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SUMS OF FINITE PRODUCTS OF LEGENDRE AND LAGUERRE POLYNOMIALS BY CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we will consider sums of finite products of Legendre and Laguerre polynomials. Then we represent each of those sums of finite products as linear combinations of the four kinds of Chebyshev polynomials.

1. Introduction and preliminaries

First, we fix some notations and recall some basic facts that will be used throughout this paper.

For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $< x >_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\ (x)_0 = 1, \tag{1.1}$$

$$< x >_n = x(x+1)\cdots(x+n-1),\ (n \geq 1),\ < x >_0 = 1. \tag{1.2}$$

The two factorial polynomials are related by

$$(x)_n = (-1)^n < -x >_n,\ < x >_n = (-1)^n(-x)_n. \tag{1.3}$$

The following identity will be useful.

$$\frac{(2n-2s)!}{(n-s)!} = \frac{2^{2n-2s}(-1)^s < \frac{1}{2} >_n}{< \frac{1}{2} - n >_s}, \tag{1.4}$$

for any integers n, s with $n \geq s \geq 0$.

The hypergeometric function ${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x)$ is defined by

$$\begin{aligned} &{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) \\ &= \sum_{n=0}^{\infty} \frac{< a_1 >_n \cdots < a_p >_n}{< b_1 >_n \cdots < b_q >_n} \frac{x^n}{n!},\ (p \leq q+1,\ |x| < 1). \end{aligned} \tag{1.5}$$

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The well-known Chu-Vandermonde formula is given by

$$ {}_2F_1(-n,a;c;1) = \frac{< c-a >_n}{< c >_n},\ (c \neq 0,\cdots,1-n). \tag{1.6} $$

In this paper, we only need very basic knowledge about Legendre, Laguerre and Chebyshev polynomials that we will recall here in below. They belong to the family of classical orthogonal polynomials of which we let the reader refer to [2,4,15,16] for full accounts . In terms of generating functions, the Legendre, Laguerre, and Chebyshev polynomials of the first, second, third, and fourth kinds are respectively given by

$$ F(t,x) = (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_n(x)t^n, \tag{1.7} $$

$$ G(t,x) = (1-t)^{-1}\exp(-\tfrac{xt}{1-t}) = \sum_{n=0}^{\infty} L_n(x)t^n, \tag{1.8} $$

$$ \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{1.9} $$

$$ \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{1.10} $$

$$ \frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \tag{1.11} $$

$$ \frac{1}{1-xt-t^2} = \sum_{n=0}^{\infty} W_n(x)t^n. \tag{1.12} $$

They are explicitly given as in the following.

$$\begin{aligned} P_n(x) &= {}_2F_1(-n, n+1; 1; \tfrac{1-x}{2}) \\ &= \frac{1}{2^n} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n}{l} \binom{2n-2l}{n} x^{n-2l}, \ (n \geq 0), \end{aligned} \tag{1.13}$$

$$\begin{aligned} L_n(x) &= {}_1F_1(-n; 1; x) \\ &= \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} \frac{1}{(n-l)!} x^{n-l}, \ (n \geq 0), \end{aligned} \tag{1.14}$$

$$\begin{aligned} T_n(x) &= {}_2F_1(-n, n; \tfrac{1}{2}; \tfrac{1-x}{2}) \\ &= \frac{n}{2} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \frac{1}{n-l} \binom{n-l}{l} (2x)^{n-2l}, \ (n \geq 1), \end{aligned} \tag{1.15}$$

$$\begin{aligned} U_n(x) &= (n+1) {}_2F_1(-n, n+2; \tfrac{3}{2}; \tfrac{1-x}{2}) \\ &= \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n-l}{l} (2x)^{n-2l}, \ (n \geq 0), \end{aligned} \tag{1.16}$$

$$\begin{aligned} V_n(x) &= {}_2F_1(-n, n+1; \tfrac{1}{2}; \tfrac{1-x}{2}) \\ &= \sum_{l=0}^{n} \binom{2n-l}{l} 2^{n-l} (x-1)^{n-l}, \ (n \geq 0), \end{aligned} \tag{1.17}$$

$$\begin{aligned} W_n(x) &= (2n+1) {}_2F_1(-n, n+1; \tfrac{3}{2}; \tfrac{1-x}{2}) \\ &= (2n+1) \sum_{l=0}^{n} \frac{2^{n-l}}{2n-2l+1} \binom{2n-l}{l} (x-1)^{n-l}, \ (n \geq 0). \end{aligned} \tag{1.18}$$

The Chebyshev polynomials of the first, second, third and fourth kinds are given by the Rodrigues' formulas.

