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SOME IDENTITIES OF THE DEGENERATE CHANGHEE NUMBERS OF SECOND KIND ARISING FROM DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the degenerate Changhee numbers of second kind and derive a family of differential equations satisfied by the generating function of these numbers. As an immediate application, we derive an identity expressing the higher-order degenerate Changhee numbers of second kind in terms of the degenerate Changhee numbers of second kind.

1. Introduction

As is well known, the Changhee polynomials are defined by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \text{ (see [3, 5]).} \tag{1.1}$$

When $x = 0$, $Ch_n = Ch_n(0), (n \geq 0)$, are called the Changhee numbers.

In [14], H.-I. Kwon et al. introduced the degenerate Changhee polynomials are defined by the generating function to be

$$\frac{2\lambda}{2\lambda + \log(1+\lambda t)}(1 + \log(1+\lambda t)^{\frac{1}{\lambda}})^x = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.2}$$

When $x = 0$, $Ch_{n,\lambda} = Ch_{n,\lambda}(0), (n \geq 0)$, are called the degenerate Changhee numbers.

We recall the Stirling numbers of the first kind $S_1(n, m)$ are defined by

$$(\log(1+t))^m = (m!)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}, \text{ (see [1, 3, 5]).} \tag{1.3}$$

In [3], G.-W. Jang et al. introduced the degenerate Changhee numbers of second kind are defined by the generating function to be

$$\frac{2}{1 + \left(1 + \lambda\log(1+t)\right)^{\frac{1}{\lambda}}} = \sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!}. \tag{1.4}$$

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From (1.4), we have

$$
\begin{aligned}
2 &= \Big(\sum_{m=0}^{\infty} Ch_{m,\lambda}\frac{t^m}{m!}\Big)\Big(1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}\Big)\\
&= \sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!} + \Big(\sum_{m=0}^{\infty} Ch_{m,\lambda}\frac{t^m}{m!}\Big)\Big(\sum_{l=0}^{\infty}\binom{\frac{1}{\lambda}}{l}\lambda^l\big(\log(1+t)\big)^l\Big)\\
&= \sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!} + \Big(\sum_{m=0}^{\infty} Ch_{m,\lambda}\frac{t^m}{m!}\Big)\Big(\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{1}{l!}\big(\log(1+t)\big)^l\Big)\\
&= \sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!} + \Big(\sum_{m=0}^{\infty} Ch_{m,\lambda}\frac{t^m}{m!}\Big)\Big(\sum_{l=0}^{\infty}(1)_{l,\lambda}\sum_{k=l}^{\infty}S_1(k,l)\frac{t^k}{k!}\Big)\\
&= \sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!} + \Big(\sum_{m=0}^{\infty} Ch_{m,\lambda}\frac{t^m}{m!}\Big)\Big(\sum_{k=0}^{\infty}\sum_{l=0}^{k}(1)_{l,\lambda}S_1(k,l)\frac{t^k}{k!}\Big)\\
&= \sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!} + \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(1)_{l,\lambda}S_1(k,l)Ch_{n-k,\lambda}\Big)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\Big(Ch_{n,\lambda} + \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(1)_{l,\lambda}S_1(k,l)Ch_{n-k,\lambda}\Big)\frac{t^n}{n!},
\end{aligned}
\tag{1.5}
$$

where $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.
By (1.5), we get

$$
Ch_{n,\lambda} + \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(1)_{l,\lambda}S_1(k,l)Ch_{n-k,\lambda} = 2\delta_{n,0}. \tag{1.6}
$$

For $r \in \mathbb{N}$, the higher-order degenerate changhee numbers of second kind are defined by the generating function to be

$$
\Big(\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\Big)^r = \sum_{n=0}^{\infty} Ch_{n,\lambda}^{(r)}\frac{t^n}{n!}. \tag{1.7}
$$

From (1.4), we have

$$
\begin{aligned}
\Big(\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\Big)^r &= \Big(\sum_{l_1=0}^{\infty} Ch_{l_1,\lambda}\frac{t^{l_1}}{l_1!}\Big)\times\cdots\times\Big(\sum_{l_r=0}^{\infty} Ch_{l_r,\lambda}\frac{t^{l_r}}{l_r!}\Big)\\
&= \sum_{n=0}^{\infty}\Big(\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,l_2,\cdots l_r}Ch_{l_1,\lambda}\times\cdots\\
&\quad\times Ch_{l_r,\lambda}\Big)\frac{t^n}{n!}.
\end{aligned}
\tag{1.8}
$$

