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FOURIER SERIES OF FUNCTIONS RELATED TO TWO VARIABLE HIGHER-ORDER FUBINI POLYNOMIALS

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ABSTRACT. In this paper, we consider three types of functions related to two variable higher-order Fubini polynomials and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

1. INTRODUCTION

The two variable Fubini polynomials $F_m^{(r)}(x;y)$ of order $r(r \in \mathbb{Z}_{\geq 0})$ are defined by

$$\frac{e^{xt}}{(1-y(e^t-1))^r} = \sum_{m=0}^{\infty} F_m^{(r)}(x;y)\frac{t^m}{m!}, \text{ (see [6,8,13])}. \tag{1.1}$$

Unless otherwise stated, throughout this paper y will be an arbitrary but fixed nonzero real number so that $F_m^{(r)}(x;y)$ are polynomials in x, for each fixed $0 \neq y \in \mathbb{R}$.

Kargin introduced in [6] the two variable Fubini polynomials $F_m(x;y) = F_m^{(1)}(x;y)$. For $x = 0$, $F_m^{(r)}(y) = F_m^{(r)}(0;y)$ are called the Fubini polynomials of order r, and $F_m^{(r)} = F_m^{(r)}(1) = F_m^{(r)}(0;1)$ the Fubini numbers of order r.

From (1.1), we see that

$$\begin{aligned} &\frac{d}{dx}F_m^{(r)}(x;y) = mF_{m-1}^{(r)}(x;y), \ (m \geq 1),\\ &F_m^{(r)}(x+1;y) = \frac{y+1}{y}F_m^{(r)}(x;y) - \frac{1}{y}F_m^{(r-1)}(x;y), \ (m \geq 0). \end{aligned} \tag{1.2}$$

These in turn imply that

$$\begin{aligned} F_m^{(r)}(1;y) =& \frac{y+1}{y}F_m^{(r)}(y) - \frac{1}{y}F_m^{(r-1)}(y),\\ \int_0^1 F_m^{(r)}(x;y) =& \frac{1}{m+1}\left(F_{m+1}^{(r)}(1;y) - F_{m+1}^{(r)}(y)\right)\\ =& \frac{1}{(m+1)y}\left(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y)\right). \end{aligned} \tag{1.3}$$

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For any real number x, the fractional part of x is denoted by

$$< x >= x - [x] \in [0,1). \tag{1.4}$$

The Bernoulli polynomials $B_m(x)$ are given by

$$\frac{t}{e^t-1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}, \text{ (see [3,5,7,9,14].} \tag{1.5}$$

We also need the following facts about Bernoulli functions $B_m(< x >)$:
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \text{ (see [1,10–12,15,16,18]),} \tag{1.6}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.7}$$

In this paper, we will consider the following three types of functions $\alpha_m(< x >; y)$, $\beta_m(< x >; y)$, and $\gamma_m(< x >; y)$ related to two variable higher-order Fubini polynomials. We will derive their Fourier series expansions and also express them in terms of Bernoulli functions.

(1) $\alpha_m(< x >; y) = \sum_{k=0}^{m} F_k^{(r)}(< x >; y) < x >^{m-k}$, $(m \geq 1)$;
(2) $\beta_m(< x >; y) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} F_k^{(r)}(< x >; y) < x >^{m-k}$, $(m \geq 1)$;
(3) $\gamma_m(< x >; y) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(< x >; y) < x >^{m-k}$, $(m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1,2,17,18].

As to $\gamma_m(< x >; y)$, we note that the following polynomial identity follows immediately from (4.24) and (4.28).

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(x; y) x^{m-k} \\ &= \frac{1}{m} \sum_{k=0}^{m} \binom{m}{k} \Big\{ \Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} \\ &\times \Big(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\Big) \Big\} B_k(x), \end{aligned}$$

,

where, for each integer $l \geq 2$,

$$\Lambda_l(y) = \sum_{k=1}^{l-1} \frac{1}{k(l-k)y} \big((y+1)F_k^{(r)}(y) - F_k^{(r-1)}(y)\big),$$

with $\Lambda_1(y) = 0$, and $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers. Similar polynomial identities for $\alpha_m(< x >; y)$ and $\beta_m(< x >; y)$ follow from (2.15) and (2.18), (3.15) and (3.19), respectively. For some recent related works, we let the reader refer to [4,11,12,15,16].

