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GENERALIZED QUASI-PRIMARY RINGS

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Abstract. In this paper, the structure of commutative rings with identity all of whose ideals are quasi-primary, called generalized quasi-primary rings, is studied and several equivalent conditions to such rings are considered. Equivalently, a generalized quasi-primary ring may be viewed as a ring whose the set of radical ideals forms a chain. It is proved that an Artinian local ring R is a generalized quasi-primary ring and the converse is true if R is a non-domain Noetherian ring.

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1. Introduction

Throughout this paper all rings are commutative with identity. A ring in which all ideals are primary is called a generalized primary ring (or GP-ring for short) . This class of rings was introduced and studied by Satyanarayana, where some characterizations of Noetherian generalized primary domains were also given [9]. For instance, Satyanarayana proved that a Noetherian domain R is generalized primary if and only if R is local and $dimR = 1$ [9, Theorem 4.5]. Later, Chaudhuri in [1] improved Satyanarayana's results. For example, she showed in [1, Theorem 2.4] that the Noetherian condition in [9, Theorem 4.5] is redundant . In fact, Chaudhuri proved that a ring R is generalized primary if and only if it is a zero-dimensional local ring or a one-dimensional local domain. According to some generalizations of primary ideals from commutative rings to a non-commutative setting, Gorton et al. [4] developed the structure of generalized primary rings, in particular considering these rings under various chain conditions.

A proper ideal I of a ring R is called quasi-primary if $rs \in I$, for $r, s \in R$, implies that $r \in \sqrt{I}$ or $s \in \sqrt{I}$. Indeed, I is a quasi-primary ideal of R if and only if $\sqrt{I}$ is a prime ideal of R [3, Definition 2, p. 176]. Clearly every primary ideal is quasi-primary, but the converse is not true generally (See for example [7, Exercise 11(b) page 56]).

Definition 1.1. *We say that a ring R is a generalized quasi-primary ring (or GQ-ring for short) if every ideal of R is quasi-primary.*

It is clear that every GP-ring is a GQ-ring, but the converse is not necessarily true (Example 2.3). We provide conditions under which these notions coincide. For example, if R satisfies the ascending chain condition for prime ideals, or in particular R is a Noetherian ring, then R is a GP-ring if and only if R is a GQ-ring. In this case, for every non-zero proper ideal I of R, $\sqrt{I} = \sqrt{Ra}$ for some $a \in I$ (Theorem 2.4). It is also proved that if R is an Artinian local ring, then R is a GQ-ring and

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the converse is true provided R is a non-domain Noetherian ring (Corollary 2.2 and Theorem 2.5). If we replace the Artinian property by the condition that $\cap_{n=1}^{\infty} P^n$ is a prime ideal of R in Theorem 2.5, then Noetherian property will be restricted to ascending chain condition for prime ideals (Theorem 2.8).

In [9, Theorem 4.12], it has been proved that a Noetherian GP-ring R is a discrete valuation ring if and only if the unique maximal ideal of R is principal. Also, in [1, Corollary 2.5], it has been shown that a GP-ring R is a discrete valuation ring or a special primary ring if and only if the unique maximal ideal M of R is principal.

2. Generalized quasi-primary rings

We start with an elementary lemma that will be useful in analyzing the structure of GQ-rings.

Lemma 2.1. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a GQ-ring;*
(2) *For any proper ideals I and J of R, $I \subseteq \sqrt{J}$ or $J \subseteq \sqrt{I}$;*
(3) *The set of radical of ideals of R is totally ordered under inclusion;*
(4) *The set of prime ideals of R is totally ordered under inclusion;*
(5) *Every proper ideal of R has a unique minimal prime ideal.*

Proof. (1)⇒(2). Suppose that R is a GQ-ring. Let I and J be two proper ideals of R. Then $I \cap J$ is a quasi-primary ideal of R. Since $IJ \subseteq I \cap J$, we have $I \subseteq \sqrt{I \cap J}$ or $J \subseteq \sqrt{I \cap J}$, that is $I \subseteq \sqrt{J}$ or $J \subseteq \sqrt{I}$.