By (1.7) and (1.8), we get

$$
Ch_{n,\lambda}^{(r)} = \sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,l_2,\cdots l_r}Ch_{l_1,\lambda}\times\cdots\times Ch_{l_r,\lambda}. \tag{1.9}
$$

Recently, several researchers studied some identities of special polynomials arising from differential equations(see [2,4,6-13,15-17]). Also, G.-W. Jang et al. introduced the degenerate Changhee numbers and polynomials of the second kind(see [3]). They investigated some interesting identities of these numbers and polynomials. In this paper, we investigate some explicit identities and properties of the degenerate Changhee numbers of the second kind arising from the differential equations.

2. Some identities of degenerate Changhee numbers of second kind arising from differential equations.

In this section, we investigate some identities on the degenerate Changhee numbers of second kind arising from differential equations.

Let

$$F = F(t) = \frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}. \tag{2.1}$$

Taking the derivatives of (2.1) with respect to t, we have

$$\begin{aligned} F^{(1)} &= \frac{2}{\Big(1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}\Big)^2}\Big(-\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}-1}\frac{1}{(1+t)}\Big) \\ &= \frac{1}{(1+t)}\frac{1}{\big(1+\lambda\log(1+t)\big)}\frac{-2\Big(1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}-1\Big)}{\Big(1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}\Big)^2} \\ &= \frac{1}{(1+t)}\frac{1}{\big(1+\lambda\log(1+t)\big)}\Big(\frac{1}{2}F^2-F\Big). \end{aligned} \tag{2.2}$$

Multiple $2(1+t)\big(1+\lambda\log(1+t)\big)$ on the both sides of (2.2), we get

$$F^2-2F = 2(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)}. \tag{2.3}$$

From (2.3), we have

$$\begin{aligned} 2FF^{(1)} &= 2F^{(1)} + 2\big(1+\lambda\log(1+t)\big)F^{(1)} + 2\lambda F^{(1)} \\ &\quad + 2(1+t)\big(1+\lambda\log(1+t)\big)F^{(2)} \\ &= (2+2\lambda)F^{(1)} + 2\big(1+\lambda\log(1+t)\big)F^{(1)} \\ &\quad + 2(1+t)\big(1+\lambda\log(1+t)\big)F^{(2)}. \end{aligned} \tag{2.4}$$

Multiple $2(1+t)\big(1+\lambda\log(1+t)\big)$ on the both sides of (2.4), we obtain

$$\begin{aligned} 2F(F^2-2F) &= (4+4\lambda)(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} + 4(1+t) \\ &\quad \times \big(1+\lambda\log(1+t)\big)^2F^{(1)} + 4(1+t)^2\big(1+\lambda\log(1+t)\big)^2F^{(2)}. \end{aligned} \tag{2.5}$$

From (2.3) and (2.5), we have

$$\begin{aligned} 2F^3 &= 4F^2 + (4+4\lambda)(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} + 4(1+t) \\ &\quad \times \big(1+\lambda\log(1+t)\big)^2F^{(1)} + 4(1+t)^2\big(1+\lambda\log(1+t)\big)^2F^{(2)} \\ &= 8F + (12+4\lambda)(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} + 4(1+t) \\ &\quad \times \big(1+\lambda\log(1+t)\big)^2F^{(1)} + 4(1+t)^2\big(1+\lambda\log(1+t)\big)^2F^{(2)}. \end{aligned} \tag{2.6}$$

By (2.6), we get

$$\begin{aligned} 2!(F^3-2^2F) &= (12+4\lambda)(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} + 4(1+t) \\ &\quad \times \big(1+\lambda\log(1+t)\big)^2F^{(1)} + 4(1+t)^2\big(1+\lambda\log(1+t)\big)^2F^{(2)}. \end{aligned} \tag{2.7}$$