2. The function $\alpha_m(<x>;y)$

Let $\alpha_m(x;y) = \sum_{k=0}^{m} F_k^{(r)}(x;y)x^{m-k}$, $(m \geq 1))$. Then we will consider the function

$$\alpha_m(<x>;y) = \sum_{k=0}^{m} F_k^{(r)}(<x>;y) <x>^{m-k},\ (m \geq 1)),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(<x>;y)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m,y)} e^{2\pi i n x}, \tag{2.1}$$

where

$$A_n^{(m)} = A_n^{(m,y)} = \int_0^1 \alpha_m(<x>;y)e^{-2\pi i n x}dx = \int_0^1 \alpha_m(x;y)e^{-2\pi i n x}dx. \tag{2.2}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}
\frac{d}{dx}\alpha_m(x;y) &= \sum_{k=0}^{m}\left(kF_{k-1}^{(r)}(x;y)x^{m-k} + (m-k)F_k^{(r)}(x;y)x^{m-k-1}\right) \\
&= \sum_{k=1}^{m} kF_{k-1}^{(r)}(x)x^{m-k} + \sum_{k=0}^{m-1}(m-k)F_k^{(r)}(x;y)x^{m-k-1} \\
&= \sum_{k=0}^{m-1}(k+1)F_k^{(r)}(x;y)x^{m-k-1} + \sum_{k=0}^{m-1}(m-k)F_k^{(r)}(x;y)x^{m-k-1} \\
&= (m+1)\sum_{k=0}^{m-1} F_k^{(r)}(x;y)x^{m-1-k} \\
&= (m+1)\alpha_{m-1}(x;y).
\end{aligned} \tag{2.3}$$

This implies that

$$\frac{d}{dx}\left(\frac{\alpha_{m+1}(x;y)}{m+2}\right) = \alpha_m(x;y), \tag{2.4}$$

and

$$\int_0^1 \alpha_m(x;y)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1;y) - \alpha_{m+1}(0;y)\right). \tag{2.5}$$

For $m \geq 1$, we put

$$\begin{aligned}
\Delta_m(y) &= \alpha_m(1;y) - \alpha_m(0;y) \\
&= \sum_{k=0}^{m}\left(F_k^{(r)}(1;y) - F_k^{(r)}(y)\delta_{m,k}\right) \\
&= \sum_{k=0}^{m}\left(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)\right) - F_m^{(r)}(y) \\
&= \sum_{k=0}^{m-1}\left(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)\right) + \frac{1}{y}\left(F_m^{(r)}(y) - F_m^{(r-1)}(y)\right).
\end{aligned} \tag{2.6}$$

Now, we note that

$$\alpha_m(0;y) = \alpha_m(1;y) \Longleftrightarrow \Delta_m(y) = 0, \tag{2.7}$$

and

$$\int_0^1 \alpha_m(x;y)dx = \frac{1}{m+2}\Delta_{m+1}(y). \tag{2.8}$$

We are now ready to determine $A_n^{(m)}$.

$Case\ 1 : n \neq 0.$

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x;y)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}[\alpha_m(x;y)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \left(\frac{d}{dx}\alpha_m(x;y)\right)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left(\alpha_m(1;y) - \alpha_m(0;y)\right) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x;y)e^{-2\pi inx}dx \\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m(y). \end{aligned} \tag{2.9}$$

Thus we have derived the recursive relation

$$A_n^{(m)} = \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m(y), \tag{2.10}$$

from which by induction on m, we can immediately derive

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}(y). \tag{2.11}$$

$Case\ 2 : n = 0.$

$$A_0^{(m)} = \int_0^1 \alpha_m(x;y)dx = \frac{1}{m+2}\Delta_{m+1}(y). \tag{2.12}$$

$\alpha_m(<x>;y), (m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(<x>;y)$ is continuous for those positive integers m with $\Delta_m(y) = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m(y) \neq 0$.

Assume first that $\Delta_m(y) = 0$, for a positive integer m. Then $\alpha_m(0;y) = \alpha_m(1;y)$. Hence $\alpha_m(<x>;y)$ is piecewise C^∞ , and continuous. Thus the Fourier series of $\alpha_m(<x>;y)$ converges uniformly to $\alpha_m(<x>;y)$, and

$$\alpha_m(<x>;y) = \frac{1}{m+2}\Delta_{m+1}(y) + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}(y)\right)e^{2\pi inx}$$
$$= \frac{1}{m+2}\Delta_{m+1}(y) + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y)\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)$$
$$= \frac{1}{m+2}\Delta_{m+1}(y) + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y)B_j(<x>)$$
$$+\Delta_m(y)\times\begin{cases} B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases} \tag{2.13}$$

Now, our first result follows.

