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A new class of degenerate Frobenius-Euler-Hermite polynomials

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Abstract. In this paper, we introduce a new class of degenerate Frobenius-Euler-Hermite polynomials and investigate some identities of these polynomials. Some implicit summation formulae and symmetric identities are also derived by applying the generating functions. These results extend some known summations and identities of generalized degenerate Frobenius-Euler-Hermite polynomials studied by Pathan and Khan.

Keywords: Hermite polynomials, degenerate Frobenius-Euler polynomials, degenerate Frobenius-Euler-Hermite polynomials, Summation formulae, Symmetric identities.
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1. Introduction

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [2, 3] are defined as

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

It is easily seen from definition (1.1) that

$$H_n(2x,-1) = H_n(x)$$

and

$$H_n(x,-\frac{1}{2}) = He_n(x),$$

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials [1]. Also

$$H_n(x,0) = x^n.$$

The generating function for Hermite polynomial H_n(x,y) are given by [5, 6, 7, 9, 10]:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.2}$$

The generating function for degenerate Hermite polynomials are given by [13]:

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!}, \tag{1.3}$$

where $\lambda \neq 0$.

Since $(1+\lambda t)^{\frac{1}{\lambda}} \longrightarrow e^t$ as $\lambda \longrightarrow 0$, it is evident that (1.3) reduces to (1.2). That is $H_n(x,y)$ limiting case of $H_n(x,y;\lambda)$ when $\lambda \longrightarrow 0$.

By equating coefficients of t^n on both the sides of (1.3), the following representation of $H_n(x,y;\lambda)$ is obtained

$$H_n(x,y;\lambda) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{(-\frac{x}{\lambda})_{n-2r}(-\frac{y}{\lambda})_r(-\lambda)^{n-r}}{r!(n-2r)!}. \tag{1.4}$$

Since $\lim_{\lambda\longrightarrow 0} H_n(x,y;\lambda) = H_n(x,y)$, (1.1) is a limiting case of (1.4).

Recently, Kurt and Simsek [19] introduced the following classical Frobenius-Euler polynomial as follows:

The classical Frobenius-Euler polynomial $H_n^{(\alpha)}(x;u)$ of order α is defined by means of the following generating function:

$$\left(\frac{1-u}{e^t-u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{1.5}$$

where u is an algebraic number and $\alpha \in \mathbb{Z}$.

Observe that $H_n^{(1)}(x,u) = H_n(x,u)$, which denotes the Frobenius-Euler polynomials, $H_n^{(\alpha)}(0;u) = H_n^{(\alpha)}(u)$, which denotes the Frobenius-Euler numbers of order α and $H_n(x;-1) = E_n(x)$, which denotes the Euler polynomials (c.f [1-27]).

In [4], Carlitz introduced the degenerate Bernoulli polynomials are defined by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda,x)\frac{t^n}{n!}. \tag{1.6}$$

When $x=0$, $\beta_n^{(\alpha)}(\lambda) = \beta_n^{(\alpha)}(\lambda,0)$ are called the degenerate Bernoulli numbers.

From (1.6), we have

$$\lim_{\lambda\longrightarrow 0}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

where $B_n(x)$ are called the Bernoulli polynomials (see [21-25]).

Kim et al. [14;p.4.,Eq.(2.13)] introduced the degenerate Frobenius-Euler polynomials are defined by

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x;u)\frac{t^n}{n!}. \tag{1.7}$$

So that

$$h_{n,\lambda}(x;u) = \sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) h_{m,\lambda}(u)\left(\frac{x}{\lambda}\right)_{n-m},\ (n \geq 0).$$

When $x=0$, $h_{n,\lambda}(u) = h_{n,\lambda}(0;u)$, are called degenerate Frobenius-Euler numbers.