Continuing this process, we obtain

$$N!(F^{N+1}-2^NF) = \sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)(1+t)^k\big(1+\lambda\log(1+t)\big)^jF^{(k)}. \tag{2.8}$$

Let us take the derivatives of (2.8) with respect to t, we have

$$\begin{aligned} (N+1)!F^NF^{(1)} &= \sum_{k=1}^{N}\sum_{j=k}^{N} ka_{k,j}(N)(1+t)^{k-1}\big(1+\lambda\log(1+t)\big)^jF^{(k)} \\ &\quad + \sum_{k=1}^{N}\sum_{j=k}^{N} j\lambda a_{k,j}(N)(1+t)^{k-1}\big(1+\lambda\log(1+t)\big)^{j-1}F^{(k)} \\ &\quad + \sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{j}F^{(k+1)} \\ &\quad + N!2^NF^{(1)}. \end{aligned} \tag{2.9}$$

Multiple $2(1+t)\big(1+\lambda\log(1+t)\big)$ on the both sides of (2.9), we get

$$\begin{aligned} (N+1)!F^N(F^2-2F) &= N!2^{N+1}(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} \\ &\quad + \sum_{k=1}^{N}\sum_{j=k}^{N} 2ka_{k,j}(N)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{j+1}F^{(k)} \\ &\quad + \sum_{k=1}^{N}\sum_{j=k}^{N} 2j\lambda a_{k,j}(N)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{j}F^{(k)} \\ &\quad + \sum_{k=1}^{N}\sum_{j=k}^{N} 2a_{k,j}(N)(1+t)^{k+1}\big(1+\lambda\log(1+t)\big)^{j+1}F^{(k+1)}. \end{aligned} \tag{2.10}$$

From (2.8) and (2.10), we have

$$
\begin{aligned}
(N+1)!F^{N+2} &= 2(N+1)!F^{N+1} + N!2^{N+1}(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} \\
&+ \sum_{k=1}^{N}\sum_{j=k}^{N} 2ka_{k,j}(N)(1+t)^k\big(1+\lambda\log(1+t)\big)^{j+1}F^{(k)} \\
&+ \sum_{k=1}^{N}\sum_{j=k}^{N} 2j\lambda a_{k,j}(N)(1+t)^k\big(1+\lambda\log(1+t)\big)^{j}F^{(k)} \\
&+ \sum_{k=1}^{N}\sum_{j=k}^{N} 2a_{k,j}(N)(1+t)^{k+1}\big(1+\lambda\log(1+t)\big)^{j+1}F^{(k+1)} \\
&= (N+1)!2^{N+1}F + \sum_{k=1}^{N}\sum_{j=k}^{N} 2(N+1)a_{k,j}(N)(1+t)^k \\
&\times \big(1+\lambda\log(1+t)\big)^{j}F^{(k)} + N!2^{N+1}(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} \\
&+ \sum_{k=1}^{N}\sum_{j=k}^{N} 2ka_{k,j}(N)(1+t)^k\big(1+\lambda\log(1+t)\big)^{j+1}F^{(k)} \\
&+ \sum_{k=1}^{N}\sum_{j=k}^{N} 2j\lambda a_{k,j}(N)(1+t)^k\big(1+\lambda\log(1+t)\big)^{j}F^{(k)} \\
&+ \sum_{k=1}^{N}\sum_{j=k}^{N} 2a_{k,j}(N)(1+t)^{k+1}\big(1+\lambda\log(1+t)\big)^{j+1}F^{(k+1)}.
\end{aligned} \tag{2.11}
$$