Theorem 2.1. *For each positive integer l, we let*

$$\Delta_l(y) = \sum_{k=0}^{l-1}\left(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)\right) + \frac{1}{y}\left(F_l^{(r)}(y) - F_l^{(r-1)}(y)\right).$$

Assume that $\Delta_m(y) = 0$, for a positive integer m. Then we have the following.
(a) $\sum_{k=0}^{m}F_k^{(r)}(<x>;y)<x>^{m-k}$ has the Fourier series expansion

$$\sum_{k=0}^{m}F_k^{(r)}(<x>;y)<x>^{m-k}$$
$$=\frac{1}{m+2}\Delta_{m+1}(y) + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}(y)\right)e^{2\pi inx}, \tag{2.14}$$

for all $x\in\mathbb{R}$. Here the convergence is uniform.

$$(b)\sum_{k=0}^{m}F_k^{(r)}(<x>;y)<x>^{m-k} = \frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y)B_j(<x>), \tag{2.15}$$

for all $x\in\mathbb{R}$.

Assume next that $\Delta_m(y)\neq 0$, for a positive integer m. Then $\alpha_m(0;y)\neq\alpha_m(1;y)$. Hence $\alpha_m(<x>;y)$ is piecewise C^∞ , and discontinuous with jump discontinuities at integers.
The Fourier series of $\alpha_m(<x>;y)$ converges pointwise to $\alpha_m(<x>;y)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0;y)+\alpha_m(1;y)\right) = \alpha_m(0;y) + \frac{1}{2}\Delta_m(y), \tag{2.16}$$

for $x\in\mathbb{Z}$.

Now, our second result follows.

Theorem 2.2. *For each positive integer l, we let*

$$\Delta_l(y) = \sum_{k=0}^{l-1} \left(\frac{y+1}{y} F_k^{(r)}(y) - \frac{1}{y} F_k^{(r-1)}(y) \right) + \frac{1}{y} \left(F_l^{(r)}(y) - F_l^{(r-1)}(y) \right).$$

Assume that $\Delta_m(y) \neq 0$, for a positive integer m. Then we have the following.

$$(a) \frac{1}{m+2} \Delta_{m+1}(y) + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1}(y) \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} F_k^{(r)}(< x >; y) < x >^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ F_m^{(r)}(y) + \frac{1}{2}\Delta_m(y), & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.17}$$

$$(b) \frac{1}{m+2} \sum_{j=0}^{m} \binom{m+2}{j} \Delta_{m-j+1}(y) B_j(< x >)$$
$$= \sum_{k=0}^{m} F_k^{(r)}(< x >; y) < x >^{m-k}, \ \text{for } x \notin \mathbb{Z};$$
$$\frac{1}{m+2} \sum_{j=0, j\neq 1}^{m} \binom{m+2}{j} \Delta_{m-j+1}(y) B_j(< x >)$$
$$= F_m^{(r)}(y) + \frac{1}{2}\Delta_m(y), \ \text{for } x \in \mathbb{Z}. \tag{2.18}$$

3. The function $\beta_m(< x >; y)$

Let $\beta_m(x; y) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} F_k^{(r)}(x; y) x^{m-k}$, $(m \geq 1)$. Then we will consider the function

$$\beta_m(< x >; y) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} F_k^{(r)}(< x >; y) < x >^{m-k}, \ (m \geq 1),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >; y)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m,y)} e^{2\pi i n x}, \tag{3.1}$$

where

$$B_n^{(m)} = B_n^{(m,y)} = \int_0^1 \beta_m(< x >; y) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x; y) e^{-2\pi i n x} dx. \tag{3.2}$$

To proceed further, we need to note the following.

$$\begin{aligned}
\frac{d}{dx}\beta_m(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}F_{k-1}^{(r)}(x;y)x^{m-k} + \frac{(m-k)}{k!(m-k)!}F_k^{(r)}(x;y)x^{m-k-1}\right\} \\
&= \sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}F_{k-1}^{(r)}(x;y)x^{m-k} \\
&+ \sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}F_k^{(r)}(x;y)x^{m-k-1} \\
&= \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}F_k^{(r)}(x;y)x^{m-1-k} \\
&+ \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}F_k^{(r)}(x;y)x^{m-1-k} \\
&= 2\beta_{m-1}(x;y).
\end{aligned} \tag{3.3}$$

