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SUMS OF PRODUCTS OF TWO VARIABLE HIGHER-ORDER FUBINI FUNCTIONS ARISING FROM FOURIER SERIES

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Abstract. In this paper, we study three types of sums of products of two variable higher-order Fubini functions and derive their Fourier series expansions. Furthermore, we express each of them in terms of Bernoulli functions from which the corresponding polynomial identities easily follow.

1. Introduction

The two variable Fubini polynomials $F_m^{(r)}(x;y)$ of order r $(r \in \mathbb{Z}_{\geq 0})$ are defined by

$$\frac{e^{xt}}{(1-y(e^t-1))^r} = \sum_{m=0}^{\infty} F_m^{(r)}(x;y)\frac{t^m}{m!}, \text{ (see [5, 6, 8, 9])}. \tag{1.1}$$

Unless otherwise stated, throughout this paper y will be an arbitrary but fixed nonzero real number so that $F_m^{(r)}(x;y)$ are polynomials in x, for each fixed $0 \neq y \in \mathbb{R}$. For $x=0$, $F_m^{(r)}(y) = F_m^{(r)}(0,y)$ are called the Fubini polynomials of order r, and $F_m^{(r)} = F_m^{(r)}(1) = F_m^{(r)}(0;1)$ the Fubini numbers of order r. We can easily show from (1.1) that

$$F_m^{(r)}(y) = \sum_{k=0}^{m}(r+k-1)_k S_2(m,k)y^k, \tag{1.2}$$

where $S_2(m,k)$ are the Stirling numbers of the second kind. Then (1.1) and (1.2) together yield the following expression of $F_m^{(r)}(x;y)$:

$$F_m^{(r)}(x;y) = \sum_{l=0}^{m}\left(\sum_{k=0}^{m-l}\binom{m}{l}(r+k-1)_k S_2(m-l,k)y^k\right)x^l. \tag{1.3}$$

The family of Fubini type numbers and polynomials $F_m(x;y) = F_m^{(1)}(x;y)$ (see also [5,6,8,9]). From (1.1), we immediately see that

$$\frac{d}{dx}F_m^{(r)}(x;y) = mF_{m-1}^{(r)}(x;y), \quad (m \geq 1), \tag{1.4}$$

$$F_m^{(r)}(x+1;y) = \frac{y+1}{y}F_m^{(r)}(x;y) - \frac{1}{y}F_m^{(r-1)}(x;y), \quad (r \geq 1, m \geq 0). \tag{1.5}$$

In turn, (1.4) and (1.5) imply that

$$F_m^{(r)}(1;y) = \frac{y+1}{y}F_m^{(r)}(y) - \frac{1}{y}F_m^{(r-1)}(y), \quad (r \geq 1, m \geq 0), \tag{1.6}$$

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$$\int_0^1 F_m^{(r)}(x;y)dx = \frac{1}{m+1}\left(F_{m+1}^{(r)}(1;y) - F_{m+1}^{(r)}(y)\right) = \frac{1}{(m+1)y}\left(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y)\right). \tag{1.7}$$

For any real number x, the fractional part of x is dented by

$$< x >= x - \lfloor x \rfloor \in [0,1). \tag{1.8}$$

The Bernoulli polynomials $B_m(x)$ are given by

$$\frac{t}{e^t - 1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}, \text{ (see [3])}. \tag{1.9}$$

About the Bernoulli functions $B_m(< x >)$ we will need the following elementary facts:
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \tag{1.10}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.11}$$

Here $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$.

Let r and s be fixed positive integers, and let y and z be fixed nonzero real numbers. Then in this paper, we will consider the following three-types of functions $\alpha_m(< x >; y, z)$, $\beta_m(< x >; y, z)$, and $\gamma_m(< x >; y, z)$ given by sums of products of two variable higher-order Fubini functions. We will find their Fourier series expansions and express each of them in terms of Bernoulli functions from which the corresponding polynomial identities will easily follow.

(1) $\alpha_m(< x >; y, z) = \sum_{k=0}^{m} F_k^{(r)}(< x >; y)F_{m-k}^{(s)}(< x >; z)$, $(m \geq 1)$;
(2) $\beta_m(< x >; y, z) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}F_k^{(r)}(< x >; y)F_{m-k}^{(s)}(< x >; z)$, $(m \geq 1)$;
(3) $\gamma_m(< x >; y, z) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}F_k^{(r)}(< x >; y)F_{m-k}^{(s)}(< x >; z)$, $(m \geq 2)$.

For elementary facts about Fourier series expansions, we let the reader to refer to [10, 11, 13]. It is noteworthy that the next polynomial identity follows immediately from the Fourier expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k (< x >) B_{m-k} (< x >)$ and after some simple modification of that (see [12]):

$$\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} B_{2k}(x) B_{2m-2k}(x) + \frac{2}{2m-1} B_1(x) B_{2m-1}(x) \tag{1.12}$$
$$= \frac{1}{m}\sum_{k=1}^{m} \frac{1}{2k}\binom{2m}{2k} B_{2k}B_{2m-2k}(x) + \frac{1}{m}H_{2m-1}B_{2m}(x) + \frac{2}{2m-1}B_1(x)B_{2m-1}, \quad (m \geq 2),$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers.

We can derive a slightly different version of the well-known Miki's identity (see [2, 14]) by letting $x = 0$ in (1.12):

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k\,(2m-2k)} B_{2k} B_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m} \frac{1}{2k}\binom{2m}{2k} B_{2k} B_{2m-2k} + \frac{1}{m} H_{2m-1} B_{2m}, \quad (m \geq 2)\,.\end{aligned} \tag{1.13}$$

In addition, we can derive the famous Faber-Pandharipande-Zagier identity by setting $x = \frac{1}{2}$ in (1.12) (see [2]):

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k\,(2m-2k)} \overline{B}_{2k} \overline{B}_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m} \frac{1}{2k}\binom{2m}{2k} B_{2k} \overline{B}_{2m-2k} + \frac{1}{m} H_{2m-1} \overline{B}_{2m}, \quad (m \geq 2)\,,\end{aligned} \tag{1.14}$$

where $\overline{B}_m = \left(\frac{1-2^{m-1}}{2^{m-1}}\right) B_m = \left(2^{1-m}-1\right) B_m = B_m\left(\frac{1}{2}\right)$. Some related works can be found in [1,8-10,12].

