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EXISTENCE OF SOLUTIONS FOR BOUNDARY VALUE PROBLEMS VIA FIXED POINT METHOD

BENTERKI ABDESSALEM AND ROUAKI MOHAMED

Abstract. The aim of this paper is to study the existence of solutions for the following two-point boundary value problem of second order differential equation

$$(1)\qquad \begin{cases} -\varepsilon u'' = f(x,u) - \lambda & x \in (0,1) \\ \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(1) + \delta u'(1) = 0 \end{cases}$$

where f be a continuous real function. The constants $\varepsilon, \lambda, \alpha, \beta, \gamma$ and δ are such that

$$(2)\quad \varepsilon > 0, \lambda, \beta, \delta \geqslant 0,\ \alpha + \beta > 0,\ \gamma + \delta > 0,\ k := \alpha\gamma + \alpha\delta + \beta\gamma > 0.$$

The proof of our main result is based upon a local fixed point theorem in the setting of partial metric spaces.

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1. Introduction

Many studies concerning the existence of positive, non-positive, nodal, radial or non-radial solutions for second-order differential equations subject to various boundary conditions, have appeared during the previous ten decades; see for example [1, 2, 3, 4, 5, 6]. The mathematicians have found their way to establish a rich collection of results for differential equations via sub-super solution methods, variational methods, degree theory, fixed point theory, shooting methods, quadrature methods, Maximum principles, bifurcation theory.

In this study, we are interesting about the fixed point technique. Several researchers have studied the different problems represented like (1) by using different types of fixed point theorems like Sadovskii, Schauder, Browder, contraction, Leray-Schauder nonlinear alternative and fixed point theorem on cones.

In what follows, we use the local fixed point theorem for set-valued and single-valued mappings presented in [7] on the setting of complete partial metric spaces which generalized and extended several known results in the literature as [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

Recall that the notion of a partial metric spaces is introduced by Matthews in [8], which is a generalization of usual metric spaces in which the self-distance for any point need not be equal to zero.

The aim of the paper is to provide the local existence of solutions for the following boundary value problem

$$(3)\qquad \begin{cases} -\varepsilon u'' = f(x,u) - \lambda & x \in (0,1) \\ \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(1) + \delta u'(1) = 0 \end{cases}$$

where f be a continuous real function and the constants $\varepsilon, \lambda, \alpha, \beta, \gamma$ and δ satisfying the conditions (2). We give also the necessary condition to have uniqueness of solutions for the problem mentioned below. Let us recall some problems for the one-dimensional case can rewritten as (3) with different choices on the constants $\varepsilon, \lambda, \alpha, \beta, \gamma$ and δ and the function $f(x,u)$.

Castro and Shivaji, in [22], introduced the semipositone problem of the form

$$(4)\qquad \begin{cases} -u'' = \mu f(u) & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

and consider, in [1], the problem

$$(5)\qquad \begin{cases} -u'' = f(u) - g(x) - \lambda & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

Ammar Khodja in [23] studied problem of the form

$$(6)\qquad \begin{cases} -u'' = u^2 - \lambda & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

In [2], Addou and Benmezaï determined the exact number of solutions of quasilinear problem

$$(7)\qquad \begin{cases} -u'' = |u|^p - \lambda & x \in (0,1) \text{ and } p > 1 \\ u(0) = 0 = u(1) \end{cases}$$

Rouaki in [3, 6] studied the nodal radial solutions for the problem

$$(8)\qquad \begin{cases} -u'' = f(u) - g(x) - \lambda & x \in (0,1) \\ u'(0) = 0 = u(1) \end{cases}$$

The paper is organized as follows: in section 2, we give some definitions and recall a few preliminary results. In section 3, we set our main result and the necessary conditions to have a local solutions (not necessary positives) to the boundary value problems mentioned below by using the local fixed point theorem in the setting of partial metric space.

2. Notations and preliminary results

Before we state and prove our main result, we give the following preliminaries used in our subsequent discussions.

Definition 2.1. *Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:*

(P_1): $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$;
(P_2): $p(x,x) \leqslant p(x,y)$;
(P_3): $p(x,y) = p(y,x)$;
(P_4): $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is then called a partial metric space. Thus a metric space is precisely a partial metric space for which each self distance $p(x, x) = 0$.