These imply that

$$\frac{d}{dx}\left(\frac{1}{2}\beta_{m+1}(x;y)\right) = \beta_m(x;y), \tag{3.4}$$

and

$$\int_0^1 \beta_m(x;y)dx = \frac{1}{2}\Big(\beta_{m+1}(1;y) - \beta_{m+1}(0;y)\Big). \tag{3.5}$$

For $m \geq 1$, we put

$$\begin{aligned}
\Omega_m(y) &= \beta_m(1;y) - \beta_m(0;y) \\
&= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\big(F_k^{(r)}(1;y) - F_k^{(r)}(y)\delta_{m,k}\big) \\
&= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)\right) - \frac{1}{m!}F_m^{(r)}(y) \\
&= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\big(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)\big) \\
&+ \frac{1}{m!y}\big(F_m^{(r)}(y) - F_m^{(r-1)}(y)\big).
\end{aligned} \tag{3.6}$$

We note that

$$\beta_m(0;y) = \beta_m(1;y) \Longleftrightarrow \Omega_m(y) = 0, \tag{3.7}$$

and

$$\int_0^1 \beta_m(x;y)dx = \frac{1}{2}\Omega_{m+1}(y). \tag{3.8}$$

We now would like to determine the Fourier coefficients $B_n^{(m)}$.
Case $1 : n \neq 0$

$$
\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x;y) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\beta_m(x;y)e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 (\frac{d}{dx}\beta_m(x;y))e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\Big(\beta_m(1;y) - \beta_m(0;y)\Big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x;y)e^{-2\pi i n x}dx \\
&= \frac{2}{2\pi i n}B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m(y),
\end{aligned} \tag{3.9}
$$

from which by induction on m we can easily show that

$$
B_n^{(m)} = -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}(y). \tag{3.10}
$$

Case 2 : $n = 0$

$$
B_0^{(m)} = \int_0^1 \beta_m(x;y)dx = \frac{1}{2}\Omega_{m+1}(y). \tag{3.11}
$$

$\beta_m(< x >; y)$ $(m \geq 1)$ is piecewise C^{∞}. Moreover, $\beta_m(< x >; y)$ is continuous for those positive integers m with $\Omega_m(y) = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m(y) \neq 0$.

Assume first that $\Omega_m(y) = 0$, for a positive integer m. Then $\beta_m(0; y) = \beta_m(1; y)$. Hence $\beta_m(< x >; y)$ is piecewise C^{∞}, and continuous. Thus the Fourier series of $\beta_m(< x >; y)$ converges uniformly to $\beta_m(< x >; y)$, and

$$
\begin{aligned}
&\beta_m(< x >; y) \\
&= \frac{1}{2}\Omega_{m+1}(y) + \sum_{n=-\infty, n\neq 0}^{\infty}\Big(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}(y)\Big)e^{2\pi i n x} \\
&= \frac{1}{2}\Omega_{m+1}(y) + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y)\Big(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi n x}}{(2\pi i n)^j}\Big) \\
&= \frac{1}{2}\Omega_{m+1}(y) + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y)B_j(< x >) \\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.12}
$$

We are now ready to state our first result.

Theorem 3.1. *For each positive integer l, we let*

$$
\begin{aligned}
\Omega_l(y) = &\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)) \\
&+ \frac{1}{l!y}\big(F_l^{(r)}(y) - F_l^{(r-1)}(y)\big).
\end{aligned} \tag{3.13}
$$

Assume that $\Omega_m(y) = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} F_k^{(r)}(<x>; y) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} \frac{1}{k!(m-k)!} F_k^{(r)}(<x>; y) <x>^{m-k}\\&= \frac{1}{2}\Omega_{m+1}(y) + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}(y)\Big) e^{2\pi i n x},\end{aligned} \tag{3.14}$$

for all $x \in \mathbb{R}$. Here the convergence is uniform.

$$\begin{aligned}(b) &\sum_{k=0}^{m} \frac{1}{k!(m-k)!} F_k^{(r)}(<x>; y) <x>^{m-k}\\&= \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}(y) B_j(<x>),\end{aligned} \tag{3.15}$$

for all $x \in \mathbb{R}$.

Assume next that $\Omega_m(y) \neq 0$, for a positive integer m. Then, $\beta_m(0; y) \neq \beta_m(1; y)$. Hence $\beta_m(<x>; y)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\beta_m(<x>; y)$ converges pointwise to $\beta_m(<x>; y)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0; y) + \beta_m(1; y)) = \beta_m(0; y) + \frac{1}{2}\Omega_m(y), \tag{3.16}$$

for $x \in \mathbb{Z}$.