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}, \tag{1.19}$$

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!} (1-x^2)^{-\frac{1}{2}} \frac{d^n}{dx^n} (1-x^2)^{n+\frac{1}{2}}, \tag{1.20}$$

$$(1-x)^{-\frac{1}{2}} (1+x)^{\frac{1}{2}} V_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n} (1-x)^{n-\frac{1}{2}} (1+x)^{n+\frac{1}{2}}, \tag{1.21}$$

$$(1-x)^{\frac{1}{2}} (1+x)^{-\frac{1}{2}} W_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n} (1-x)^{n+\frac{1}{2}} (1+x)^{n-\frac{1}{2}}. \tag{1.22}$$

As is well known, they have the following orthogonalities with respect to various weight functions.

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_n(x)T_m(x)=\frac{\pi}{\mathcal{E}_n}\delta_{n,m}, \tag{1.23}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}U_n(x)U_m(x)=\frac{\pi}{2}\delta_{n,m}, \tag{1.24}$$

$$\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}V_n(x)V_m(x)=\pi\delta_{n,m}, \tag{1.25}$$

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}W_n(x)W_m(x)=\pi\delta_{n,m}, \tag{1.26}$$

where

$$\mathcal{E}_n=\begin{cases}1, & \text{if } n=0,\\ 2, & \text{if } n\geq 1,\end{cases} \qquad \delta_{n,m}=\begin{cases}0, & \text{if } n\neq m,\\ 1, & \text{if } n=m.\end{cases} \tag{1.27}$$

For convenience, we let

$$\alpha_{n,r}(x)=\sum_{i_1+i_2+\cdots+i_{2r+1}=n}P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x),\ (n,r\geq 0), \tag{1.28}$$

$$\beta_{n,r}(x)=\sum_{i_1+i_2+\cdots+i_{r+1}=n}L_{i_1}(\tfrac{x}{r+1})L_{i_2}(\tfrac{x}{r+1})\cdots L_{i_{r+1}}(\tfrac{x}{r+1}),\ (n,r\geq 0). \tag{1.29}$$

Note here that both $\alpha_{n,r}(x)$ and $\beta_{n,r}(x)$ are polynomials of degree n. Here in this paper, we would like to investigate the sums of finite products of Legendre and Laguerre polynomials in (1.28) and (1.29). Then we will express both of them as linear combinations of the four kinds of Chebyshev polynomials $T_n(x), U_n(x), V_n(x)$, and $W_n(x)$. This will be done with explicit computations, using Propositions 3 and 4. We remark here that the formulas in Proposition 3 can be derived from the orthogonalities in (1.23)-(1.26), Rodrigues' formulas (1.19)-(1.22), and integration by parts.

The next two theorems are our main results.

Theorem 1.1. *Let n, r be nonnegative integers. Then we have the following.*

$$\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_{2r+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x)\\
&=\frac{2^r}{(2r-1)!!n!}\\
&\quad\times\sum_{j=0}^{[\frac{n}{2}]}\binom{n}{j}\mathcal{E}_{n-2j}(j+r-\tfrac{1}{2})_j(n+r-j-\tfrac{1}{2})_{n+r-j}T_{n-2j}(x) \qquad (1.30)\\
&=\frac{2^r}{(2r-1)!!(n+1)!}\\
&\quad\times\sum_{j=0}^{[\frac{n}{2}]}(n-2j+1)\binom{n+1}{j}(j+r-\tfrac{3}{2})_j(n+r-j-\tfrac{1}{2})_{n+r-j}U_{n-2j}(x) \qquad (1.31)\\
&=\frac{2^r}{(2r-1)!!n!}\\
&\quad\times\sum_{j=0}^{n}\binom{n}{[\frac{j}{2}]}([\tfrac{j}{2}]+r-\tfrac{1}{2})_{[\frac{j}{2}]}(n+r-[\tfrac{j}{2}]-\tfrac{1}{2})_{n+r-[\frac{j}{2}]}V_{n-j}(x) \qquad (1.32)\\
&=\frac{2^r}{(2r-1)!!n!}\\
&\quad\times\sum_{j=0}^{n}(-1)^j\binom{n}{[\frac{j}{2}]}([\tfrac{j}{2}]+r-\tfrac{1}{2})_{[\frac{j}{2}]}(n+r-[\tfrac{j}{2}]-\tfrac{1}{2})_{n+r-[\frac{j}{2}]}W_{n-j}(x). \qquad (1.33)
\end{aligned}$$