Example 2.2. *The following functions* p_i $(i \in \{1, 2\})$ *define a partial metrics for each* X

$$p_1(x, y) = \max\{x, y\} \qquad x, y \in X = \mathbb{R}^+$$
$$p_2(x, y) = |x - y| + c \qquad x, y \in X = \mathbb{R} \text{ and } c \geqslant 0$$

Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$. The closed p-ball of radius r centered at x is denoted by $\overline{B_p}(x, r)$ where

$$\overline{B_p}(x, r) = \{y \in X : p(x, y) \leqslant p(x, x) + r\}.$$

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

Let (X, p) be a partial metric space. Then:

- A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$.
- A sequence $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \to +\infty} p(x_n, x_m)$.
- (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

Lemma 2.3. *Let* (X, p) *be a partial metric space.*

(a): $\{x_n\}$ *is a Cauchy sequence in* (X, p) *if and only if it is a Cauchy sequence in the metric space* (X, p^s).

(b): *A partial metric space* (X, p) *is complete if and only if the metric space* (X, p^s) *is complete. Furthermore,*

$$\lim_{n \to +\infty} p^s(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m) \tag{9}$$

where x *is a limit of* $\{x_n\}$ *in* (X, p^s).

Lemma 2.4. [8] *Let* (X, p) *be a partial metric space. Then*

(a): *if* $p(x, y) = 0$*, then* $x = y$*. But if* $x = y$*, then* $p(x, y)$ *may not be zero;*

(b): *if* $x \neq y$*, then* $p(x, y) > 0$.

Let $C^p(X)$ be the family of all nonempty and closed subsets of the partial metric space (X, p). For $x \in X$ and $A, B \in C^p(X)$, we define

$$p(x, A) = \inf\{p(x, a), a \in A\}, \tag{10}$$

$$\delta_p(A, B) = \sup\{p(a, B), a \in A\}, \tag{11}$$

with the convention

$$\delta_p(\emptyset, B) = 0. \tag{12}$$

Lemma 2.5. [15] *Let (X,p) be a partial metric space and $A \subset X$. Then $p(a,A) = p(a,a) \Leftrightarrow a \in \overline{A}$. Moreover, $p(a,A) = 0 \Leftrightarrow p(a,a) = 0$ and $a \in \overline{A}$*

where $\overline{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X,p) if and only if $\overline{A} = A$.

For further information about properties of partial metric spaces one can refer to [7, 8].

In what follows, we denote J an interval on $\mathbb{R}^+$ containing 0, that are an interval of the form $[0,a)$, $[0,a]$ or $[0,+\infty)$.

Definition 2.6. *A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function (or (c)-comparison) on J if*

$$s(t) := \sum_{n=0}^{\infty} \varphi^n(t) \text{ is convergent for all } t \in J \tag{13}$$

where φ^n denotes the n-th iteration of the function φ and $\varphi^0(t) = t$ i.e.

$\varphi^0(t) = t$, $\varphi^1(t) = \varphi(t)$, $\varphi^2(t) = \varphi(\varphi(t)), \ldots, \varphi^n(t) = \varphi(\varphi^{n-1}(t))$.

Note that the functions s and φ satisfy, for each $t \in J$, the functional equation

$$s(t) = t + s(\varphi(t)) \tag{14}$$

and (with the exception of some cases) we have :

$$\varphi(t) = s^{-1}(s(t) - t). \tag{15}$$

Example 2.7. *The following functions are Bianchini-Grandolfi gauge functions on J*

- **(a):** $\varphi(t) = \lambda t$ $(0 < \lambda < 1)$ *on* $J = [0,+\infty[$;
- **(b):** $\varphi(t) = c\, t^r$ $(c > 0, r > 1)$ *on* $J = [0,R[$, *where* $R = (1/c)^{1/(r-1)}$;
- **(c):** *Every convex function φ on an interval J such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t \in J \setminus \{0\}$*
- **(d):** $\varphi(t) = \sinh^{-1}(\sinh(t) - t)$ *on* $J = [0,+\infty[$
- **(e):** $\varphi(t) = \dfrac{t^2}{2\sqrt{t^2+a^2}}$ *where $a \geqslant 0$ and* $J = [0,+\infty[$.