From (2.11), we have

$$
\begin{aligned}
(N+1)!(F^{N+2} - 2^{N+1}F) &= N!2^{N+1}(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} \\
&+ \sum_{k=1}^{N}\sum_{j=k}^{N} \big(2(N+1)+2j\lambda\big)a_{k,j}(N)(1+t)^k \\
&\times \big(1+\lambda\log(1+t)\big)^{j}F^{(k)} + \sum_{k=1}^{N}\sum_{j=k+1}^{N+1} 2ka_{k,j-1}(N) \\
&\times (1+t)^k\big(1+\lambda\log(1+t)\big)^{j}F^{(k)} + \sum_{k=2}^{N+1}\sum_{j=k}^{N+1} 2 \\
&\times a_{k-1,j-1}(N)(1+t)^k\big(1+\lambda\log(1+t)\big)^{j}F^{(k)}
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
&= \Big(N!2^{N+1} + \big(2(N+1)+2\lambda\big)a_{1,1}(N)\Big)\\
&\times (1+t)\big(1+\lambda\log(1+t)\big)F^{(1)}\\
&+ \sum_{j=2}^{N}\Big(\big(2(N+1)+2j\lambda\big)a_{1,j}(N) + 2a_{1,j-1}(N)\Big)\\
&\times (1+t)\big(1+\lambda\log(1+t)\big)^{j}F^{(1)}\\
&+ 2a_{1,N}(N)(1+t)\big(1+\lambda\log(1+t)\big)^{N+1}F^{(1)}\\
&+ 2a_{N,N}(N)(1+t)^{N+1}\big(1+\lambda\log(1+t)\big)^{N+1}F^{(N+1)}\\
&+ \sum_{k=2}^{N}\bigg\{\Big(\big(2(N+1)+2k\lambda\big)a_{k,k}(N) + 2a_{k-1,k-1}(N)\Big)\\
&\times (1+t)^{k}\big(1+\lambda\log(1+t)\big)^{k}F^{(k)}\\
&+ \big(2ka_{k,N}(N) + 2a_{k-1,N}(N)\big)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{N+1}\\
&\times F^{(k)} + \sum_{j=k+1}^{N}\Big(\big(2(N+1)+2j\lambda\big)a_{k,j}(N) + 2ka_{k,j-1}(N)\\
&+ 2a_{k-1,j-1}(N)\Big)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{j}F^{(k)}\bigg\}.
\end{aligned}
$$

By replacing N by $N+1$ in (2.8), we have

$$
\begin{aligned}
(N+1)!(F^{N+2} - 2^{N+1}F) &= \sum_{k=1}^{N+1}\sum_{j=k}^{N+1} a_{k,j}(N+1)(1+t)^{k}\\
&\times \big(1+\lambda\log(1+t)\big)^{j}F^{(k)}\\
&= a_{1,1}(N+1)(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)}\\
&+ \sum_{j=2}^{N} a_{1,j}(N+1)(1+t)\big(1+\lambda\log(1+t)\big)^{j}F^{(1)}\\
&+ a_{1,N+1}(N+1)(1+t)\big(1+\lambda\log(1+t)\big)^{N+1}F^{(1)}\\
&+ a_{N+1,N+1}(N+1)(1+t)^{N+1}\big(1+\lambda\log(1+t)\big)^{N+1}F^{(N+1)}\\
&+ \sum_{k=2}^{N}\bigg\{a_{k,k}(N+1)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{k}F^{(k)}\\
&+ a_{k,N+1}(N+1)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{N+1}F^{(k)}\\
&+ \sum_{j=k+1}^{N} a_{k,j}(N+1)(1+t)^{k}\big(1+\lambda\log(1+t)\big)^{j}F^{(k)}.
\end{aligned} \tag{2.13}
$$

Comparing the coefficients of (2.12) and (2.13), we have

$$a_{1,1}(N+1) = N!2^{N+1} + 2(N+1+\lambda)a_{1,1}(N). \tag{2.14}$$

$$a_{1,N+1}(N+1) = 2a_{1,N}(N). \tag{2.15}$$

$$a_{N+1,N+1}(N+1) = 2a_{N,N}(N). \tag{2.16}$$

$$a_{k,j}(N+1) = 2(N+1+j\lambda)a_{k,j}(N) + 2ka_{k,j-1}(N) + 2a_{k-1,j-1}(N). \tag{2.17}$$

(2.17) holds for $2 \le k \le N$ and $k+1 \le j \le N$.