Here $(2r-1)!!$ is the double factorial given by

$$(2r-1)!!=(2r-1)(2r-3)\cdots 1,(r\geq 1),\ (-1)!!=1. \qquad (1.34)$$

Theorem 1.2. *Let n, r be nonnegative integers. Then we have the following.*

$$\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_{r+1}=n} L_{i_1}(\tfrac{x}{r+1})L_{i_2}(\tfrac{x}{r+1})\cdots L_{i_{r+1}}(\tfrac{x}{r+1})\\
&=\sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k\mathcal{E}_k\sum_{j=0}^{[\frac{n-k}{2}]}\frac{\binom{n+r}{n-k-2j}(\frac{1}{4})^j}{(k+j)!j!}T_k(x) \qquad (1.35)\\
&=\sum_{k=0}^{n}(k+1)\left(-\frac{1}{2}\right)^k\sum_{j=0}^{[\frac{n-k}{2}]}\frac{\binom{n+r}{n-k-2j}(\frac{1}{4})^j}{(k+j+1)!j!}U_k(x) \qquad (1.36)
\end{aligned}$$

$$= \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^{k} \sum_{j=0}^{n-k} \frac{\binom{n+r}{n-k-j}(-\frac{1}{2})^{j}}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!} V_k(x) \tag{1.37}$$

$$= \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^{k} \sum_{j=0}^{n-k} \frac{\binom{n+r}{n-k-j}(\frac{1}{2})^{j}}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!} W_k(x). \tag{1.38}$$

Before closing this section, we would like to mention some of the previous papers which have been written along the some line as the present one. In [10], the sums of finite products of Legendre and Laguerre polynomials were expressed as linear combinations of Bernoulli polynomials. The same had been done for the sums of finite products of Bernoulli, Euler and Genocchi polynomials in [1,12,13]. All of these were derived from the Fourier series expansions of the functions closely connected with those various sums of finite products. Here we recall that Bernoulli polynomials are not orthogonal polynomials but Appell polynomials. Some various polynomials and sums of products of polynomials were expressed in terms of Legendre polynomials in [3,9] and of Laguerre polynomials in [7]. For a related work, we let the reader refer to [5]. The papers [6,8] are about representation of polynomials by Chebyshev polynomials, as it will be done in this paper. Also, the reader may refer to [14] for some other applications of Chebyshev polynomials.

2. Proof of Theorem 1.1

Here in this section we are going to prove Theorem 1.1. For the proofs of Theorems 1.1 and 1.2, we will need Propositions 2.1 and 2.2. They are stated in [11]. The results (a) and (b) in Proposition 2.1 are respectively from (1.24) and (1.36) of [8], while (c) and (d) are stated respectively in the equations (1.23) and (1.38) of [6].

Proposition 2.1. Let $g(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then we have the following

$$(a) \quad q(x) = \sum_{k=0}^{n} C_{k,1} T_k(x), \text{ where}$$

$$C_{k,1} = \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx.$$

$$(b) \quad q(x) = \sum_{k=0}^{n} C_{k,2} U_k(x), \text{ where}$$

$$C_{k,2} = \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}} dx.$$

$$(c)\quad q(x)=\sum_{k=0}^{n}C_{k,3}V_k(x),\ \text{where}$$
$$C_{k,3}=\frac{(-1)^k k!2^k}{(2k)!\pi}\int_{-1}^{1}q(x)\frac{d^k}{dx^k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx.$$
$$(d)\quad q(x)=\sum_{k=0}^{n}C_{k,4}W_k(x),\ \text{where}$$
$$C_{k,4}=\frac{(-1)^k k!2^k}{(2k)!\pi}\int_{-1}^{1}q(x)\frac{d^k}{dx^k}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}dx.$$

We observe that, in case of $k=0$, the following integrals are the moments of the four kinds of Chebyshev polynomials.