(2)⇒(3)⇒(4)⇒(5)⇒(1) is obvious. □

Corollary 2.2. *Any GQ-ring has a unique maximal ideal.*

Clearly every GP-ring is a GQ-ring, but the converse is not true in general, as the following example shows.

Example 2.3. *A ring R is said to be an almost multiplication ring if each quasi-primary ideal of R is a power of its radical* (*See* [7, p. 216]). *Let R be an almost multiplication ring which is not a GP-domain. Then $dim R > 1$. Let P be a proper ideal R which is not a minimal prime ideal of the ideal* (0). *Suppose that Q is a minimal prime ideal of* (0) *contained in P. It is shown that the ring R_P has only two proper prime ideals PR_P and QR_P* [7, See the proof of Theorem 9.23]. *Therefore, R_P is a GQ-ring which is not a GP-ring.*

A ring R is said to satisfy the ascending chain condition for prime ideals if any strictly ascending chain of prime ideals $P_1 \subset P_2 \subset P_3 \subset \cdots$ is stationary.

Theorem 2.4. *Let R be a local ring with the unique maximal ideal M. Consider the following statements:*

(1) *For every non-zero proper ideal I of R, $\sqrt{I} = \sqrt{Ra}$ for some $a \in I$;*
(2) *$M = \sqrt{Rp}$ for some $p \in M$;*
(3) *R is a GP-ring;*
(4) *R is a GQ-ring.*

Then, (1) ⇒ (2) ⇒ (3) ⇒ (4). *Moreover, if R is a ring satisfying the ascending chain condition for prime ideals, then* (4) ⇒ (1).

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clearly true.

(2) $\Rightarrow$ (3). Let I be a proper ideal of R and $M = \sqrt{Rp}$, for some $p \in R$. If $a \in I$, then for some positive integer k, a^k divisible by p, so that by successive division by p, we have $a^k = up^t$, $(t \geq 1, k \geq 1)$, u unit. Hence $Ra^k = Rp^t$. Therefore $p \in \sqrt{Ra^k} = \sqrt{Ra} \subseteq \sqrt{I}$ and hence $\sqrt{I} = M$. Thus I is an M-primary ideal of R.

Now let R be a GQ-ring satisfying the ascending chain condition for prime ideals.

(4) $\Rightarrow$ (1). Let I be a non-zero proper ideal of R. By Lemma 2.1, we must have a chain of prime ideals $\sqrt{Ra_1} \subseteq \sqrt{Ra_2} \subseteq \cdots$ where $a_i \in I$ for $i \geq 1$. By our assumption, there is a largest prime ideal $\sqrt{Ra_j}$ among $\sqrt{Ra_1}, \sqrt{Ra_2}, \cdots$ and hence $\sqrt{I} = \sqrt{Ra_j}$. □

A ring R is called a Max ring provided that every non-zero R-module has a maximal submodule. These rings have been characterized in [5]. Also a subset A of a ring R is called T-nilpotent, if for any sequence of elements $\{a_1, a_2, a_3, \cdots\} \subseteq A$, there exists an integer $n \geq 1$ such that $a_1 a_2 \cdots a_n = 0$ (See [6, Definition 23.13]).

Let R be a commutative ring with identity and M be an R-module. We say that M satisfies the weak Nakayama property, if $IM = M$, where I is an ideal of R, implies that for any $x \in M$ there exists $a \in I$ such that $(a-1)x = 0$. In [8], the present authors introduced this family of modules and investigated some important properties of them. It has been proved that if R is a local ring with the maximal ideal M, then R is a Max ring if and only if M is T-nilpotent if and only if every R-module satisfies the weak Nakayama property [8, Theorem 3.4]. Here is the connection between a local GQ-ring R and R-modules satisfying the weak Nakayama property.

Theorem 2.5. *Let R be a local ring. Consider the following statements:*

(1) *R is an Artinian ring;*
(2) *Every R-module satisfies the weak Nakayama property;*
(3) *R is a Max ring;*
(4) *M is T-nilpotent;*
(5) *M is nil;*
(6) *The radical of every proper ideal of R is maximal;*
(7) *$dimR = 0$;*
(8) *R is a GP-ring;*
(9) *R is a GQ-ring.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9). *Moreover, if R is a Noetherian ring which is not a domain, then* (9) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) follows from [8, Theorem 3.6].