2. Fourier series of functions of the first type

In this section, we will derive the Fourier series of functions which are given by sums of products of two variable higher-order Fubini functions. Let

$$\alpha_m(x; y, z) = \sum_{k=0}^{m} F_k^{(r)}(x; y) F_{m-k}^{(s)}(x; z), \quad (m \geq 1).$$

Then we will consider the function

$$\alpha_m(< x >; y, z) = \sum_{k=0}^{m} F_k^{(r)}(< x >; y) F_{m-k}^{(s)}(< x >; z), \quad (m \geq 1), \tag{2.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\alpha_m(< x >; y, z)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m,y,z)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned}A_n^{(m)} = A_n^{(m,y,z)} &= \int_0^1 \alpha_m(< x >; y, z) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x; y, z) e^{-2\pi i n x} dx.\end{aligned} \tag{2.3}$$

Next, we need to observe the following.

$$\begin{aligned}\frac{d}{dx}\alpha_m(x;y,z) &= \sum_{k=0}^{m}\left\{kF_{k-1}^{(r)}(x;y)F_{m-k}^{(s)}(x;z)+(m-k)F_k^{(r)}(x;y)F_{m-k-1}^{(s)}(x;z)\right\}\\ &= \sum_{k=1}^{m}kF_{k-1}^{(r)}(x;y)F_{m-k}^{(s)}(x;z)+\sum_{k=0}^{m-1}(m-k)F_k^{(r)}(x;y)F_{m-k-1}^{(s)}(x;z)\\ &= \sum_{k=0}^{m-1}(k+1)F_k^{(r)}(x;y)F_{m-k-1}^{(s)}(x;z)+\sum_{k=0}^{m-1}(m-k)F_k^{(r)}(x;y)F_{m-k-1}^{(s)}(x;z)\\ &= (m+1)\sum_{k=0}^{m-1}F_k^{(r)}(x;y)F_{m-k-1}^{(s)}(x;z)\\ &= (m+1)\alpha_{m-1}(x;y,z).\end{aligned} \tag{2.4}$$

From (2.2) , we obtain

$$\frac{d}{dx}\left(\frac{\alpha_{m+1}(x;y,z)}{m+2}\right)=\alpha_m(x;y,z), \tag{2.5}$$

and

$$\int_0^1\alpha_m(x;y,z)dx=\frac{1}{m+2}(\alpha_{m+1}(1;y,z)-\alpha_{m+1}(0;y,z)). \tag{2.6}$$

For $m\geq 1$, we set

$$\begin{aligned}&\Delta_m(y,z)=\alpha_m(1;y,z)-\alpha_m(0;y,z)\\ &=\sum_{k=0}^{m}(F_k^{(r)}(1;y)F_{m-k}^{(s)}(1;z)-F_k^{(r)}(y)F_{m-k}^{(s)}(z))\\ &=\sum_{k=0}^{m}\left(\left(\frac{y+1}{y}F_k^{(r)}(y)-\frac{1}{y}F_k^{(r-1)}(y)\right)\left(\frac{z+1}{z}F_{m-k}^{(s)}(z)-\frac{1}{z}F_{m-k}^{(s-1)}(z)\right)-F_k^{(r)}(y)F_{m-k}^{(s)}(z)\right)\\ &=\frac{1}{yz}\sum_{k=0}^{m}\Big((y+z+1)F_k^{(r)}(y)F_{m-k}^{(s)}(z)+F_k^{(r-1)}(y)F_{m-k}^{(s-1)}(z)\\ &\qquad -(y+1)F_k^{(r)}(y)F_{m-k}^{(s-1)}(z)-(z+1)F_k^{(r-1)}(y)F_{m-k}^{(s)}(z)\Big).\end{aligned} \tag{2.7}$$

Then we see that

$$\alpha_m(0;y,z)=\alpha_m(1;y,z)\iff\Delta_m(y,z)=0, \tag{2.8}$$

and

$$\int_0^1\alpha_m(x;y,z)dx=\frac{1}{m+2}\Delta_{m+1}(y,z). \tag{2.9}$$

We now would like to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$
\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x;y,z)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left[\alpha_m(x;y,z)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \frac{d}{dx}(\alpha_m(x;y,z))e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\alpha_m(1;y,z) - \alpha_m(0;y,z)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x;y,z)e^{-2\pi inx}dx \\
&= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m(y,z).
\end{aligned}
\tag{2.10}
$$

Thus we have shown that

$$
A_n^{(m)} = \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m(y,z), \quad (n \neq 0). \tag{2.11}
$$

An easy induction on m of the recurrence relation (2.11) gives the following expression.

$$
A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}(y,z). \tag{2.12}
$$

Case 2: $n = 0$.

$$
A_0^{(m)} = \int_0^1 \alpha_m(x;y,z)dx = \frac{1}{m+2}\Delta_{m+1}(y,z). \tag{2.13}
$$

$\alpha_m(< x >;y,z), (m \geq 1)$ is piecewise C^∞. Further, $\alpha_m(< x >;y,z)$ is continuous for those positive integers m with $\Delta_m(y,z) = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m(y,z) \neq 0$. Assume first that $\Delta_m(y,z) = 0$, for a positive integer m. Then $\alpha_m(0;y,z) = \alpha_m(1;y,z)$. Hence $\alpha_m(< x >;y,z)$ is piecewise C^∞, and continuous. It follows that the Fourier series of $\alpha_m(< x >;y,z)$ converges uniformly to $\alpha_m(< x >;y,z)$, and

$$
\begin{aligned}
&\alpha_m(< x >;y,z) \\
&= \frac{1}{m+2}\Delta_{m+1}(y,z) + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}(y,z)\right)e^{2\pi inx} \\
&= \frac{1}{m+2}\Delta_{m+1}(y,z) + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y,z)\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\
&= \frac{1}{m+2}\Delta_{m+1}(y,z) + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y,z)B_j(< x >) \\
&\quad + \Delta_m(y,z) \times \begin{cases} B_1(< x >), & \text{for } \ x \in \mathbb{Z}^c, \\ 0, & \text{for } \ x \in \mathbb{Z}. \end{cases}
\end{aligned}
\tag{2.14}
$$

We are now ready to state our first result.