Proposition 2.2. For any nonnegative integers m and k, we have the following.

$$(a)\quad \int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^m dx = \begin{cases} 0, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0 \pmod 2.\end{cases}$$

$$(b)\quad \int_{-1}^{1}(1-x^2)^{k+\frac{1}{2}}x^m dx = \begin{cases} 0, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k+2)!\pi}{2^{m+2k+2}(\frac{m}{2}+k+1)!(\frac{m}{2})!(k+1)!}, & \text{if } m\equiv 0 \pmod 2.\end{cases}$$

$$(c)\quad \int_{-1}^{1}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}x^m dx = \begin{cases} \frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0 \pmod 2.\end{cases}$$

$$(d)\quad \int_{-1}^{1}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}x^m dx = \begin{cases} -\frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0 \pmod 2.\end{cases}$$

As was shown in [10], we can obtain the following lemma by differentiating (1.7).

Lemma 2.3. Let n, r be nonnegative integers. Then we have the following identity.

$$\sum_{i_1+i_2+\cdots+i_{2r+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x) = \frac{1}{(2r-1)!!}P_{n+r}^{(r)}(x), \tag{2.1}$$

where the sum is over all nonnegative integers $i_1, i_2, \cdots i_{2r+1}$, with $i_1 + i_2 + \cdots + i_{2r+1} = n$.

The rth derivative of (1.13) is given by

$$P_n^{(r)}(x) = \frac{1}{2^n}\sum_{l=0}^{[\frac{n-r}{2}]}(-1)^l\binom{n}{l}\binom{2n-2l}{n}(n-2l)_r x^{n-2l-r}. \tag{2.2}$$

In particular, we have

$$P_{n+r}^{(r+k)}(x) = \frac{1}{2^{n+r}}\sum_{l=0}^{[\frac{n-k}{2}]}(-1)^l\binom{n+r}{l}\binom{2n+2r-2l}{n+r}(n+r-2l)_{r+k}x^{n-2l-k}. \tag{2.3}$$

Here we will show only (1.31) and (1.33) in Theorem 1.1, as (1.30) and (1.32) can be proved similarly to those.

with $\alpha_{n,r}(x)$ as in (1.28), we let

$$\alpha_{n,r}(x) = \sum_{k=0}^{n} C_{k,2}U_k(x). \tag{2.4}$$

Then, from (b) of Proposition 2.1, (2.1), (2.3), and integration by parts k times , we have

$$\begin{aligned}
C_{k,2} &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\alpha_{n,r}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx \\
&= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi(2r-1)!!}\int_{-1}^{1}P_{n+r}^{(r)}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx \\
&= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi(2r-1)!!}\int_{-1}^{1}P_{n+r}^{(r+k)}(x)(1-x^2)^{k+\frac{1}{2}}dx \\
&= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi(2r-1)!!2^{n+r}}\sum_{l=0}^{[\frac{n-k}{2}]}(-1)^l\binom{n+r}{l}\binom{2n+2r-2l}{n+r} \\
&\quad\times (n+r-2l)_{r+k}\int_{-1}^{1}x^{n-2l-k}(1-x^2)^{k+\frac{1}{2}}dx.
\end{aligned} \tag{2.5}$$

From (2.5), (b) of Proposition 2.2, and after some simplifications, we get

$$C_{k,2} = \frac{(k+1)}{(2r-1)!!2^{2n+r}} \sum_{l=0}^{[\frac{n-k}{2}]} (-4)^l \binom{n+r}{l} \binom{2n+2r-2l}{n+r} (n+r-2l)_{r+k} \times \begin{cases} 0, & \text{if } k \not\equiv n \pmod 2, \\ \frac{(n-2l-k)!}{(\frac{n+k}{2}-l+1)!(\frac{n-k}{2}-l)!}, & \text{if } k \equiv n \pmod 2. \end{cases} \tag{2.6}$$