Note that

$$h_{n,\lambda}(\frac{1}{u}) - uh_{n,\lambda}(u) = \begin{cases} 1-u, & \text{if n} = 0 \\ 0, & \text{if } n > 0. \end{cases}$$

Kim et al. [14] proved that

$$h_{n,\lambda}^{(\alpha)}(x;u) = \sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) h_{m,\lambda}^{(\alpha)}(u)\left(\frac{x}{\lambda}\right)_{n-m},\ (n \geq 0), \tag{1.8}$$

where

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}^{(\alpha)}(x;u)\frac{t^n}{n!}. \tag{1.9}$$

Pathan and Khan [22] introduced the following generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$ as follows:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{1.10}$$

Setting $\alpha=1$ in (1.10), the result reduces to known result of Dattoli et al. [8;p.386(1.6)]:

$$\left(\frac{t}{e^t-1}\right) e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n(x,y)\frac{t^n}{n!}. \tag{1.11}$$

Also, we recall here the following definitions:

The Stirling number of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l,(n\geq 0); \tag{1.12}$$

and the Stirling number of the second kind is defined by generating function

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}. \tag{1.13}$$

A generalized falling factorial sum $\tau_k(n;\lambda)$ can be defined by the generating function [27]:

$$\sum_{k=0}^{\infty}\tau_k(n;\lambda)\frac{t^k}{k!}=\frac{1-(-(1+\lambda t)^{\frac{(n+1)}{\lambda}})}{1+(1+\lambda t)^{\frac{1}{\lambda}}}, \tag{1.14}$$

where $\lim_{\lambda\longrightarrow 0}\tau_k(n;\lambda)=T_k(n)$.

In this paper, we consider the degenerate Frobenius-Euler-Hermite polynomials and investigate some properties and identities of these polynomials.

2. Generalized degenerate Frobenius-Euler-Hermite polynomials

For $\lambda,u\in\mathbb{C}$ and $\alpha\in\mathbb{N}$ with $u\neq 1$, we consider generalized degenerate Frobenius-Euler-Hermite polynomials of order α which are given by the generating function:

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)\frac{t^n}{n!}. \tag{2.1}$$

So that

$$ {}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)=\sum_{m=0}^{n}\binom{n}{m}h_{n-m,\lambda}^{(\alpha)}(u)H_m(x,y;\lambda),n\geq 0. \tag{2.2}$$

When $x=y=0$ in (2.1), $h_{n,\lambda}^{(\alpha)}(u)={}_Hh_{n,\lambda}^{(\alpha)}(0,0;u)$ are called generalized degenerate Frobenius-Euler numbers of order α.
Note that

$$\lim_{\lambda\longrightarrow 0}\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\left(\frac{1-u}{e^t-u}\right)^{\alpha}e^{xt+yt^2}$$

$$=\sum_{n=0}^{\infty}{}_HE_n^{(\alpha)}(x,y;u)\frac{t^n}{n!}. \tag{2.3}$$

From (2.1) and (2.3), we have

$$\lim_{\lambda \longrightarrow 0} {}_H h_{n,\lambda}^{(\alpha)}(x, y; u) = {}_H E_n^{(\alpha)}(x, y; u), \tag{2.4}$$

where ${}_H E_n^{(\alpha)}(x, y; u)$ are called generalized Frobenius-Euler-Hermite polynomials of order α.

For $\alpha = 1$, (2.1) reduces to

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} {}_H h_{n,\lambda}(x, y; u)\frac{t^n}{n!}. \tag{2.5}$$

By using equations (2.1) and (2.5), we state the following theorem:

Theorem 2.1. For $n \geq 0$, we have

$$\frac{1}{1-u}\left\{{}_H h_{n,\lambda}^{(\alpha)}(x+1, y; u) - {}_H h_{n,\lambda}^{(\alpha)}(x, y; u)\right\} = {}_H h_{n,\lambda}^{(\alpha-1)}(x, y; u). \tag{2.6}$$

Proof. We observe that

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^{\alpha}(1+\lambda t)^{\frac{x+1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} - \left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^{\alpha-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1-u)$$

$$\sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha)}(x+1, y; u)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha)}(x, y; u)\frac{t^n}{n!} = (1-u)\sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha-1)}(x, y; u)\frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the result (2.6).

Remark 2.1. For $y = 0$, Theorem 2.1 reduces to known result of Kim et al. [14;p.6.,Theorem 2.7].

