(U_m) = V_G(U_{m+i})$ for all positive integers i. Thus $\mathcal{X}_G$ is Noetherian.

$(ii) \Leftrightarrow (iv)$ It is Theorem 3.7.

$(ii) \Leftrightarrow (iii)$ One can prove it with the standard argument by using Theorem 3.3. □

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