Theorem 2.1. *For each positive integer l, we let*

$$
\begin{aligned}
\Delta_l(y,z) = \frac{1}{yz}\sum_{k=0}^{l}\Big((y+z+1)F_k^{(r)}(y)F_{l-k}^{(s)}(z) + F_k^{(r-1)}(y)F_{l-k}^{(s-1)}(z) \\
-(y+1)F_k^{(r)}(y)F_{l-k}^{(s-1)}(z) - (z+1)F_k^{(r-1)}(y)F_{l-k}^{(s)}(z)\Big).
\end{aligned}
\tag{2.15}
$$

Assume that $\Delta_m(y,z)=0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m} F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z)\\&=\frac{1}{m+2}\Delta_{m+1}(y,z)+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}(y,z)\right)e^{2\pi inx\pi},\end{aligned}\tag{2.16}$$

for all $x\in\mathbb{R}$. Here the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=0}^{m} F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z)\\&=\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y,z)B_j(<x>),\end{aligned}\tag{2.17}$$

for all $x\in\mathbb{R}$.

Assume next that $\Delta_m(y,z)\neq 0$, for a positive integer m. Then $\alpha_m(0;y,z)\neq\alpha_m(1;y,z)$. Thus $\alpha_m(<x>:y,z)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. It follows that the Fourier series of $\alpha_m(<x>;y,z)$ converges pointwise to $\alpha_m(<x>;y,z)$, for $x\in\mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0;y,z)+\alpha_m(1;y,z))=\alpha_m(0;y,z)+\frac{1}{2}\Delta_m(y,z),\tag{2.18}$$

for $x\in\mathbb{Z}$.

We are now going to state our second result.

Theorem 2.2. *For each positive integer l, we let*

$$\begin{aligned}\Delta_l(y,z)=\frac{1}{yz}\sum_{k=0}^{l}&\Big((y+z+1)F_k^{(r)}(y)F_{l-k}^{(s)}(z)+F_k^{(r-1)}(y)F_{l-k}^{(s-1)}(z)\\&-(y+1)F_k^{(r)}(y)F_{l-k}^{(s-1)}(z)-(z+1)F_k^{(r-1)}(y)F_{l-k}^{(s)}(z)\Big).\end{aligned}\tag{2.19}$$

Assume that $\Delta_m(y,z)\neq 0$, for a positive integer m. Then the following holds.

(a)

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}(y,z)+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}(y,z)\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z), & \text{for}\quad x\in\mathbb{Z}^c,\\ \sum_{k=0}^{m}F_k^{(r)}(y)F_{m-k}^{(s)}(z)+\frac{1}{2}\Delta_m(y,z), & \text{for}\quad x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{2.20}$$

(b)

$$\begin{aligned}&\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y,z)B_j(<x>)\\&=\sum_{k=0}^{m}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z),\quad\text{for } x\in\mathbb{Z}^c;\end{aligned}\tag{2.21}$$

$$\begin{aligned}
&\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\Delta_{m-j+1}(y,z)B_j(<x>) \\
&=\sum_{k=0}^{m}F_k^{(r)}(y)F_{m-k}^{(s)}(z)+\frac{1}{2}\Delta_m(y,z),\ x\in\mathbb{Z}.
\end{aligned} \tag{2.22}$$

3. Fourier series of functions of the second type

Let $\beta_m(x;y,z)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(x;y)F_{m-k}^{(s)}(x;z)$, $(m\geq 1)$. Then we will consider the function

$$\beta_m(<x>;y,z)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z),\ \ (m\geq 1), \tag{3.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\beta_m(<x>;y,z)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m,y,z)}e^{2\pi inx}, \tag{3.2}$$

where

$$\begin{aligned}
B_n^{(m)}=B_n^{(m,y,z)}&=\int_0^1\beta_m(<x>;y,z)e^{-2\pi inx}dx \\
&=\int_0^1\beta_m(x;y,z)e^{-2\pi inx}dx.
\end{aligned} \tag{3.3}$$

We now need to observe the following.

$$\begin{aligned}
&\frac{d}{dx}\beta_m(x;y,z) \\
&=\sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}F_{k-1}^{(r)}(x;y)F_{m-k}^{(s)}(x;z)+\frac{m-k}{k!(m-k)!}F_k^{(r)}(x;y)F_{m-k-1}^{(s)}(x;z)\right\} \\
&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}F_{k-1}^{(r)}(x;y)F_{m-k}^{(s)}(x;z)+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}F_k^{(r)}(x;y)F_{m-k-1}^{(s)}(x;z) \\
&=\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}F_k^{(r)}(x;y)F_{m-1-k}^{(s)}(x;z)+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}F_k^{(r)}(x;y)F_{m-1-k}^{(s)}(x;z) \\
&=2\beta_{m-1}(x;y,z).
\end{aligned} \tag{3.4}$$

These imply that

$$\frac{d}{dx}\left(\frac{1}{2}\beta_{m+1}(x;y,z)\right)=\beta_m(x;y,z), \tag{3.5}$$

and

$$\int_0^1\beta_m(x;y,z)dx=\frac{1}{2}(\beta_{m+1}(1;y,z)-\beta_{m+1}(0;y,z)). \tag{3.6}$$