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows from [8, Theorem 3.4].

(4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (6) Let M be the maximal ideal of R. Let $a \in M$. Then there is a positive integer n such that $a^n = 0$, since M is nil. It follows that $a \in I$ for every proper ideal I of R. Thus $\sqrt{I} = M$ for every proper ideal I of R.

(6) $\Rightarrow$ (7) is evident.

(7) $\Rightarrow$ (8) follows from [1, Theorem2.4].

(8) $\Rightarrow$ (9) is clear.

Now let R be a Noetherian GQ-ring which is not a domain.

(9) $\Rightarrow$ (1). By Theorem 2.4, R is a GP-ring. It follows from [1, Thoerem 2.4] that R is a local zero-dimensional ring and hence the proof is completed. $\square$

In Theorem 2.5, the hypothesis that R is Noetherian is essential, as the following example shows.

Example 2.6. *Let $\mathbb{Q}$ be the field of rational numbers and R be the commutative $\mathbb{Q}$-algebra generated by $x_1, x_2, \cdots$ with the relations $x_i x_j = 0$ for $i \neq j$, and $x_i^3 = 0$ for all i. It is easy to see that R is a non-Noetherian local ring with the maximal ideal $M = \underset{i\in\mathbb{N}}{\oplus}(\mathbb{Q}x_i \oplus \mathbb{Q}x_i^2)$. Since $x_i x_j x_k = 0$ for all i, j, k, we have $M^3 = 0$, so R is a GQ-ring by Theorem 2.5.*

Lemma 2.7. *Let R be a ring such that every prime ideal of R is idempotent. Then the following are equivalent.*

(1) *R is a GQ-ring;*
(2) *Every proper ideal of R is prime;*
(3) *$R = 0$ or R is a field;*
(4) *R is a GP-ring.*

Proof. (1) $\Rightarrow$ (2). Let P be any proper ideal of R and $ab \in P$ for some $a, b \in R$. By Lemma 2.1, we may assume that $\sqrt{Ra} \subseteq \sqrt{Rb}$. Hence, by hypothesis, $Ra \subseteq \sqrt{Ra} = (\sqrt{Ra})^2 \subseteq \sqrt{Ra}\sqrt{Rb} \subseteq \sqrt{Ra} \cap \sqrt{Ra} = \sqrt{Rab} \subseteq P$, so we have $a \in P$, as required.

(2) $\Rightarrow$ (3). Let R be a commutative ring in which all proper ideals are prime. Then R is a domain. If R is not a field, there is a non-unit element $a \neq 0$ of R. Then Ra^2 is prime, so we must have $a = a^2 b$ for some $b \in R$. But then $ab = 1$, a contradiction.

(3) $\Rightarrow$ (4) $\Rightarrow$ (1) are clear. $\square$

Theorem 2.8. *Let R be a local domain and for any non-zero prime ideal P of R, $\cap_{n=1}^{\infty} P^n$ is a prime ideal of R. Then the following statements are equivalent:*

(1) *For any non-zero prime ideal P of R, $J = \cap_{n=1}^{\infty} P^n$ is properly contained in P and if Q is a prime ideal of R which is properly contained in P, then $Q \subseteq J$;*
(2) *R satisfies the ascending chain condition for prime ideals and R is a GQ-ring;*
(3) *R is a GP-ring.*

Proof. (1) $\Rightarrow$ (2). Let $P_1 \subset P_2 \subset \cdots$ be an ascending chain of prime ideals of R. Then $P = \bigcup P_i$ is a prime ideal of R. If $P \neq P_i$ for all i, then $P_i \subseteq \cap_{n=1}^{\infty} P^n$ for all i; and hence we have $P = \bigcup P_i \subseteq \cap_{n=1}^{\infty} P^n \subset P$, a contradiction. Therefore, R satisfies the ascending chain condition for prime ideals.