Combining (2.4) and (2.6), and using (1.3) and (1.4), we have

$$\begin{aligned} \alpha_{n,r}(x) &= \frac{1}{(2r-1)!!2^{2n+r}} \sum_{\substack{0 \le k \le n \\ k \equiv n \pmod 2}} \sum_{l=0}^{[\frac{n-k}{2}]} (-4)^l \binom{n+r}{l} \binom{2n+2r-2l}{n+r} \\ &\quad \times (n+r-2l)_{r+k} \frac{(k+1)(n-2l-k)!}{(\frac{n+k}{2}-l+1)!(\frac{n-k}{2}-l)!} U_k(x) \\ &= \frac{1}{(2r-1)!!2^{2n+r}} \sum_{\substack{0 \le k \le n \\ k \equiv n \pmod 2}} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-4)^l (2n+2r-2l)!}{l!(n+r-l)!} \\ &\quad \times \frac{(k+1)}{(\frac{n+k}{2}-l+1)!(\frac{n-k}{2}-l)!} U_k(x) \\ &= \frac{1}{(2r-1)!!2^{2n+r}} \sum_{j=0}^{[\frac{n}{2}]} \sum_{l=0}^{j} \frac{(-4)^l (2n+2r-2l)!(n-2j+1)}{l!(n+r-l)!(n-j-l+1)!(j-l)!} \\ &= \frac{2^r (n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n-2j+1)}{(n-j+1)!j!} \\ &\quad \times \sum_{l=0}^{j} \frac{<-j>_l <j-n-1>_l}{<\frac{1}{2}-n-r>_l \, l!} U_{n-2j}(x) \\ &= \frac{2^r (n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n-2j+1)}{(n-j+1)!j!} \\ &\quad \times {}_2F_1(-j, j-n-1; \tfrac{1}{2}-n-r; 1) U_{n-2j}(x). \end{aligned} \tag{2.7}$$

From (1.6) and (2.7), we finally obtain

$$\begin{aligned}\alpha_{n,r}(x) &= \frac{2^r(n+r-\frac{1}{2})_{n+r}}{(2r-1)!!}\sum_{j=0}^{[\frac{n}{2}]}\frac{(n-2j+1)<\frac{3}{2}-j-r>_j}{(n-j+1)!j!<\frac{1}{2}-n-r>_j}U_{n-2j}(x)\\ &= \frac{2^r}{(2r-1)!!(n+1)!}\sum_{j=0}^{[\frac{n}{2}]}(n-2j+1)\binom{n+1}{j}\\ &\quad\times(j+r-\tfrac{3}{2})_j(n+r-j-\tfrac{1}{2})_{n+r-j}U_{n-2j}(x).\end{aligned} \tag{2.8}$$

This shows (1.31), as we wanted.
Next, we show the equation (1.33). With $\alpha_{n,r}(x)$ as in (1.28), we set

$$\alpha_{n,r}(x) = \sum_{k=0}^{n} C_{k,4}W_k(x). \tag{2.9}$$

Then, from (d) of Proposition 2.1, (2.1), (2.3) and integration by parts k times , we have

$$\begin{aligned}C_{k,4} &= \frac{k!2^k}{(2k)!\pi(2r-1)!!2^{n+r}}\sum_{l=0}^{[\frac{n-k}{2}]}(-1)^l\binom{n+r}{l}\binom{2n+2r-2l}{n+r}\\ &\quad\times(n+r-2l)_{r+k}\int_{-1}^{1}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}x^{n-2l-k}dx.\end{aligned} \tag{2.10}$$

From (2.10), (d) of Proposition 2.2 and after some simplifications, we have

$$\begin{aligned}C_{k,4} &= \frac{1}{(2r-1)!!2^{2n+r}}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-4)^l(2n+2r-2l)!}{l!(n+r-l)!}\\ &\quad\times\begin{cases}-\frac{n-k-2l+1}{2(\frac{n+k+1}{2}-l)!(\frac{n-k+1}{2}-l)!}, & \text{if } k\not\equiv n \pmod 2,\\ \frac{1}{(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!}, & \text{if } k\equiv n \pmod 2.\end{cases}\end{aligned} \tag{2.11}$$