The needed fixe point theorem reads as follow

Theorem 2.8. [7] *Let (X,p) be a complete partial metric space. Let $\overline{x} \in X$ and $r > 0$ such that $\phi : \overline{B_p}(\overline{x},r) \to C^p(X)$ be a set-valued mapping. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a increasing and continuous function such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim_{t \downarrow 0} \varphi(t) = 0$. If there exists $\alpha \in J$ such that the following two conditions hold:*

- **(a):** $p(\overline{x}, \phi(\overline{x})) < \alpha$ *where* $s(\alpha) \leqslant p(\overline{x},\overline{x}) + r$
- **(b):** $\delta_p(\phi(x) \cap \overline{B_p}(\overline{x},r), \phi(y)) \leqslant \varphi\,(p(x,y)) \quad \forall x, y \in \overline{B_p}(\overline{x},r)$,

then ϕ has a fixed point x^* in $\overline{B_p}(\overline{x},r)$. If ϕ is a single valued mapping and $p(\overline{x},\overline{x}) + 2r \in J$, then x^* is the unique fixed point of ϕ in $\overline{B_p}(\overline{x},r)$.

Observe that no boundedness assumption or relatively compactness on the values and also the continuous, completely continuous or condensing assumptions on the single-valued (set-valued) mapping are not required (see e.g. [7, Example 3.3])

3. Existence of solutions

In this section we will use our existence principle, Theorem 2.8, to establish an existence result for (3). Let $X = C([0,1])$ be the space of all continuous real functions defined on $I = [0,1]$ endowed with partial metric

$$p(u,v) = \|u - v\| + c \qquad \forall u, v \in X,$$

where $\|u\| = \sup_{x\in I} |u(x)|$ and $c \geqslant 0$. Since

$$p^s(u,v) = 2p(u,v) - p(u,u) - p(v,v) = 2\|u - v\|,$$

so by Lemma 2.3, (X, p) is complete since the metric space (X, p^s) is complete. It is well known that $u^* \in C([0,1]) \cap C^2((0,1))$ is a solution of problem (3) if and only if $u^* \in C([0,1])$ is a solution of the following nonlinear integral equation:

$$u(x) = \frac{1}{\varepsilon}\int_0^1 G(x,s)[f(s,u(s)) - \lambda]ds \quad x \in I \tag{16}$$

where $G(x,s)$ is the Green function of the second-order Sturm-Liouville boundary value problem

$$\begin{cases} -z''(x) = 0, & x \in (0,1); \\ \alpha z(0) - \beta z'(0) = 0, \ \gamma z(1) + \delta z'(1) = 0 \end{cases} \tag{17}$$

It is known that [24, 25]

$$G(x,s) = \frac{1}{k}\begin{cases} (\beta + \alpha s)[\delta + \gamma(1-x)], & 0 \leqslant s \leqslant x \leqslant 1, \\ (\beta + \alpha x)[\delta + \gamma(1-s)], & 0 \leqslant x \leqslant s \leqslant 1. \end{cases} \tag{18}$$

and then for all $x \in I$, we have

$$\begin{aligned} \int_0^1 G(x,s)ds &= \int_0^x G(x,s)ds + \int_x^1 G(x,s)ds \\ &= \frac{1}{k}\left(\int_0^x (\beta + \alpha s)[\delta + \gamma(1-x)]ds + \int_x^1 (\beta + \alpha x)[\delta + \gamma(1-s)]ds\right) \\ &= \frac{1}{2k}\left(\beta\gamma + 2\beta\delta + (\alpha\gamma + 2\alpha\delta)x - kx^2\right) \end{aligned}$$

which implies that

$$\sup_{x\in I}\int_0^1 G(x,s)ds = \frac{1}{8k^2}\left(4k(\beta\gamma + 2\beta\delta) + (\alpha\gamma + 2\alpha\delta)^2\right) := M \neq 0.$$

Let us define a nonlinear operator $A : X \to X$ by

$$A(u)(x) = \frac{1}{\varepsilon}\int_0^1 G(x,s)[f(s,u(s)) - \lambda]ds \qquad \text{for all } u \in X. \tag{19}$$

Thus the existence of solutions to problem (16) is equivalent to the existence of fixed point to nonlinear operator A.