Now suppose that R is not a GQ-ring. Then, by Lemma 2.1, there exist prime ideals P_1, P_2 of R such that $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Since R satisfies the ascending chain condition for prime ideals, there exists a prime ideal J, maximal with respect to the properties $P_1 \subseteq J$, $P_2 \nsubseteq J$. Since $P_2 \nsubseteq J$, J is not the maximal ideal of R and there exists a prime ideal P_α properly containing J. If $\{P_\alpha\}$ is the set of all such prime ideals, then $J \neq \cap P_\alpha$ since $P_2 \subseteq \cap P_\alpha$ and $P_2 \nsubseteq J$. Therefore, by Zorn's lemma, there is a prime ideal P_0 minimal with respect to the property that $P_0 \supset J$. Therefore, $J \subseteq \cap_{n=1}^{\infty}(P_0)^n \subset P_0$ implies $J = \cap_{n=1}^{\infty}(P_0)^n$. On the other hand, if $P_2 = P_0$, then $P_2 \supseteq J$ and so $P_2 \supseteq P_1$, since $J \supseteq P_1$. This is a contradiction to $P_2 \nsupseteq P_1$ and $P_1 \nsupseteq P_2$. Now the relation $P_2 \subset P_0$ means $P_2 \subseteq \cap_{n=1}^{\infty}(P_0)^n = J$, a contradiction to the choice of J.

(2) $\Rightarrow$ (3) follows from Theorem 2.4.

(3) $\Rightarrow$ (1). Let R be a GP-domain with the unique non-zero prime ideal M. If $\cap_{n=1}^{\infty} M^n = M$, then M is idempotent and hence, by Lemma 2.7, R is a field. This completes the proof. In the other case, since by [1, Lemma 2.1] R is one dimensional, if $\cap_{n=1}^{\infty} M^n \subset M$, then $\cap_{n=1}^{\infty} M^n = 0$ and hence the item (1) holds. □

Theorem 2.9. *Let R be a GQ-domain and I a proper ideal of R. If $J = \cap_{n=1}^{\infty} I^n$ contains every proper ideal of R which is properly contained in $\sqrt{I}$, then J is a prime ideal of R.*

Proof. Let R be a GQ-ring. Let a and b be elements of R such that $a \notin J$, $b \notin J$. Then $a \notin I^n, b \notin I^m$ for some positive integers n and m. If $Ra = \sqrt{I}$, then $ab \notin I^{n+m}$ and so $ab \notin J$, as required. For, if otherwise, $ab \in I^{n+m} \subseteq Ra^{n+m}$, then there exists $c \in R$ such that $ab = ca^{n+m}$ and hence $b = ca^{n-1}a^m \in \sqrt{I^m} = \sqrt{I}$. Now, if $Rb \subset \sqrt{I}$, then $Rb \subseteq J$, a contradiction to the choice of b. Thus $Rb = \sqrt{I}$. It concludes that $Ra = \sqrt{I} = Rb$ and so these ideals are prime. So we have $\sqrt{I} = Ra \cap Rb = RaRb = Rab \subseteq I^{n+m} \subseteq I^m \subseteq I \subseteq \sqrt{I}$. It is a contradiction to $b \notin I^m$. Now if $Ra \subset \sqrt{I}$, then by our assumption, we must have $Ra \subseteq J$ and so $a \in I^n$, a contradiction. Thus we may assume that $Ra \nsubseteq \sqrt{I}$. By the same argument we can assume that $Rb \nsubseteq \sqrt{I}$ and thus by Lemma 2.1 we have $I \subseteq \sqrt{Ra}$ and $I \subseteq \sqrt{Rb}$. So $\sqrt{I} \subseteq \sqrt{Ra} \cap \sqrt{Rb} = \sqrt{Rab}$. Actually, we have $\sqrt{I} \subset \sqrt{Rab}$. If $\sqrt{I} = \sqrt{Rab}$, then $Rab \subseteq \sqrt{I}$ and hence $Ra \subseteq \sqrt{I}$ or $Rb \subseteq \sqrt{I}$, it is a contradiction; thus $\sqrt{I} \subset \sqrt{Rab}$. Hence $ab \notin I^{n+m}$. For, since $\sqrt{I} \subset \sqrt{Rab}$, there exists an element x of R such that $x \in \sqrt{Rab} \backslash \sqrt{I}$. If $Rab \subseteq I^{n+m}$, then $x^t = cab \in I^{n+m}$, for some positive integer t and $c \in R$, a contradiction. Therefore, $ab \notin J$ and we conclude that J is a prime ideal of R. □