Now, we are ready to state our second result.

Theorem 3.2. *For each positive integer l, we let*

$$\begin{aligned}\Omega_l(y) = &\sum_{k=0}^{l-1} \frac{1}{k!(l-k)!}\Big(\frac{y+1}{y} F_k^{(r)}(y) - \frac{1}{y} F_k^{(r-1)}(y)\Big)\\&+ \frac{1}{l! y}\big(F_l^{(r)}(y) - F_l^{(r-1)}(y)\big).\end{aligned} \tag{3.17}$$

Assume that $\Omega_m(y) \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}(a)\ &\frac{1}{2}\Omega_{m+1}(y) + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}(y)\Big) e^{2\pi i n x}\\&= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} F_k^{(r)}(<x>; y) <x>^{m-k}, & \text{for } x \notin \mathbb{Z},\\ \frac{1}{m!} F_m^{(r)}(y) + \frac{1}{2}\Omega_m(y), & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{3.18}$$

$$\begin{aligned}(b)\ &\sum_{j=0}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y)B_j(<x>)\\ &=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(<x>;y)<x>^{m-k}, \quad \text{for } x\notin\mathbb{Z};\\ &\sum_{j=0,j\neq1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y)B_j(<x>)\\ &=\frac{1}{m!}F_m^{(r)}(y)+\frac{1}{2}\Omega_m(y), \quad \text{for } x\in\mathbb{Z}.\end{aligned} \tag{3.19}$$

4. The function $\gamma_m(<x>;y)$

Let $\gamma_m(x;y)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(x;y)x^{m-k}$, $(m\geq 2)$. Then we will consider the function

$$\gamma_m(<x>;y)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(<x>;y)<x>^{m-k},\ (m\geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(<x>;y)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m,y)}e^{2\pi inx}, \tag{4.1}$$

where

$$C_n^{(m)}=C_n^{(m,y)}=\int_0^1\gamma_m(<x>;y)e^{-2\pi inx}dx=\int_0^1\gamma_m(x;y)e^{-2\pi inx}dx. \tag{4.2}$$

To proceed further, we need to observe the following.

$$\begin{aligned}\frac{d}{dx}\gamma_m(x;y)&=\sum_{k=1}^{m-1}\frac{1}{m-k}F_{k-1}^{(r)}(x;y)x^{m-k}+\sum_{k=1}^{m-1}\frac{1}{k}F_k^{(r)}(x;y)x^{m-k-1}\\ &=\sum_{k=0}^{m-2}\frac{1}{m-1-k}F_k^{(r)}(x;y)x^{m-1-k}+\sum_{k=1}^{m-1}\frac{1}{k}F_k^{(r)}(x;y)x^{m-1-k}\\ &=\sum_{k=1}^{m-2}\Big(\frac{1}{m-1-k}+\frac{1}{k}\Big)F_k^{(r)}(x;y)x^{m-1-k}+\frac{1}{m-1}x^{m-1}+\frac{1}{m-1}F_{m-1}^{(r)}(x;y)\\ &=(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}F_k^{(r)}(x;y)x^{m-1-k}+\frac{1}{m-1}x^{m-1}+\frac{1}{m-1}F_{m-1}^{(r)}(x;y)\\ &=(m-1)\gamma_{m-1}(x;y)+\frac{1}{m-1}x^{m-1}+\frac{1}{m-1}F_{m-1}^{(r)}(x;y).\end{aligned} \tag{4.3}$$

From this, we immediately see that

$$\Big(\frac{1}{m}\big(\gamma_{m+1}(x;y)-\frac{1}{m(m+1)}x^{m+1}-\frac{1}{m(m+1)}F_{m+1}^{(r)}(x;y)\big)\Big)'=\gamma_m(x;y), \tag{4.4}$$

and

$$\begin{aligned}
&\int_0^1 \gamma_m(x;y)dx \\
&= \frac{1}{m}\Big[\gamma_{m+1}(x;y) - \frac{1}{m(m+1)}x^{m+1} - \frac{1}{m(m+1)}F_{m+1}^{(r)}(x;y)\Big]_0^1 \\
&= \frac{1}{m}\Big(\gamma_{m+1}(1;y) - \gamma_{m+1}(0;y) - \frac{1}{m(m+1)} - \frac{1}{m(m+1)}(F_{m+1}^{(r)}(1;y) - F_{m+1}^{(r)}(y))\Big) \\
&= \frac{1}{m}\Big(\gamma_{m+1}(1;y) - \frac{1}{m(m+1)} - \frac{1}{m(m+1)y}(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y))\Big)
\end{aligned} \tag{4.5}$$