From (2.3) and (2.8), we get

$$\begin{aligned} F^2 - 2F &= \sum_{k=1}^{1}\sum_{j=k}^{1} a_{k,j}(1)(1+t)^k\big(1+\lambda\log(1+t)\big)^j F^{(k)} \\ &= a_{1,1}(1)(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)} \\ &= 2(1+t)\big(1+\lambda\log(1+t)\big)F^{(1)}. \end{aligned} \tag{2.18}$$

By (2.18), we get

$$a_{1,1}(1) = 2. \tag{2.19}$$

From (2.14), we have

$$\begin{aligned} a_{1,1}(N+1) &= N!2^{N+1} + 2(N+1+\lambda)a_{1,1}(N) \\ &= N!2^{N+1} + 2(N+1+\lambda)\big((N-1)!2^N + 2(N+\lambda) \\ &\times a_{1,1}(N-1)\big) \\ &= N!2^{N+1} + (N-1)!2^{N+1}(N+1+\lambda) + 2^2(N+1+\lambda)_2 \\ &\times a_{1,1}(N-1) \\ &= \cdots \\ &= N!2^{N+1} + (N-1)!2^{N+1}(N+1+\lambda)_1 + \cdots + 2^N \\ &\times (N+1+\lambda)_N a_{1,1}(1). \end{aligned} \tag{2.20}$$

From (2.19) and (2.20), we get

$$\begin{aligned} a_{1,1}(N+1) &= N!2^{N+1} + (N-1)!2^{N+1}(N+1+\lambda)_1 + \cdots + 2^{N+1} \\ &\times (N+1+\lambda)_N \\ &= 2^{N+1}\sum_{l=0}^{N}(N-l)!(N+1+\lambda)_l. \end{aligned} \tag{2.21}$$

From (2.15) and (2.19), we have

$$\begin{aligned} a_{1,N+1}(N+1) &= 2a_{1,N}(N) = 2^2 a_{1,N-1}(N-1) = \cdots = 2^N a_{1,1}(1) \\ &= 2^{N+1}. \end{aligned} \tag{2.22}$$

Continuing this process, we get the following theorem.

Theorem 2.1. *Let $N \in \mathbb{N}$, then the following differential equations*

$$N!(F^{N+1} - 2^N F) = \sum_{k=1}^{N} \sum_{j=k}^{N} a_{k,j}(N)(1+t)^k \big(1 + \lambda \log(1+t)\big)^j F^{(k)},$$

have a solution for $F(t) = \frac{2}{1+\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}}$, *where*

$$a_{1,1}(N) = 2^N \sum_{l=0}^{N-1} (N-1-l)!(N+\lambda)_l.$$

$$a_{1,N}(N) = 2^N.$$

$$\begin{aligned} a_{1,j}(N) = &\sum_{i_1=0}^{N-j} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{j-1}=0}^{i_{j-2}} \sum_{l=0}^{i_{j-1}} 2^N (N+j\lambda)_{N-j-i_1} \\ &\times (j-1+(j-1)\lambda+i_1)_{i_1-i_2} \cdots (2+2\lambda+i_{j-2})_{i_{j-2}-i_{j-1}} \\ &\times (i_{j-1}-l)!(i_{j-1}+1+\lambda)_l. \end{aligned}$$

$$a_{N,N}(N) = 2^N.$$

$$\begin{aligned} a_{k,k}(N) = &\sum_{s_1=0}^{N-k} \sum_{s_2=0}^{s_1} \cdots \sum_{s_{k-1}=0}^{s_{k-2}} \sum_{l=0}^{s_{k-1}} 2^N (N+k\lambda)_{N-k-s_1} \\ &\times (k-1+(k-1)\lambda+s_1)_{s_1-s_2} \cdots (2+2\lambda+s_{k-2})_{s_{k-2}-s_{k-1}} \\ &\times (s_{k-1}-l)!(s_{k-1}+1+\lambda)_l. \end{aligned}$$

$$a_{k,j}(N) = 2(N+j\lambda)a_{k,j}(N-1) + 2ka_{k,j-1}(N-1) + 2a_{k-1,j-1}(N-1).$$

$$\begin{aligned} a_{k,N}(N) = &\sum_{m_1=0}^{N-k} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{k-1}=0}^{m_{k-2}} 2^N k^{N-k-m_1} \\ &\times (k-1)^{m_1-m_2} \cdots 2^{m_{k-2}-m_{k-1}}. \end{aligned}$$