For $m \geq 1$, we set

$$\begin{aligned}
&\Omega_m(y,z) = \beta_m(1;y,z) - \beta_m(0;y,z) \\
&= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}(F_k^{(r)}(1;y)F_{m-k}^{(s)}(1;z) - F_k^{(r)}(y)F_{m-k}^{(s)}(z)) \\
&= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\left(\left(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)\right)\left(\frac{z+1}{z}F_{m-k}^{(s)}(z) - \frac{1}{z}F_{m-k}^{(s-1)}(z)\right) - F_k^{(r)}(y)F_{m-k}^{(s)}(z)\right) \\
&= \frac{1}{yz}\sum_{k=0}^{m} \frac{1}{k!(m-k)!}\Big((y+z+1)F_k^{(r)}(y)F_{m-k}^{(s)}(z) + F_k^{(r-1)}(y)F_{m-k}^{(s-1)}(z) \\
&\quad -(y+1)F_k^{(r)}(y)F_{m-k}^{(s-1)}(z) - (z+1)F_k^{(r-1)}(y)F_{m-k}^{(s)}(z)\Big).
\end{aligned} \tag{3.7}$$

Here we note that

$$\beta_m(0;y,z) = \beta_m(1;y,z) \Longleftrightarrow \Omega_m(y,z) = 0, \tag{3.8}$$

and

$$\int_0^1 \beta_m(x;y,z)dx = \frac{1}{2}\Omega_{m+1}(y,z). \tag{3.9}$$

Now, we want to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x;y,z)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left[\beta_m(x;y,z)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \left(\frac{d}{dx}\beta_m(x;y,z)\right)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\beta_m(1;y,z) - \beta_m(0;y,z)) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x;y,z)e^{-2\pi inx}dx \\
&= \frac{2}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m(y,z).
\end{aligned} \tag{3.10}$$

Thus we have shown that

$$B_n^{(m)} = \frac{2}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m(y,z),\ (n \neq 0). \tag{3.11}$$

From (3.11) we obtain the following expression by induction on m.

$$B_n^{(m)} = -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}(y,z). \tag{3.12}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x;y,z)dx = \frac{1}{2}\Omega_{m+1}(y,z). \tag{3.13}$$

$\beta_m(< x >;y,z), (m \geq 1)$ is piecewise C^∞. In addition, $\beta_m(< x >;x,y)$ is continuous for those positive integers m with $\Omega_m(y,z) = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m(y,z) \neq 0$.

Assume first that $\Omega_m(y,z)=0$, for a positive integer m. Then $\beta_m(0;y,z)=\beta_m(1;y,z)$. Hence $\beta_m(<x>;y,z)$ is piecewise C^∞, and continuous. The Fourier series of $\beta_m(<x>;y,z)$ converges uniformly to $\beta_m(<x>;y,z)$, and

$$\begin{aligned}\beta_m(<x>;y,z)&=\frac{1}{2}\Omega_{m+1}(y,z)+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}(y,z)\right)e^{2\pi inx}\\&=\frac{1}{2}\Omega_{m+1}(y,z)+\sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y,z)\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\&=\frac{1}{2}\Omega_{m+1}(y,z)+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y,z)B_j(<x>)+\Omega_m(y,z)\times\begin{cases}B_1(<x>), & \text{for } x\in\mathbb{Z}^c,\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{3.14}$$

We are now ready to state our first result.

Theorem 3.1. *For each positive integer l, we let*

$$\begin{aligned}\Omega_l(y,z)=\frac{1}{yz}\sum_{k=0}^{l}\frac{1}{k!(l-k)!}&\Big((y+z+1)F_k^{(r)}(y)F_{l-k}^{(s)}(z)+F_k^{(r-1)}(y)F_{l-k}^{(s-1)}(z)\\&-(y+1)F_k^{(r)}(y)F_{l-k}^{(s-1)}(z)-(z+1)F_k^{(r-1)}(y)F_{l-k}^{(s)}(z)\Big).\end{aligned}\tag{3.15}$$

Assume that $\Omega_m(y,z)=0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z)\\&=\frac{1}{2}\Omega_{m+1}(y,z)+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}(y,z)\right)e^{2\pi inx},\end{aligned}\tag{3.16}$$

for all $x\in\mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z)\\&=\sum_{j=0,j\neq 1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y,z)B_j(<x>),\end{aligned}\tag{3.17}$$

for all $x\in\mathbb{R}$.

Assume next that $\Omega_m(y,z)\neq 0$, for a positive integer m. Then $\beta_m(0;y,z)\neq\beta_m(1;y,z)$. Hence $\beta_m(<x>;y,z)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. It follows that the Fourier series of $\beta_m(<x>;y,z)$ converges pointwise to $\beta_m(<x>;y,z)$, for $x\in\mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0;y,z)+\beta_m(1;y,z))=\beta_m(0;y,z)+\frac{1}{2}\Omega_m(y,z),\tag{3.18}$$

for $x\in\mathbb{Z}$. Now, we are ready to state our second result.

Theorem 3.2. *For each positive integer l, we let*

$$\Omega_l(y,z)=\frac{1}{yz}\sum_{k=0}^{l}\frac{1}{k!(l-k)!}\Big((y+z+1)F_k^{(r)}(y)F_{l-k}^{(s)}(z)+F_k^{(r-1)}(y)F_{l-k}^{(s-1)}(z)$$
$$-(y+1)F_k^{(r)}(y)F_{l-k}^{(s-1)}(z)-(z+1)F_k^{(r-1)}(y)F_{l-k}^{(s)}(z)\Big). \tag{3.19}$$

Assume that $\Omega_m(y,z)\neq 0$, for a positive integer m. Then we have the following.
(a)

$$\frac{1}{2}\Omega_{m+1}(y,z)+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}(y,z)\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z), & \text{for } x\in\mathbb{Z}^c,\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(y)F_{m-k}^{(s)}(z)+\frac{1}{2}\Omega_m(y,z), & \text{for } x\in\mathbb{Z}.\end{cases} \tag{3.20}$$

(b)

$$\sum_{j=0}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y,z)B_j(<x>)$$
$$=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z), \tag{3.21}$$

for $x\in\mathbb{Z}^c$;

$$\sum_{j=0,j\neq 1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}(y,z)B_j(<x>)$$
$$=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}F_k^{(r)}(y)F_{m-k}^{(s)}(z)+\frac{1}{2}\Omega_m(y,z), \tag{3.22}$$

for $x\in\mathbb{Z}$.