For $m \geq 2$, we let

$$\begin{aligned}
\Lambda_m(y) &= \gamma_m(1;y) - \gamma_m(0;y) \\
&= \gamma_m(1;y) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)y}\Big((y+1)F_k^{(r)}(y) - F_k^{(r-1)}(y)\Big).
\end{aligned} \tag{4.6}$$

For convenience, we also let $\Lambda_1(y) = 0$. Clearly, we have

$$\gamma_m(0;y) = \gamma_m(1;y) = 0 \ \Leftrightarrow \ \Lambda_m(y) = 0, \ \ for \ m \geq 2, \tag{4.7}$$

and

$$\int_0^1 \gamma_m(x;y)dx = \frac{1}{m}\left(\Lambda_{m+1}(y) - \frac{1}{m(m+1)} - \frac{1}{m(m+1)y}(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y))\right). \tag{4.8}$$

Our next task is to determine the Fourier coefficients $C_n^{(m)}$.

Case 1 : $n \neq 0$

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x;y)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big[\gamma_m(x;y)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 (\frac{d}{dx}\gamma_m(x;y))e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big(\gamma_m(1;y) - \gamma_m(0;y)\Big) \\
&+ \frac{1}{2\pi in}\int_0^1 \left((m-1)\gamma_{m-1}(x;y) + \frac{1}{m-1}x^{m-1} + \frac{1}{m-1}F_{m-1}^{(r)}(x;y)\right)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m(y) + \frac{1}{2\pi in(m-1)}\int_0^1 x^{m-1}e^{-2\pi inx}dx \\
&+ \frac{1}{2\pi in(m-1)}\int_0^1 F_{m-1}^{(r)}(x;y)e^{-2\pi inx}dx.
\end{aligned} \tag{4.9}$$

Here we recall that, for $m \geq 1$,

$$\int_0^1 x^m e^{-2\pi i n x} dx = \begin{cases} -\sum_{k=1}^{m} \frac{(m)_{k-1}}{(2\pi i n)^k}, & \text{for } n \neq 0, \\ \frac{1}{m+1}, & \text{for } n = 0. \end{cases} \tag{4.10}$$

Also, we can easily show that

$$\begin{aligned} &\int_0^1 F_m^{(r)}(x;y) e^{-2\pi i n x} dx \\ &= -\sum_{k=1}^{m} \frac{(m)_{k-1}}{(2\pi i n)^k} \Big(F_{m-k+1}^{(r)}(1;y) - F_{m-k+1}^{(r)}(y) \Big) \\ &= -\frac{1}{y} \sum_{k=1}^{m} \frac{(m)_{k-1}}{(2\pi i n)^k} \Big(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y) \Big), \ for\ n \neq 0, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &\int_0^1 F_m^{(r)}(x;y) dx \\ &= \frac{1}{m+1} \Big(F_{m+1}^{(r)}(1;y) - F_{m+1}^{(r)}(y) \Big) \\ &= \frac{1}{(m+1)y} \Big(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y) \Big). \end{aligned} \tag{4.12}$$

Thus we have shown that

$$\begin{aligned} C_n^{(m)} = &\frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m(y) - \frac{1}{2\pi i n(m-1)} \Psi_n^{(m)} \\ &- \frac{1}{2\pi i n(m-1)} \Phi_n^{(m,r)}(y), \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \Psi_n^{(m)} &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k}, \\ \Phi_n^{(m,r)}(y) &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \Big(F_{m-k}^{(r)}(1;y) - F_{m-k}^{(r)}(y) \Big). \end{aligned} \tag{4.14}$$

We can show the following by induction on m applied to (**??**)

$$\begin{aligned} C_n^{(m)} = &-\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{m-j+1}(y) - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi i n)^j (m-j)} \Psi_n^{(m-j+1)} \\ &- \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi i n)^j (m-j)} \Phi_n^{(m-j+1,r)}(y). \end{aligned} \tag{4.15}$$