To check that A satisfies all assumptions of Theorem 2.8 on the closed p-ball of radius $K(\lambda)$ centered at 0_X (the null function), we consider the following conditions:

(C1): There exist a constant $C \geqslant 0$ and $K(\lambda)$ a positive continuous function defined for $\lambda \geqslant C$

(C2): There exists an increasing and continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim_{t \downarrow 0} \varphi(t) = 0$

(C3): There exists $\alpha \in J$ such that $s(\alpha) \leqslant c + K(\lambda)$

(C4): $\|f(\cdot, 0) - \lambda\| < \dfrac{\varepsilon(\alpha - c)}{M}$

(C5): $|f(x,a) - f(x,b)| \leqslant \dfrac{\varepsilon}{M}(\varphi(|a-b|+c) - c) \quad \forall \begin{cases} x \in I, \\ |a|, |b| \leqslant K(\lambda). \end{cases}$

Hence, we state and prove our main result as follows

Theorem 3.1. *For a fixed $c \geqslant 0$, suppose that conditions (C1)-(C5) holds. Then* (1) *has at least one solution u^* in $C([0,1]) \cap C^2((0,1))$ such that $\|u^*\| \leqslant K(\lambda)$. Moreover, if $c + 2K(\lambda) \in J$ then the solution is unique.*

Proof. Let $A : X \to X$ be the sample set-valued mapping defined by

$$A(u)(x) = \frac{1}{\varepsilon} \int_0^1 G(x,s)[f(s,u(s)) - \lambda]ds \qquad \text{for all } u \in X. \tag{20}$$

First, the use of assumptions (C1)-(C4) give the following

$$\begin{aligned} p(0_X, A(0_X)) &= \sup_{x \in I} |A(0_X)(x)| + c \\ &\leqslant \frac{1}{\varepsilon} \sup_{x \in I} \left| \int_0^1 G(x,s)[f(s, 0_X(s)) - \lambda]ds \right| + c \\ &\leqslant \frac{1}{\varepsilon} \sup_{x \in I} \int_0^1 G(t,s) \, |f(s,0) - \lambda| \, ds + c \\ &\leqslant \frac{1}{\varepsilon} \|f(\cdot,0) - \lambda\| \sup_{x \in I} \int_0^1 G(x,s)ds + c \\ &< \frac{M\varepsilon(\alpha - c)}{\varepsilon M} + c = \alpha \end{aligned}$$

and $s(\alpha) \leqslant c + K(\lambda) = p(0_X, 0_X) + K(\lambda)$. Thus the condition (a) of Theorem 2.8 is satisfied.

Let $u, v \in \overline{B_p}(0_X, K(\lambda))$ then we have two cases. The first one, if $A(u) \cap \overline{B_p}(0_X, K(\lambda)) = \emptyset$ then according to convention (12) we have

$$\delta_p(A(u) \cap \overline{B_p}(0_X, K(\lambda)), A(v)) = 0 \leqslant \varphi(p(u,v)).$$

So we assume that $A(u)\cap\overline{B_p}(0_X,K(\lambda))\neq\emptyset$. From condition (C5), we have

$$\begin{aligned}
\delta_p(A(u)\cap\overline{B_p}(0_X,K(\lambda)),A(v)) &= p(A(u),A(v))\\
&= \sup_{x\in I}|A(u)(x)-A(v)(x)|+c\\
&= \frac{1}{\varepsilon}\sup_{x\in I}\left|\int_0^1 G(x,s)(f(s,u(s))-f(s,v(s)))ds\right|+c\\
&\leqslant \frac{1}{\varepsilon}\sup_{x\in I}\int_0^1 G(x,s)\left|f(s,u(s))-f(s,v(s))\right|ds+c\\
&\leqslant \frac{1}{\varepsilon}\sup_{x\in I}\int_0^1 G(x,s)(\frac{\varepsilon}{M}(\varphi(|u(s)-v(s)|+c)-c)ds+c\\
&\leqslant \frac{1}{\varepsilon}\left(\sup_{x\in I}\int_0^1 G(x,s)ds\right)\frac{\varepsilon}{M}(\varphi(\|u-v\|+c)-c)+c\\
&\leqslant \frac{M}{\varepsilon}(\frac{\varepsilon}{M}(\varphi(\|u-v\|+c)-c))+c\\
&\leqslant \varphi(p(u,v))
\end{aligned}$$