4. Fourier series of functions of the third type

Let $\gamma_m(x;y,z)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(x;y)F_{m-k}^{(s)}(x;z)$, $(m\geq 2)$. Then we will consider the function

$$\gamma_m(<x>;y,z)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z),\ (m\geq 2), \tag{4.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\gamma_m(<x>;y,z)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m,y,z)}e^{2\pi inx}, \tag{4.2}$$

where

$$C_n^{(m)} = C_n^{(m,y,z)} = \int_0^1 \gamma_m(< x >; y, z) e^{-2\pi inx} dx \\ = \int_0^1 \gamma_m(x; y, z) e^{-2\pi inx} dx. \tag{4.3}$$

We now need to observe the following.

$$\begin{aligned} &\frac{d}{dx}\gamma_m(x; y, z) \\ &= \sum_{k=1}^{m-1} \frac{1}{m-k} F_{k-1}^{(r)}(x; y) F_{m-k}^{(s)}(x; y) + \sum_{k=1}^{m-1} \frac{1}{k} F_k^{(r)}(x; y) F_{m-k-1}^{(s)}(x; z) \\ &= \sum_{k=0}^{m-2} \frac{1}{m-1-k} F_k^{(r)}(x; y) F_{m-1-k}^{(s)}(x; z) + \sum_{k=1}^{m-1} \frac{1}{k} F_k^{(r)}(x; y) F_{m-1-k}^{(s)}(x; z) \\ &= \sum_{k=1}^{m-2} \left(\frac{1}{m-1-k} + \frac{1}{k} \right) F_k^{(r)}(x; y) F_{m-1-k}^{(s)}(x; z) + \frac{1}{m-1} (F_{m-1}^{(r)}(x; y) + F_{m-1}^{(s)}(x; z)) \\ &= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} F_k^{(r)}(x; y) F_{m-1-k}^{(s)}(x; z) + + \frac{1}{m-1} \left(F_{m-1}^{(r)}(x; y) + F_{m-1}^{(s)}(x; z) \right) \\ &= (m-1)\gamma_{m-1}(x; y, z) + \frac{1}{m-1} (F_{m-1}^{(r)}(x; y) + F_{m-1}^{(s)}(x; z)). \end{aligned} \tag{4.4}$$

From (4.4), we immediately see that

$$\frac{d}{dx}\frac{1}{m} \left(\gamma_{m+1}(x; y, z) - \frac{1}{m(m+1)} F_{m+1}^{(r)}(x; y) - \frac{1}{m(m+1)} F_{m+1}^{(s)}(x; z) \right) = \gamma_m(x; y, z), \tag{4.5}$$

and

$$\begin{aligned} &\int_0^1 \gamma_m(x; y, z) dx \\ &= \frac{1}{m} \left[\gamma_{m+1}(x; y, z) - \frac{1}{m(m+1)} F_{m+1}^{(r)}(x; y) - \frac{1}{m(m+1)} F_{m+1}^{(s)}(x; z) \right]_0^1 \\ &= \frac{1}{m} \Big(\gamma_{m+1}(1; y, z) - \gamma_{m+1}(0; y, z) - \frac{1}{m(m+1)} (F_{m+1}^{(r)}(1; y) - F_{m+1}^{(r)}(y)) - \frac{1}{m(m+1)} (F_{m+1}^{(s)}(1; z) - F_{m+1}^{(s)}(z)) \Big) \\ &= \frac{1}{m} \Big(\gamma_{m+1}(1; y, z) - \gamma_{m+1}(0; y, z) - \frac{1}{m(m+1)y} (F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y)) - \frac{1}{m(m+1)z} (F_{m+1}^{(s)}(z) - F_{m+1}^{(s-1)}(z)) \Big). \end{aligned} \tag{4.6}$$

For $m \geq 2$, we let

$$
\begin{aligned}
&\Lambda_m(y,z) = \gamma_m(1;y,z) - \gamma_m(0;y,z) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (F_k^{(r)}(1;y)F_{m-k}^{(s)}(1;z) - F_k^{(r)}(y)F_{m-k}^{(s)}(z)) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left(\Big(\frac{y+1}{y}F_k^{(r)}(y) - \frac{1}{y}F_k^{(r-1)}(y)\Big)\Big(\frac{z+1}{z}F_{m-k}^{(s)}(z) - \frac{1}{z}F_{m-k}^{(s-1)}(z)\Big) - F_k^{(r)}(y)F_{m-k}^{(s)}(z) \right) \\
&= \frac{1}{yz}\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big((y+z+1)F_k^{(r)}(y)F_{m-k}^{(s)}(z) + F_k^{(r-1)}(y)F_{m-k}^{(s-1)}(z) \\
&\qquad - (y+1)F_k^{(r)}(y)F_{m-k}^{(s-1)}(z) - (z+1)F_k^{(r-1)}(y)F_{m-k}^{(s)}(z) \Big).
\end{aligned} \tag{4.7}
$$

For convenience, we also let $\Lambda_1(y,z) = 0$. We note here that

$$
\gamma_m(0;y,z) = \gamma_m(1;y,z) \Longleftrightarrow \Lambda_m(y,z) = 0, (m \geq 2), \tag{4.8}
$$

and

$$
\begin{aligned}
&\int_0^1 \gamma_m(x;y,z)dx \\
&= \frac{1}{m}\Big(\Lambda_{m+1}(y,z) - \frac{1}{m(m+1)}\left(F_{m+1}^{(r)}(1;y) - F_{m+1}^{(r)}(y)\right) - \frac{1}{m(m+1)}\left(F_{m+1}^{(s)}(1;z) - F_{m+1}^{(s)}(z)\right) \\
&= \frac{1}{m}\Big(\Lambda_{m+1}(y,z) - \frac{1}{m(m+1)y}\left(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y)\right) - \frac{1}{m(m+1)z}\left(F_{m+1}^{(s)}(z) - F_{m+1}^{(s-1)}(z)\right)\Big).
\end{aligned} \tag{4.9}
$$