Corollary 2.10. *Let R be a domain and for any proper ideal I of R, $J = \cap_{n=1}^{\infty} I^n$ is properly contained in $\sqrt{I}$ and if L is a proper ideal of R which is properly contained in $\sqrt{I}$, then $L \subseteq J$. Then R is a GQ-ring if and only if R is a GP-ring.*

Proof. Let R be a GQ-domain and P be a prime ideal of R. By Theorem 2.9 $\cap_{n=1}^{\infty} P^n$ is a prime ideal of R and thus by Theorem 2.8 R is a GP-ring. □

Recall that a ring R is said to be a Zerlegung Primideale ring (ZPI-ring) if every proper ideal of R can be written as a product of prime ideals of R. Also, a ring R is said to be special primary if R has a unique maximal ideal M and if each proper ideal of R is a power of M. In [7, Theorem 9.10], it has been proved that the following statements are equivalent:

(1) R is a ZPI-ring;
(2) R is a Noetherian ring and for each maximal ideal M of R, there are no ideals of R strictly between M and M^2.
(3) R is a direct sum of a finite number of Dedekind domains and special primary rings.

Lemma 2.11. *A ring R is a GP-ring (and thus a GQ-ring) in each of the following cases:*

(1) *If R is a local ring with a principal maximal ideal.*
(2) *If R is a local ZPI-ring.*

Proof. (1) is clear by Theorem 2.4.

(2) If $M^2 = M$, then by Nakayama's lemma $M = 0$ and so R is a field. Thus, in this case, the proof is completed. Now let $x \in M \backslash M^2$. Then $M^2 \subset M^2 + Rx \subseteq M$. Since R is a ZPI-ring, there are no ideals of R strictly between M^2 and M. Hence $M^2 + Rx = M$, and thus by Nakayama's lemma $M = Rx$. The proof is completed by part (1). □

A discrete valuation ring (not necessarily an integral domain) is a commutative ring with identity in which every ideal is principal and all the ideals form a chain under set inclusion (See [9, Definition 4.8]).

Compare the next two results with [9, Theorem 4.12] and [1, Corollary 2.5].

Theorem 2.12. *Let R be a local domain. Then the following statements are equivalent:*

(1) *R is a special primary ring;*
(2) *R is a ZPI-ring;*
(3) *R is a discrete valuation ring;*
(4) *R is a P.I.D. with a unique maximal ideal $M \neq 0$;*
(5) *R is a U.F.D. with a unique (up to associates) irreducible element t;*
(6) *R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is nonzero and principal;*
(7) *R is a Noetherian, integrally closed, integral domain that is also a local ring of Krull dimension 1 i.e., R has a unique nonzero prime ideal;*
(8) *R is a Noetherian GQ-ring and the maximal ideal of R is principal.*

Proof. Without less of generality, we may assume that R is not a field.

(1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let R be a ZPI-ring. By the proof of Lemma 2.11, the maximal ideal of R is non-zero and principal. Now by [7, Proposition 9.26], R is a discrete valuation ring.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are equivalent by [2, page 757, Theorem 7].

(3) $\Rightarrow$ (8) By the definition, the maximal ideal of a discrete valuation ring R is principal, and thus by Lemma 2.11 R is a GQ-ring.

(8) $\Rightarrow$ (1). By Lemma 2.11 R is a GP-ring, and hence by [1, Lemma 2.1] R has a unique non-zero prime ideal $M = Rt$ for some $t \in R$. Let I be a non-zero proper ideal of R. It follows from [7, Corollary 2.24] that $\cap_{n=1}^{\infty} M^n = 0$. Therefore, there exists a positive integer s such that $I \subseteq M^s$ and $I \nsubseteq M^{s+1}$. Choose $a \in I, a \notin M^{s+1}$. Then $a = ut^s$ for some $u \in R \setminus M$, and hence u is a unit of R. Hence $t^s \in I$, i.e., $I = M^s$. Thus R is a special primary ring.

□

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