Combining (2.9) and (2.11), and using (1.3), (1.4), and (1.6), we get

$$\begin{aligned}\alpha_{n,r}(x) &= \frac{1}{(2r-1)!!2^{2n+r}}\left\{\sum_{j=0}^{[\frac{n}{2}]} W_{n-2j}(x)\sum_{l=0}^{j}\frac{(-4)^l(2n+2r-2l)!}{l!(n+r-l)!(n-j-l)!(j-l)!}\right.\\ &\quad \left.-\sum_{j=0}^{[\frac{n-1}{2}]} W_{n-2j-1}(x)\sum_{l=0}^{j}\frac{(-4)^l(2n+2r-2l)!}{l!(n+r-l)!(n-j-l)!(j-l)!}\right\}\\ &= \frac{1}{(2r-1)!!2^{2n+r}}\sum_{j=0}^{n}(-1)^j W_{n-j}(x)\sum_{l=0}^{[\frac{j}{2}]}\frac{(-4)^l(2n+2r-2l)!}{l!(n+r-l)!(n-[\frac{j}{2}]-l)!([\frac{j}{2}]-l)!}\\ &= \frac{2^r(n+r-\frac{1}{2})_{n+r}}{(2r-1)!!n!}\sum_{j=0}^{n}(-1)^j\binom{n}{[\frac{j}{2}]}{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-n;\tfrac{1}{2}-n-r;1)W_{n-j}(x)\\ &= \frac{2^r}{(2r-1)!!n!}\sum_{j=0}^{n}(-1)^j\binom{n}{[\frac{j}{2}]}([\tfrac{j}{2}]+r-\tfrac{1}{2})_{[\frac{j}{2}]}\\ &\quad\times(n+r-[\tfrac{j}{2}]-\tfrac{1}{2})_{n+r-[\frac{j}{2}]}W_{n-j}(x).\end{aligned} \tag{2.12}$$

This completes the proof of (1.33).

3. Proof of Theorem 1.2

In this section, we are going to show (1.35) and (1.37) of Theorem 1.2, leaving (1.36) and (1.38) as exercises for the reader.

We start with the following fact which is stated in [10] and important for our discussion in this section.

Lemma 3.1. Let n, r be nonnegative integers. Then we have the following identity.

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} L_{i_1}(\tfrac{x}{r+1})L_{i_2}(\tfrac{x}{r+1})\cdots L_{i_{r+1}}(\tfrac{x}{r+1}) = (-1)^r L_{n+r}^{(r)}(x), \tag{3.1}$$

where the sum runs over all nonnegative integers $i_1, i_2, \cdots, i_{r+1}$, with $i_1+i_2+\cdots+i_{r+1}=n$.

From (1.14), the rth derivative of $L_n(x)$ is given by

$$L_n^{(r)}(x) = \sum_{l=0}^{n-r}(-1)^{n-l}\binom{n}{l}\frac{1}{(n-l-r)!}x^{n-l-r}. \tag{3.2}$$

Especially, we have

$$L_{n+r}^{(r+k)}(x) = \sum_{l=0}^{n-k} (-1)^{n+r-l} \binom{n+r}{l} \frac{1}{(n-k-l)!} x^{n-k-l}. \tag{3.3}$$

With $\beta_{n,r}(x)$ as in (1.29), we put

$$\beta_{n,r}(x) = \sum_{k=0}^{n} C_{k,1} T_k(x). \tag{3.4}$$

Then, from (a) of Proposition 2.1, (3.1), (3.3), and integration by parts k times, we have

$$\begin{aligned} C_{k,1} &= \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)!\pi} \int_{-1}^{1} \beta_{n,r}(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx \\ &= \frac{(-1)^k 2^k k! \mathcal{E}_k (-1)^r}{(2k)!\pi} \int_{-1}^{1} L_{n+r}^{(r)}(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx \\ &= \frac{2^k k! \mathcal{E}_k (-1)^r}{(2k)!\pi} \int_{-1}^{1} L_{n+r}^{(r+k)}(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx \\ &= \frac{2^k k! \mathcal{E}_k (-1)^r}{(2k)!\pi} \sum_{l=0}^{n-k} (-1)^{n+r-l} \binom{n+r}{l} \frac{1}{(n-k-l)!} \\ &\quad \times \int_{-1}^{1} (1-x^2)^{k-\frac{1}{2}} x^{n-k-l} dx. \end{aligned} \tag{3.5}$$