We now would like to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$.

$$
\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x;y,z)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}[\gamma_m(x;y,z)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \frac{d}{dx}\left(\gamma_m(x;y,z)\right)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1;y,z) - \gamma_m(0;y,z)) \\
&\quad + \frac{1}{2\pi in}\int_0^1 \left\{(m-1)\gamma_{m-1}(x;y,z) + \frac{1}{m-1}\left(F_{m-1}^{(r)}(x;y) + F_{m-1}^{(s)}(x;z)\right)\right\}e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m(y,z) + \frac{1}{2\pi in(m-1)}\int_0^1 F_{m-1}^{(r)}(x;y)e^{-2\pi inx}dx \\
&\quad + \frac{1}{2\pi in(m-1)}\int_0^1 F_{m-1}^{(s)}(x;y)e^{-2\pi inx}dx.
\end{aligned} \tag{4.10}
$$

In a previous paper, we showed that

$$
\begin{aligned}
&\int_0^1 F_m^{(r)}(x;y)e^{-2\pi inx}dx \\
&= -\sum_{k=1}^{m}\frac{(m)_{k-1}}{(2\pi in)^k}\left(F_{m-k+1}^{(r)}(1;y)-F_{m-k+1}^{(r)}(y)\right) \\
&= -\frac{1}{y}\sum_{k=1}^{m}\frac{(m)_{k-1}}{(2\pi in)^k}\left(F_{m-k+1}^{(r)}(y)-F_{m-k+1}^{(r-1)}(y)\right).
\end{aligned} \tag{4.11}
$$

From (4.10) and (4.11), we obtain

$$
C_n^{(m)} = \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m(y,z) - \frac{1}{2\pi in(m-1)}(\Phi_n^{(m,r)}(y)+\Phi_n^{(m,s)}(z)), \tag{4.12}
$$

where

$$
\Phi_n^{(m,r)}(y) = \sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}\left(F_{m-k}^{(r)}(1;y)-F_{m-k}^{(r)}(y)\right). \tag{4.13}
$$

By induction on m applied to (4.12) yields the following expression.

$$
C_n^{(m)} = -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1}(y,z) - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}(\Phi_n^{(m-j+1,r)}(y)+\Phi_n^{(m-j+1,s)}(z)). \tag{4.14}
$$

In order to find an explicit expression for $C_n^{(m)}$, we note that

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_n^{(m-j+1,r)}(y) \\
&= \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^k}\left(F_{m-j-k+1}^{(r)}(1;y)-F_{m-j-k+1}^{(r)}(y)\right) \\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}}\left(F_{m-j-k+1}^{(r)}(1;y)-F_{m-j-k+1}^{(r)}(y)\right) \\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{k=j+1}^{m}\frac{(m-1)_{k-2}}{(2\pi in)^k}\left(F_{m-k+1}^{(r)}(1;y)-F_{m-k+1}^{(r)}(y)\right) \\
&= \sum_{j=2}^{m}\frac{(m-1)_{k-2}}{(2\pi in)^k}\left(F_{m-k+1}^{(r)}(1;y)-F_{m-k+1}^{(r)}(y)\right)\sum_{j=1}^{k-1}\frac{1}{m-j} \\
&= \sum_{j=1}^{m}\frac{(m-1)_{k-2}}{(2\pi in)^k}\left(F_{m-k+1}^{(r)}(1;y)-F_{m-k+1}^{(r)}(y)\right)(H_{m-1}-H_{m-k}) \\
&= \frac{1}{m}\sum_{j=1}^{m}\frac{(m)_k}{(2\pi in)^k}\frac{F_{m-k+1}^{(r)}(1;y)-F_{m-k+1}^{(r)}(y)}{m-k+1}(H_{m-1}-H_{m-k}),
\end{aligned} \tag{4.15}
$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ is the harmonic number, for $m \geq 1$, and $H_0 = 0$. Recalling that $\Lambda_1(y,z) = 0$ by convention and from (4.14) and (4.15), the following expression of $C_n^{(m)} (n \neq 0)$ can be obtained.

$$
\begin{aligned}
C_n^{(m)} &= -\frac{1}{m}\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}\{\Lambda_{m-k+1}(y,z)\\
&\quad+\frac{H_{m-1}-H_{m-k}}{m-k+1}\left(F_{m-k+1}^{(r)}(1;y)-F_{m-k+1}^{(r)}(y)+F_{m-k+1}^{(s)}(1;z)-F_{m-k+1}^{(s)}(z)\right)\Big\}\\
&= -\frac{1}{m}\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}\left\{\Lambda_{m-k+1}(y,z)+\frac{H_{m-1}-H_{m-k}}{(m-k+1)y}\left(F_{m-k+1}^{(r)}(y)-F_{m-k+1}^{(r-1)}(y)\right)\right.\\
&\quad\left.+\frac{H_{m-1}-H_{m-k}}{(m-k+1)z}\left(F_{m-k+1}^{(s)}(z)-F_{m-k+1}^{(s-1)}(z)\right)\right\}.
\end{aligned}
\tag{4.16}
$$

Case 2: $n = 0$.

$$
\begin{aligned}
C_0^{(m)} &= \int_0^1 \gamma_m(x;y,z)dx\\
&= \frac{1}{m}\Big(\Lambda_{m+1}(y,z)-\frac{1}{m(m+1)}\left(F_{m+1}^{(r)}(1;y)-F_{m+1}^{(r)}(y)\right)-\frac{1}{m(m+1)}\left(F_{m+1}^{(s)}(1;z)-F_{m+1}^{(s)}(z)\right)\\
&= \frac{1}{m}\Big(\Lambda_{m+1}(y,z)-\frac{1}{m(m+1)y}\left(F_{m+1}^{(r)}(y)-F_{m+1}^{(r-1)}(y)\right)-\frac{1}{m(m+1)z}\left(F_{m+1}^{(s)}(z)-F_{m+1}^{(s-1)}(z)\right)\Big).
\end{aligned}
\tag{4.17}
$$

$\gamma_m(<x>;y,z),(m \geq 2)$ is piecewise C^∞. Further, $\gamma_m(<x>;y,z)$ is continuous for those integers $m \geq 2$ with $\Lambda_m(y,z) = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m(y,z) \neq 0$.