Corollary 2.1. For $n \geq 0$, we have

$$\frac{1}{1-u}\left\{h_{n,\lambda}^{(\alpha)}(x+1; u) - h_{n,\lambda}^{(\alpha)}(x; u)\right\} = h_{n,\lambda}^{(\alpha-1)}(x; u). \tag{2.7}$$

Theorem 2.2. For $m \geq 0$, we have

$$\sum_{n=0}^{m}\binom{m}{n} h_{m-n,\lambda}^{(\alpha)}(x; u)(-\frac{2y}{\lambda})_n(-\lambda)^n = \sum_{n=0}^{m} {}_H E_n^{(\alpha)}(x, y; u)\lambda^{m-n}S_1(m, n). \tag{2.8}$$

Proof. From equation (2.3), we have

$$\left(\frac{1-u}{e^t - u}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u)\frac{t^n}{n!}. \tag{2.9}$$

Replacing t by $log(1+\lambda t)^{\frac{1}{\lambda}}$ in above equation, we get

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{2y}{\lambda}} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u)\frac{1}{n!}\frac{1}{\lambda^n}\left(log(1+\lambda t)\right)^n$$

$$\sum_{m=0}^{\infty} h_{m,\lambda}^{(\alpha)}(x; u)\frac{t^m}{m!}\sum_{n=0}^{\infty}(-\frac{2y}{\lambda})_n\frac{(-\lambda t)^n}{n!} = \sum_{n=0}^{\infty} {}_H E_n^{(\alpha)}(x, y; u)\lambda^{-n}\sum_{m=n}^{\infty} S_1(m, n)\lambda^m\frac{t^m}{m!}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n} h_{m-n,\lambda}^{(\alpha)}(x; u)(-\frac{2y}{\lambda})_n(-\lambda)^n\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} {}_H E_n^{(\alpha)}(x, y; u)\lambda^{m-n}S_1(m, n)\frac{t^m}{m!}.$$

On equating their coefficients of $\frac{t^m}{m!}$ leads to the formula (2.8).

Remark 2.2. On setting $y = 0$ in Theorem 2.2, we get the known result of Kim et al. [14;p.5.,Theorem 2.6].

Corollary 2.2. For $m \geq 0$, we have

$$h^{(\alpha)}_{m,\lambda}(x; u) = \sum_{n=0}^{m} H^{(\alpha)}_{n}(x; u)\lambda^{m-n}S_1(m, n). \tag{2.8}$$

3. Implicit summation formulae involving generalized degenerate Frobenius-Euler-Hermite polynomials

The purpose of this section is to give some interesting generating functions, new results and relations for the generalized degenerate Frobenius-Euler-Hermite polynomials. We begin here some of these results in the following forms:

Theorem 3.1. The following implicit summation formula involving degenerate Frobenius-Euler-Hermite polynomials ${}_Hh^{(\alpha)}_{n,\lambda}(x, y; u)$ holds true:

$${}_Hh^{(\alpha)}_{n,\lambda}(x + z, y + w; u) = \sum_{m=0}^{n} \binom{n}{m} {}_Hh^{(\alpha)}_{n-m,\lambda}(x, y; u)H_m(z, w; \lambda). \tag{3.1}$$

Proof. By the definition of degenerate Frobenius-Euler-Hermite polynomials and the definition (1.3), we have

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^{\alpha} (1+\lambda t)^{\frac{x+z}{\lambda}}(1+\lambda t^2)^{\frac{y+w}{\lambda}} = \left(\sum_{n=0}^{\infty} {}_Hh^{(\alpha)}_{n,\lambda}(x, y; u)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(z, w; \lambda)\frac{t^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_Hh^{(\alpha)}_{n,\lambda}(x, y; u)H_m(z, w; \lambda)\frac{t^{m+n}}{n!m!}.$$

Replacing n by $n - m$ in above equation and comparing the coefficients of $\frac{t^n}{n!}$, we get the result (3.1).