From (3.5), (a) of Proposition 2.2, and after some simplifications, we obtain

$$\begin{aligned} C_{k,1} &= \frac{\mathcal{E}_k (-1)^n}{2^n} \sum_{l=0}^{n-k} (-2)^l \binom{n+r}{l} \\ &\quad \times \begin{cases} 0, & \text{if } l \not\equiv n-k \pmod{2}, \\ \frac{1}{(\frac{n+k-l}{2})!(\frac{n-k-l}{2})!}, & \text{if } l \equiv n-k \pmod{2}. \end{cases} \\ &= (-\tfrac{1}{2})^k \mathcal{E}_k \sum_{j=0}^{[\frac{n-k}{2}]} \frac{\binom{n+r}{n-k-2j} (\frac{1}{4})^j}{(k+j)! j!}. \end{aligned} \tag{3.6}$$

Combining (3.4) and (3.6), we finally get

$$\beta_{n,r}(x) = \sum_{k=0}^{n} (-\tfrac{1}{2})^k \mathcal{E}_k \sum_{j=0}^{[\frac{n-k}{2}]} \frac{\binom{n+r}{n-k-2j} (\frac{1}{4})^j}{(k+j)! j!} T_k(x),$$

which is what we wanted.

Next, with $\beta_{n,r}(x)$ as in (1.29), we let

$$\beta_{n,r}(x)=\sum_{k=0}^{n}C_{k,3}V_k(x). \tag{3.7}$$

By (c) of Proposition 2.1, (3.1), (3.3) and integration by parts k times, we have

$$\begin{aligned}C_{k,3}=&\frac{k!2^k(-1)^r}{(2k)!\pi}\sum_{l=0}^{n-k}(-1)^{n+r-l}\binom{n+r}{l}\frac{1}{(n-k-l)!}\\&\times\int_{-1}^{1}x^{n-k-l}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx.\end{aligned} \tag{3.8}$$

From (3.8), (c) of Proposition 2.2, and after some simplifications, we get

$$\begin{aligned}C_{k,3}=&\left(-\frac{1}{2}\right)^n\left\{\sum_{\substack{0\le l\le n-k\\ l\equiv n-k \pmod 2}}\frac{(-2)^l\binom{n+r}{l}}{(\frac{n+k-l}{2})!(\frac{n-k-l}{2})!}\right.\\&\left.+\sum_{\substack{0\le l\le n-k\\ l\not\equiv n-k \pmod 2}}\frac{(-2)^l(n-k-l+1)\binom{n+r}{l}}{2(\frac{n+k-l+1}{2})!(\frac{n-k-l+1}{2})!}\right\}\\=&\left(-\frac{1}{2}\right)^n\left\{\sum_{j=0}^{[\frac{n-k}{2}]}\frac{(-2)^{n-k-2j}\binom{n+r}{n-k-2j}}{(k+j)!j!}\right.\\&\left.+\sum_{j=0}^{[\frac{n-k-1}{2}]}\frac{(-2)^{n-k-2j-1}\binom{n+r}{n-k-2j-1}}{(k+j+1)!j!}\right\}\\=&\left(-\frac{1}{2}\right)^k\sum_{j=0}^{n-k}\frac{\binom{n+r}{n-k-j}(-\frac{1}{2})^j}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!}.\end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9), we finally arrive that

$$\beta_{n,r}(x)=\sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k\sum_{j=0}^{n-k}\frac{\binom{n+r}{n-k-j}(-\frac{1}{2})^j}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!}V_k(x).$$

This completes the proof for (1.37) of Theorem 1.2.

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