To find more explicit expression for $C_n^{(m)}$, we observe that

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Phi_n^{(m-j+1,r)}(y) \\
=&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \sum_{k=1}^{m-j} \frac{(m-j)_{k-1}}{(2\pi in)^k} \Big(F_{m-j-k+1}^{(r)}(1;y) - F_{m-j-k+1}^{(r)}(y)\Big) \\
=&\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}} \Big(F_{m-j-k+1}^{(r)}(1;y) - F_{m-j-k+1}^{(r)}(y)\Big) \\
=&\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=j+1}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^{k}} \Big(F_{m-k+1}^{(r)}(1;y) - F_{m-k+1}^{(r)}(y)\Big) \\
=&\sum_{k=2}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^{k}} \Big(F_{m-k+1}^{(r)}(1;y) - F_{m-k+1}^{(r)}(y)\Big) \sum_{j=1}^{k-1} \frac{1}{m-j} \\
=&\sum_{k=1}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^{k}} \Big(F_{m-k+1}^{(r)}(1;y) - F_{m-k+1}^{(r)}(y)\Big) (H_{m-1} - H_{m-k}) \\
=&\frac{1}{m} \sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} \frac{F_{m-k+1}^{(r)}(1;y) - F_{m-k+1}^{(r)}(y)}{m-k+1} (H_{m-1} - H_{m-k}),
\end{aligned} \tag{4.16}$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ is the harmonic number.
Similarly, we have

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Psi_n^{(m-j+1)} \\
=&\frac{1}{m} \sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} \frac{1}{m-k+1} (H_{m-1} - H_{m-k}).
\end{aligned} \tag{4.17}$$

Recalling that $\Lambda_1(y) = 0$ by convention, from (??), (??) and (??), we get the following expression of $C_n^{(m)}$, for $n \neq 0$:

$$\begin{aligned}
C_n^{(m)} =& -\frac{1}{m} \sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} \Big\{ \Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} (1 + F_{m-k+1}^{(r)}(1;y) - F_{m-k+1}^{(r)}(y)) \Big\} \\
=& -\frac{1}{m} \sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} \Big\{ \Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} \Big(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\Big) \Big\}.
\end{aligned} \tag{4.18}$$

Case 2 : $n = 0$

$$\begin{aligned}
C_0^{(m)} =& \int_0^1 \gamma_m(x;y)dx \\
=& \frac{1}{m} \Big(\Lambda_{m+1}(y) - \frac{1}{m(m+1)} (1 + F_{m+1}^{(r)}(1;y) - F_{m+1}^{(r)}(y)) \Big) \\
=& \frac{1}{m} \Big(\Lambda_{m+1}(y) - \frac{1}{m(m+1)} \Big(1 + \frac{1}{y}(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y))\Big) \Big).
\end{aligned} \tag{4.19}$$

$\gamma_m(< x >; y)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(< x >; y)$ is continuous for those integers $m \geq 2$ with $\Lambda_m(y) = 0$, and discontinuous with jump discontinuities at integers for those integer $m \geq 2$ with $\Lambda_m(y) \neq 0$.

Assume first that $\Lambda_m(y) = 0$, for some integer $m \geq 2$. Then $\gamma_m(1; y) = \gamma_m(0; y)$. Hence $\gamma_m(< x >; y)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >; y)$ converges uniformly to $\gamma_m(< x >; y)$, and

$$\begin{aligned}
&\gamma_m(< x >; y) \\
&= \frac{1}{m}\left(\Lambda_{m+1}(y) - \frac{1}{m(m+1)}\left(1 + \frac{1}{y}(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y))\right)\right) \\
&- \frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\big(\Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} \\
&\times \left(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\right)\big)\Big\}e^{2\pi i n x} \\
&= \frac{1}{m}\left(\Lambda_{m+1}(y) - \frac{1}{m(m+1)}\left(1 + \frac{1}{y}(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y))\right)\right) \\
&+ \frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}(y) + + \frac{H_{m-1} - H_{m-k}}{m-k+1} \\
&\times \left(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\right)\Big\} \\
&\times \left(-k! \sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^k}\right)
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
&= \frac{1}{m}\left(\Lambda_{m+1}(y) - \frac{1}{m(m+1)}\left(1 + \frac{1}{y}(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y))\right)\right) \\
&+ \frac{1}{m}\sum_{k=2}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} \\
&\times \left(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\right)\Big\}B_k(< x >) \\
&+ \Lambda_m(y) \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\
&= \frac{1}{m}\sum_{k=0, k\neq 1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} \\
&\times \left(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\right)\Big\}B_k(< x >) \\
&+ \Lambda_m(y) \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.21}$$

Now, we are ready to state our first result.