Assume first that $\Lambda_m(y,z) = 0$, for some integer $m \geq 2$. Then $\gamma_m(0;y,z) = \gamma_m(1;y,z)$. Hence $\gamma_m(<x>;y,z)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(<x>;y,z)$ converges uniformly to $\gamma_m(<x>;y,z)$, and

$$
\begin{aligned}
&\gamma_m(<x>;y,z)\\
&= \frac{1}{m}\left\{\Lambda_{m+1}(y,z)-\frac{1}{m(m+1)y}\left(F_{m+1}^{(r)}(y)-F_{m+1}^{(r-1)}(y)\right)-\frac{1}{m(m+1)z}\left(F_{m+1}^{(s)}(z)-F_{m+1}^{(s-1)}(z)\right)\right\}\\
&\quad-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}\Big(\Lambda_{m-k+1}(y,z)+\frac{H_{m-1}-H_{m-k}}{(m-k+1)y}(F_{m-k+1}^{(r)}(y)-F_{m-k+1}^{(r-1)}(y))\right.\\
&\quad\left.+\frac{H_{m-1}-H_{m-k}}{(m-k+1)z}(F_{m-k+1}^{(s)}(z)-F_{m-k+1}^{(s-1)}(z))\Big)\right\}e^{2\pi inx}\\
&= \frac{1}{m}\left\{\Lambda_{m+1}(y,z)-\frac{1}{m(m+1)y}\left(F_{m+1}^{(r)}(y)-F_{m+1}^{(r-1)}(y)\right)-\frac{1}{m(m+1)z}\left(F_{m+1}^{(s)}(z)-F_{m+1}^{(s-1)}(z)\right)\right\}\\
&\quad+\frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}\left\{\Lambda_{m-k+1}(y,z)+\frac{H_{m-1}-H_{m-k}}{(m-k+1)y}\left(F_{m-k+1}^{(r)}(y)-F_{m-k+1}^{(r-1)}(y)\right)\right.\\
&\quad\left.+\frac{H_{m-1}-H_{m-k}}{(m-k+1)z}\left(F_{m-k+1}^{(s)}(z)-F_{m-k+1}^{(s-1)}(z)\right)\right\}\left(-k!\sum_{k=-\infty,k\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}\right)
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&= \frac{1}{m} \sum_{k=0, k\neq 1}^{m} \binom{m}{k} \Bigg\{ \Lambda_{m-k+1}(y,z) + \frac{H_{m-1}-H_{m-k}}{(m-k+1)y} \left(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y) \right) \\
&\qquad + \frac{H_{m-1}-H_{m-k}}{(m-k+1)z} \left(F_{m-k+1}^{(s)}(z) - F_{m-k+1}^{(s-1)}(z) \right) \Bigg\} B_k(<x>) \\
&\qquad + \Lambda_m(y,z) \times \begin{cases} B_1(<x>), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.19}
$$

We are now going to state our first result.

Theorem 4.1. *For each integer $l \geq 2$, we let*

$$
\begin{aligned}
\Lambda_l(y,z) = \frac{1}{yz} \sum_{k=1}^{l-1} \frac{1}{k(l-k)} \Big(&(y+z+1) F_k^{(r)}(y) F_{l-k}^{(s)}(z) + F_k^{(r-1)}(y) F_{l-k}^{(s-1)}(z) \\
&-(y+1) F_k^{(r)}(y) F_{l-k}^{(s-1)}(z) - (z+1) F_k^{(r-1)}(y) F_{l-k}^{(s)}(z) \Big),
\end{aligned} \tag{4.20}
$$

with $\Lambda_1(y,z)=0$. Assume that $\Lambda_m(y,z)=0$, for some integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(<x>;y) F_{m-k}^{(s)}(<x>;z)$ has the Fourier series expansion

$$
\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(<x>;y) F_{m-k}^{(s)}(<x>;z) \\
&= \frac{1}{m} \left\{ \Lambda_{m+1}(y,z) - \frac{1}{m(m+1)y} \left(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y) \right) - \frac{1}{m(m+1)z} \left(F_{m+1}^{(s)}(z) - F_{m+1}^{(s-1)}(z) \right) \right\} \\
&\quad - \frac{1}{m} \sum_{n=-\infty, n\neq 0}^{\infty} \Bigg\{ \sum_{k=1}^{m} \frac{(m)_k}{(2\pi i n)^k} \Big(\Lambda_{m-k+1}(y,z) + \frac{H_{m-1}-H_{m-k}}{(m-k+1)y} \big(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y) \big) \\
&\quad + \frac{H_{m-1}-H_{m-k}}{(m-k+1)z} \big(F_{m-k+1}^{(s)}(z) - F_{m-k+1}^{(s-1)}(z) \big) \Big) \Bigg\} e^{2\pi i n x},
\end{aligned} \tag{4.21}
$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$
\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} F_k^{(r)}(<x>;y) F_{m-k}^{(s)}(<x>;z) \\
&= \frac{1}{m} \sum_{k=0, k\neq 1}^{m} \binom{m}{k} \Bigg\{ \Lambda_{m-k+1}(y,z) + \frac{H_{m-1}-H_{m-k}}{(m-k+1)y} \left(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y) \right) \\
&\quad + \frac{H_{m-1}-H_{m-k}}{(m-k+1)z} \left(F_{m-k+1}^{(s)}(z) - F_{m-k+1}^{(s-1)}(z) \right) \Bigg\} B_k(<x>),
\end{aligned} \tag{4.22}
$$

for all $x \in \mathbb{R}$.