By (1.4),we have

$$\begin{aligned} F^{(k)} &= \Big(\frac{d}{dt}\Big)^k F(t) \\ &= \Big(\frac{d}{dt}\Big)^k \sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} Ch_{n+k,\lambda}\frac{t^n}{n!}. \end{aligned} \tag{2.23}$$

From (2.8) and (2.23), we have

$$\begin{aligned} &\sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)(1+t)^k\big(1+\lambda\log(1+t)\big)^j F^{(k)} \\ &= \sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)\Big(\sum_{m=0}^{\infty}(k)_m\frac{t^m}{m!}\Big)\Big(\sum_{l_1=0}^{\infty}(j)_{l_1}\lambda^{l_1}\frac{1}{l_1!}\big(log(1+t)\big)^{l_1}\Big)F^{(k)} \\ &= \sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)\Big(\sum_{m=0}^{\infty}(k)_m\frac{t^m}{m!}\Big)\Big(\sum_{l_1=0}^{\infty}(j)_{l_1}\lambda^{l_1}\sum_{l_2=l_1}^{\infty}S_1(l_2,l_1)\frac{t^{l_2}}{l_2!}\Big)F^{(k)} \\ &= \sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)\Big(\sum_{m=0}^{\infty}(k)_m\frac{t^m}{m!}\Big)\Big(\sum_{l_2=0}^{\infty}\sum_{l_1=0}^{l_2}(j)_{l_1}\lambda^{l_1}S_1(l_2,l_1)\frac{t^{l_2}}{l_2!}\Big)F^{(k)} \\ &= \sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)\sum_{l_3=0}^{\infty}\Big(\sum_{l_2=0}^{l_3}\sum_{l_1=0}^{l_2}\binom{l_3}{l_2}(j)_{l_1}\lambda^{l_1}S_1(l_2,l_1)(k)_{l_3-l_2}\Big)\frac{t^{l_3}}{l_3!} \\ &\times\Big(\sum_{l_4=0}^{\infty}Ch_{l_4+k,\lambda}\frac{t_4^l}{l_4!}\Big) \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{N}\sum_{j=k}^{N} a_{k,j}(N)\sum_{n=0}^{\infty}\Big(\sum_{l_3=0}^{n}\sum_{l_2=0}^{l_3}\sum_{l_1=0}^{l_2}\binom{n}{l_3}\binom{l_3}{l_2}(j)_{l_1}\lambda^{l_1}S_1(l_2,l_1)(k)_{l_3-l_2} \\ &\times Ch_{n-l_3+k,\lambda}\Big)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{N}\sum_{j=k}^{N}\sum_{l_3=0}^{n}\sum_{l_2=0}^{l_3}\sum_{l_1=0}^{l_2}\binom{n}{l_3}\binom{l_3}{l_2}a_{k,j}(N)(j)_{l_1}\lambda^{l_1}S_1(l_2,l_1)(k)_{l_3-l_2} \\ &\times Ch_{n-l_3+k,\lambda}\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.24}$$

Also, by (1.4), (1.7) and (2.8), we have

$$\begin{aligned} N!(F^{N+1}-2^NF) &= N!\Big(\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(N+1)}\frac{t^n}{n!} - \sum_{n=0}^{\infty}2^NCh_{n,\lambda}\frac{t^n}{n!}\Big) \\ &= \sum_{n=0}^{\infty}N!\Big(Ch_{n,\lambda}^{(N+1)} - 2^NCh_{n,\lambda}\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.25}$$

From (2.24) and (2.25), we get the following theorem.

Theorem 2.2. *Let $N \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, we have*

$$Ch_{n,\lambda}^{(N+1)} = 2^N Ch_{n,\lambda} + \frac{1}{N!} \sum_{k=1}^{N} \sum_{j=k}^{N} \sum_{l_3=0}^{n} \sum_{l_2=0}^{l_3} \sum_{l_1=0}^{l_2} \binom{n}{l_3} \binom{l_3}{l_2} a_{k,j}(N)(j)_{l_1} \times \lambda^{l_1} S_1(l_2, l_1)(k)_{l_3-l_2} Ch_{n-l_3+k,\lambda},$$

where $a_{k,j}(N)$ are the same as the numbers mentioned in Theorem 2.1.

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