Theorem 3.2. The following implicit summation formula involving degenerate Frobenius-Euler-Hermite polynomials ${}_Hh^{(\alpha)}_{n,\lambda}(x, y; u)$ holds true:

$${}_Hh^{(\alpha)}_{n,\lambda}(x, y; u) = \sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]} h^{(\alpha)}_{m,\lambda}(u)(-\frac{x}{\lambda})_{n-m-2j}(-\lambda)^{n-m-j}(-\frac{y}{\lambda})_j\frac{n!}{m!j!(n-2j-m)!}. \tag{3.2}$$

Proof. Applying the definition (2.1) to the term $\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{\alpha}$ and expanding the function $(1 + \lambda t)^{\frac{x}{\lambda}}(1 + \lambda t^2)^{\frac{y}{\lambda}}$ at $t = 0$ yields

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u}\right)^{\alpha} (1 + \lambda t)^{\frac{x}{\lambda}}(1 + \lambda t^2)^{\frac{y}{\lambda}}$$

$$= \left(\sum_{m=0}^{\infty} h^{(\alpha)}_{m,\lambda}(u)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(-\frac{x}{\lambda})_n\frac{(-\lambda t)^n}{n!}\right)\left(\sum_{j=0}^{\infty}(-\frac{y}{\lambda})_j\frac{(-\lambda t^2)^j}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} h^{(\alpha)}_{m,\lambda}(u)(-\frac{x}{\lambda})_{n-m}(-\lambda)^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}(-\frac{y}{\lambda})_j\frac{(-\lambda t^2)^j}{j!}\right).$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty} {}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}h_{m,\lambda}^{(\alpha)}(u)(-\frac{x}{\lambda})_{n-m-2j}(-\lambda)^{n-m-j}(-\frac{y}{\lambda})_j\right)\frac{t^n}{(n-2j)!j!}. \tag{3.3}$$

Equating their coefficients of $\frac{t^n}{n!}$, we get the result (3.2).

Theorem 3.3. The following implicit summation formula involving degenerate Frobenius-Euler-Hermite polynomials ${}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)$ holds true:

$${}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)=\sum_{m=0}^{n}\binom{n}{m}(-\frac{z}{\lambda})_{n-m}(-\lambda)^{n-m}{}_Hh_{m,\lambda}^{(\alpha)}(x-z,y;u). \tag{3.4}$$

Proof. By exploiting the generating function (2.1), we can write the equation

$$\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)\frac{t^n}{n!}=\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{\alpha}(1+\lambda t)^{\frac{x-z}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1+\lambda t)^{\frac{z}{\lambda}} \tag{3.5}$$

$$=\left(\sum_{m=0}^{\infty}{}_Hh_{m,\lambda}^{(\alpha)}(x-z,y;u)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(-\frac{z}{\lambda})_n\frac{(-\lambda t)^n}{n!}\right).$$

Replacing n by $n-m$ in above equation and equating their coefficients of $\frac{t^n}{n!}$ leads to formula (3.4).

Theorem 3.4. The following implicit summation formula involving degenerate Frobenius-Euler-Hermite polynomials ${}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)$ holds true:

$${}_Hh_{n,\lambda}^{(\alpha)}(x+1,y;u)=\sum_{r=0}^{n}\binom{n}{r}(-\frac{1}{\lambda})_r(-\lambda)^r{}_Hh_{n-r,\lambda}^{(\alpha)}(x,y;u). \tag{3.6}$$

Proof. By the definition of degenerate Frobenius-Euler-Hermite polynomials, we have

$$\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}(x+1,y;u)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)\frac{t^n}{n!}$$

$$=\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1)$$

$$=\left(\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}(-\frac{1}{\lambda})_r\frac{(-\lambda t)^r}{r!}\right)+\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_Hh_{n-r,\lambda}^{(\alpha)}(x,y;u)(-\frac{1}{\lambda})_r(-\lambda)^r\frac{t^n}{(n-r)!r!}+\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)\frac{t^n}{n!}.$$

Finally, equating the coefficients of like powers of $\frac{t^n}{n!}$, we get our desired result (3.6).