Thus all conditions are satisfied and then A has at least a fixed point u^* in $\overline{B_p}(0_X,K(\lambda))$ i.e., $p(u^*,0_X)\leqslant p(0_X,0_X)+K(\lambda)\Leftrightarrow\|u^*\|\leqslant K(\lambda)$. Since A be a single valued and if $c+2K(\lambda)\in J$, i.e., $p(0_X,0_X)+2K(\lambda)\in J$ then u^* is the unique solution. □

Example 3.2. *We consider the problem*

$$\begin{cases} -\varepsilon u''=u^2-\lambda & x\in(0,1)\\ u(0)=u(1)=0 \end{cases} \tag{21}$$

where $\varepsilon=1$ *and* $0<\lambda\leqslant\frac{3}{4}$.

To show (21) has a solution we apply Theorem 3.1 with the following specifications

$$f(x,u)=u^2, M=\frac{1}{8}, c=0,$$

Take

$$\varphi(t)=\begin{cases}(\sqrt{t}-t)^2, & 0\leqslant t\leqslant\frac{1}{4};\\ t^4+\frac{15}{256}, & t\geqslant\frac{1}{4}.\end{cases}$$

which is increasing, continuous and Bianchini-Grandolfi gauge function on $J=\left[0,\frac{1}{4}\right]$ such that $s(t)=\sqrt{t}$ and $\lim_{t\downarrow 0}\varphi(t)=0$. Note that φ is not a contraction when $t\to 0$. Choose

$$C=0,\alpha=\frac{\lambda}{4}\in J,K(\lambda)=s(\alpha)=\frac{1}{2}\sqrt{\lambda}$$

To see that condition (C5) of Theorem 3.1 holds we take $a,b\in\mathbb{R}$ such that $|a|,|b|\leqslant K(\lambda)$ for $0<\lambda\leqslant\frac{3}{4}$.

(a): If $|a-b|+c \leqslant \frac{1}{4}$ then

$$\begin{aligned} |f(x,a)-f(x,b)| &= |a^2-b^2| \\ &\leqslant 8(\sqrt{|a-b|}-|a-b|)^2 \\ &= \frac{\varepsilon}{M}(\varphi(|a-b|+c)-c) \end{aligned} \tag{22}$$

(b): else if $|a-b|+c \geqslant \frac{1}{4}$ then

$$\begin{aligned} |f(x,a)-f(x,b)| &= |a^2-b^2| \\ &\leqslant 8(|a-b|^4+\frac{15}{256}) \\ &= \frac{\varepsilon}{M}(\varphi(|a-b|+c)-c) \end{aligned} \tag{23}$$

Thus, all conditions (C1)-(C5) are satisfied and then Theorem 3.1 now guarantees that (21) has a solution $u \in C([0,1]) \cap C^2((0,1))$ such that

$$\|u\| \leqslant \frac{1}{2}\sqrt{\lambda}.$$

Moreover, for $\lambda \in \left]0, \frac{1}{16}\right]$ we have $c+2K(\lambda) = \sqrt{\lambda} \in J$ and then the solution is unique. Consequently, for $\lambda \in \left]\frac{1}{16}, \frac{3}{4}\right]$ the problem (21) has at least two solutions $u, u^* \in C([0,1]) \cap C^2((0,1))$ such that $\max\{\|u\|, \|u^*\|\} \leqslant \frac{1}{2}\sqrt{\lambda}$.

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LAMDA-RO Laboratory, Department of Mathematics, University of Blida, Algeria.
Email address: benterki.abdessalem@univ-blida.dz

LAMDA-RO Laboratory, Department of Mathematics, University of Blida, Algeria.
Email address: mrouaki@univ-blida.dz