Theorem 4.1. *For each integer $l \geq 2$, we let*

$$\Lambda_l(y) = \sum_{k=1}^{l-1} \frac{1}{k(l-k)y}\big((y+1)F_k^{(r)}(y) - F_k^{(r-1)}(y)\big), \tag{4.22}$$

with $\Lambda_1(y) = 0$. Assume that $\Lambda_m(y) = 0$, for some integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(<x>; y) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(<x>; y) <x>^{m-k} \\
&= \frac{1}{m}\left(\Lambda_{m+1}(y) - \frac{1}{m(m+1)}\Big(1 + \frac{1}{y}(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y))\Big)\right) \\
&- \frac{1}{m} \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{\sum_{k=1}^{m} \frac{(m)_k}{(2\pi i n)^k}\Big(\Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} \\
&\times \Big(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\Big)\Big)\Big\} e^{2\pi i n x},
\end{aligned} \tag{4.23}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}
(b)\ &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(<x>; y) <x>^{m-k} \\
&= \frac{1}{m} \sum_{k=0, k\neq 1}^{m} \binom{m}{k} \Big\{\Lambda_{m-k+1}(y) + \frac{H_{m-1} - H_{m-k}}{m-k+1} \\
&\times \Big(1 + \frac{1}{y}(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y))\Big)\Big\} B_k(<x>),
\end{aligned} \tag{4.24}$$

for all $x \in \mathbb{R}$.

Assume next that $\Lambda_m(y) \neq 0$, for some integer $m \geq 2$. Then $\gamma_m(1; y) \neq \gamma_m(0; y)$, and hence $\gamma_m(<x>; y)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>; y)$ converges pointwise to $\gamma_m(<x>; y)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0; y) + \gamma_m(1; y)) = \frac{1}{2}\Lambda_m(y), \tag{4.25}$$

for $x \in \mathbb{Z}$.

We are now ready to state our second result.

Theorem 4.2. *For each integer $l \geq 2$, we let*

$$\Lambda_l(y) = \sum_{k=1}^{l-1} \frac{1}{k(l-k)y}\big((y+1)F_k^{(r)}(y) - F_k^{(r-1)}(y)\big), \tag{4.26}$$

with $\Lambda_1(y) = 0$. Assume that $\Lambda_m(y) \neq 0$, for an integer $m \geq 2$. Then we have the following.

$$\begin{aligned}
&(a)\ \frac{1}{m}\left(\Lambda_{m+1}(y)-\frac{1}{m(m+1)}\Big(1+\frac{1}{y}(F^{(r)}_{m+1}(y)-F^{(r-1)}_{m+1}(y))\Big)\right)\\
&-\frac{1}{m}\sum_{n=-\infty,n\neq0}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}\Big(\Lambda_{m-k+1}(y)+\frac{H_{m-1}-H_{m-k}}{m-k+1}\\
&\times\Big(1+\frac{1}{y}(F^{(r)}_{m-k+1}(y)-F^{(r-1)}_{m-k+1}(y))\Big)\Big)\Big\}e^{2\pi inx}\\
&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F^{(r)}_k(<x>;y)<x>^{m-k}, & \text{for } x\notin\mathbb{Z},\\ \frac{1}{2}\Lambda_m(y), & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}\tag{4.27}$$

$$\begin{aligned}
&(b)\ \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}(y)+\frac{H_{m-1}-H_{m-k}}{m-k+1}\\
&\times\Big(1+\frac{1}{y}(F^{(r)}_{m-k+1}(y)-F^{(r-1)}_{m-k+1}(y))\Big)\Big\}B_k(<x>)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F^{(r)}_k(<x>;y)<x>^{m-k},\ \text{for } x\notin\mathbb{Z};\\
&\frac{1}{m}\sum_{k=0,k\neq1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}(y)+\frac{H_{m-1}-H_{m-k}}{m-k+1}\\
&\times\Big(1+\frac{1}{y}(F^{(r)}_{m-k+1}(y)-F^{(r-1)}_{m-k+1}(y))\Big)\Big\}B_k(<x>)\\
&=\frac{1}{2}\Lambda_m(y),\ \text{for } x\in\mathbb{Z}.
\end{aligned}\tag{4.28}$$

Remark. In this paper, we studied three types of functions which are related to two variable higher-order Fubini functions and gave Fourier series expansions of those functions. In addition, we expressed each of them in terms of Bernoulli functions. The Fourier series expansions of the two variable higher-order Fubini functions are useful in computing the special values of zeta and multiple zeta functions. It is expected that Fourier series of the two variable higher-order Fubini functions will find some applications in connection with certain zeta functions and higher-order Bernoulli numbers and polynomials.

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