Theorem 3.5. The following implicit summation formula involving degenerate Frobenius-Euler-Hermite polynomials ${}_Hh_{n,\lambda}^{(\alpha)}(x,y;u)$ holds true:

$${}_Hh_{n,\lambda}^{(\alpha+\beta)}(x+y,z+w;u)=\sum_{m=0}^{n}\binom{n}{m}{}_Hh_{m,\lambda}^{(\beta)}(y,w;u){}_Hh_{n-m,\lambda}^{(\alpha)}(x,z;u). \tag{3.7}$$

Proof. Applying Definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha+\beta)}(x+y, z+w; u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha)}(x, z; u)\frac{t^n}{n!} \sum_{m=0}^{\infty} {}_H h_{m,\lambda}^{(\beta)}(y, w; u)\frac{t^m}{m!}.$$

Replacing n by $n-m$ in the above equation, we get

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} {}_H h_{m,\lambda}^{(\beta)}(y, w; u)\frac{t^m}{m!} {}_H h_{n-m,\lambda}^{(\alpha)}(x, z; u) \right) \frac{t^n}{n!}.$$

Now equating the coefficients of like powers of $\frac{t^n}{n!}$ in the above equation, we get the result (3.7).

4. Identities for degenerate Frobenius-Euler-Hermite polynomials

In this section, we give general symmetry identities for the degenerate Frobenius-Euler-Hermite polynomials ${}_H h_{n,\lambda}^{(\alpha)}(x, y; u)$ by applying the generating functions (2.1) and (1.8). The results extend some known identities of Khan [11-13], Young [27], Pathan and Khan [21-25]. Throughout this section α will taken as an arbitrary real or complex parameter.

Theorem 4.1. For all integers $a > 0, b > 0, n \geq 0$ and $\alpha \in \mathbb{C}$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_H h_{n-k,\lambda}^{(\alpha)}(bx, b^2y; u) {}_H h_{k,\lambda}^{(\alpha)}(ax, a^2y; u)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k {}_H h_{n-k,\lambda}^{(\alpha)}(ax, a^2y; u) {}_H h_{k,\lambda}^{(\alpha)}(bx, b^2y; u). \tag{4.1}$$

Proof. Start with

$$g(t) = \left(\frac{(1-u)^2}{((1+\lambda t)^{\frac{a}{\lambda}} - u)((1+\lambda t)^{\frac{b}{\lambda}} - u)} \right)^{\alpha} (1+\lambda t)^{\frac{abx}{\lambda}} (1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}}. \tag{4.2}$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$g(t) = \left(\sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha)}(bx, b^2y; u)\frac{(at)^n}{n!} \right) \left(\sum_{k=0}^{\infty} {}_H h_{k,\lambda}^{(\alpha)}(ax, a^2y; u)\frac{(bt)^k}{k!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_H h_{n-k,\lambda}^{(\alpha)}(bx, b^2y; u) {}_H h_{k,\lambda}^{(\alpha)}(ax, a^2y; u) \right) \frac{t^n}{n!}.$$

On the similar lines we can show that

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k {}_H h_{n-k,\lambda}^{(\alpha)}(ax, a^2y; u) {}_H h_{k,\lambda}^{(\alpha)}(bx, b^2y; u) \right) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at our desired result.

Remark 4.1. By setting $b = 1$ in Theorem (4.1), we immediately get the following corollary.

Corollary 4.1. For all integers $a>0$, $n\geq 0$ and $\alpha\in\mathbb{C}$, the following identity holds true:

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{n-k}{}_Hh_{n-k,\lambda}^{(\alpha)}(x,y;u){}_Hh_{k,\lambda}^{(\alpha)}(ax,a^2y;u)$$
$$=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{k}{}_Hh_{n-k,\lambda}^{(\alpha)}(ax,a^2y;u){}_Hh_{k,\lambda}^{(\alpha)}(x,y;u). \tag{4.3}$$

Theorem 4.2. For all integers $a>0, b>0$, $n\geq 0$ and $\alpha\in\mathbb{C}$, the following identity holds true:

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{n-k}b^k{}_Hh_{n-k,\lambda}^{(\alpha)}\left(bx,b^2z;u\right)\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\tau_{i,\lambda}(a-1;u)h_{k-i,\lambda}^{(\alpha)}(ay;u)$$
$$=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{k}b^{n-k}{}_Hh_{n-k,\lambda}^{(\alpha)}\left(ax,a^2z;u\right)\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\tau_{i,\lambda}(b-1;u)h_{k-i,\lambda}^{(\alpha)}(by;u), \tag{4.4}$$

where generalized falling factorial sum $\tau_{k,\lambda}(u;n)$ is given by (1.14).

Proof. We now use

$$g(t)=\frac{(1-u)^{2\alpha}(-u-(-(1+\lambda t)^{\frac{ab}{\lambda}}))(1+\lambda t)^{\frac{ab(x+y)}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}}}{((1+\lambda t)^{\frac{a}{\lambda}}-u)^{\alpha}((1+\lambda t)^{\frac{b}{\lambda}}-u)^{\alpha+1}}, \tag{4.5}$$

to find that

$$g(t)=\left(\frac{1-u}{(1+\lambda t)^{\frac{a}{\lambda}}-u}\right)^{\alpha}(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}}\left(\frac{(-u-(-(1+\lambda t)^{\frac{ab}{\lambda}}))}{(1+\lambda t)^{\frac{b}{\lambda}}-u}\right)$$
$$\times\left(\frac{1-u}{(1+\lambda t)^{\frac{b}{\lambda}}-u}\right)^{\alpha}(1+\lambda t)^{\frac{aby}{\lambda}}$$
$$=\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}\left(bx,b^2z;u\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\tau_{n,\lambda}(a-1;u)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(ay;u)\frac{(bt)^n}{n!}. \tag{4.6}$$

Using a similar plan, we get

$$g(t)=\sum_{n=0}^{\infty}{}_Hh_{n,\lambda}^{(\alpha)}\left(ax,a^2z;u\right)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\tau_{n,\lambda}(b-1;u)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(by;u)\frac{(at)^n}{n!}. \tag{4.7}$$

Finally (4.4) follows after an appropriate change of summation index and comparison of the coefficients of $\frac{t^n}{n!}$.

Remark 4.2. for $\alpha=1$ in Theorem (4.2), we immediately get the following corollary.

Corollary 4.2. For all integers $a>0, b>0$, $n\geq 0$ and $\alpha\in\mathbb{C}$, the following identity holds true:

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{n-k}b^k{}_Hh_{n-k,\lambda}\left(bx,b^2z;u\right)\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\tau_{i,\lambda}(a-1;u)h_{k-i,\lambda}(ay;u)$$
$$=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{k}b^{n-k}{}_Hh_{n-k,\lambda}\left(ax,a^2z;u\right)\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\tau_{i,\lambda}(b-1;u)h_{k-i,\lambda}(by;u). \tag{4.8}$$

Theorem 4.3. For all integers $a>0, b>0$, $n\geq 0$ and $\alpha\in\mathbb{C}$, the following identity holds true:

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{k}b^{n-k}\sum_{i=0}^{a-1}{}_Hh_{k,\lambda}^{(\alpha)}\left(bx+\frac{b}{a}i,b^2z;u\right)h_{n-k,\lambda}^{(\alpha)}(ay;u)$$

$$= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k \sum_{i=0}^{b-1} {}_H h_k^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; u \right) h_{n-k,\lambda}^{(\alpha)}(by; u). \tag{4.9}$$

Proof. From (4.5), $g(t)$ can also be expanded as

$$g(t) = \left(\sum_{i=0}^{a-1} \sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha)} \left(bx + \frac{b}{a} i, b^2 z; u \right) \frac{(at)^n}{n!} \right) \left(\sum_{n=0}^{\infty} h_{n,\lambda}^{(\alpha)}(ay; u) \frac{(bt)^n}{n!} \right). \tag{4.10}$$

Using a similar plan, we get

$$g(t) = \left(\sum_{i=0}^{b-1} \sum_{n=0}^{\infty} {}_H h_{n,\lambda}^{(\alpha)} \left(ax + \frac{a}{b} i, a^2 z; u \right) \frac{(bt)^n}{n!} \right) \left(\sum_{n=0}^{\infty} h_{n,\lambda}^{(\alpha)}(by; u) \frac{(at)^n}{n!} \right). \tag{4.11}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at our desired result.

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