Assume next that $\Lambda_m(y,z) \neq 0$, for some integer $m \geq 2$. Then $\gamma_m(0;y,z) \neq \gamma_m(1;y,z)$. Hence $\gamma_m(<x>;y,z)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. It follows that the Fourier series of $\gamma_m(<x>;y,z)$ converges pointwise to $\gamma_m(<x>;y,z)$, for $x \in \mathbb{Z}^c$, and that it converges to

$$
\frac{1}{2}(\gamma_m(0;y,z) + \gamma_m(1;y,z)) = \gamma_m(0;y,z) + \frac{1}{2}\Lambda_m(y,z), \tag{4.23}
$$

for $x \in \mathbb{Z}$. Now, we are going to state our second result.

Theorem 4.2. *For each integer $l \geq 2$, we let*

$$\begin{aligned}\Lambda_l(y,z) = \frac{1}{yz}\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\Big((y+z+1)F_k^{(r)}(y)F_{l-k}^{(s)}(z) + F_k^{(r-1)}(y)F_{l-k}^{(s-1)}(z)\\ -(y+1)F_k^{(r)}(y)F_{l-k}^{(s-1)}(z) - (z+1)F_k^{(r-1)}(y)F_{l-k}^{(s)}(z)\Big),\end{aligned} \tag{4.24}$$

with $\Lambda_1(y,z) = 0$. Assume that $\Lambda_m(y,z) \neq 0$, for an integer $m \geq 2$. Then we have the following.
(a)

$$\begin{aligned}&\frac{1}{m}\left\{\Lambda_{m+1}(y,z) - \frac{1}{m(m+1)y}\Big(F_{m+1}^{(r)}(y) - F_{m+1}^{(r-1)}(y)\Big) - \frac{1}{m(m+1)z}\Big(F_{m+1}^{(s)}(z) - F_{m+1}^{(s-1)}(z)\Big)\right\}\\ &-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\left(\Lambda_{m-k+1}(y,z) + \frac{H_{m-1}-H_{m-k}}{(m-k+1)y}\Big(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y)\Big)\right.\right.\\ &\left.\left.+\frac{H_{m-1}-H_{m-k}}{(m-k+1)z}\Big(F_{m-k+1}^{(s)}(z) - F_{m-k+1}^{(s-1)}(z)\Big)\right)\right\}e^{2\pi i n x},\\ &=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z), & \text{for } x \in \mathbb{Z}^c,\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(y)F_{m-k}^{(s)}(z) + \frac{1}{2}\Lambda_m(y,z), & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{4.25}$$

(b)

$$\begin{aligned}&\frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\left(\Lambda_{m-k+1}(y,z) + \frac{H_{m-1}-H_{m-k}}{(m-k+1)y}\Big(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y)\Big)\right.\\ &\quad\left.+\frac{H_{m-1}-H_{m-k}}{(m-k+1)z}\Big(F_{m-k+1}^{(s)}(z) - F_{m-k+1}^{(s-1)}(z)\Big)\right\}B_k(<x>)\\ &=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(<x>;y)F_{m-k}^{(s)}(<x>;z), \quad \text{for } x \in \mathbb{Z}^c;\end{aligned} \tag{4.26}$$

$$\begin{aligned}&\frac{1}{m}\sum_{k=0,k\neq 1}^{m}\binom{m}{k}\left\{\Lambda_{m-k+1}(y,z) + \frac{H_{m-1}-H_{m-k}}{(m-k+1)y}\Big(F_{m-k+1}^{(r)}(y) - F_{m-k+1}^{(r-1)}(y)\Big)\right.\\ &\quad\left.+\frac{H_{m-1}-H_{m-k}}{(m-k+1)z}\Big(F_{m-k+1}^{(s)}(z) - F_{m-k+1}^{(s-1)}(z)\Big)\right\}B_k(<x>)\\ &=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(y)F_{m-k}^{(s)}(z) + \frac{1}{2}\Lambda_m(y,z), \quad \text{for } x \in \mathbb{Z}.\end{aligned} \tag{4.27}$$

Corollary 4.3. *For each integer $l \geq 2$, we let*

$$\begin{aligned}\Lambda_l(y,z) = \frac{1}{yz}\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\Big((y+z+1)F_k^{(r)}(y)F_{l-k}^{(s)}(z) + F_k^{(r-1)}(y)F_{l-k}^{(s-1)}(z)\\ -(y+1)F_k^{(r)}(y)F_{l-k}^{(s-1)}(z) - (z+1)F_k^{(r-1)}(y)F_{l-k}^{(s)}(z)\Big),\end{aligned} \tag{4.28}$$

with $\Lambda_1(y,z)=0$. Then we have the following polynomial identity.

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(x;y)F_{m-k}^{(s)}(x;z)\\&=\frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\left\{\Lambda_{m-k+1}(y,z)+\frac{H_{m-1}-H_{m-k}}{(m-k+1)y}\left(F_{m-k+1}^{(r)}(y)-F_{m-k+1}^{(r-1)}(y)\right)\right.\\&\left.\quad+\frac{H_{m-1}-H_{m-k}}{(m-k+1)z}\left(F_{m-k+1}^{(s)}(z)-F_{m-k+1}^{(s-1)}(z)\right)\right\}B_k(x)\end{aligned}\tag{4.29}$$

Remark 4.4. *Corollary 4.3 also follows from the following known fact(see [9]). Let $p(x)\in\mathbb{R}[x]$ be a polynomial of degree m. Then*

$$p(x)=\sum_{k=0}^{m}a_kB_k(x),\quad a_k\in\mathbb{R},\tag{4.30}$$

where

$$a_0=\int_0^1p(x)dx,\ a_k=\frac{1}{k!}\left(p^{(k-1)}(1)-p^{(k-1)}(0)\right),\ \ (1\le k\le m).\tag{4.31}$$

For $p(x)=\gamma_m(x;y,z)$ and using (4.4), we easily see that, for $0\le k\le m$,

$$\begin{aligned}p^{(k)}(x)&=\left(\frac{d}{dx}\right)^kp(x)\\&=(m-1)_k\gamma_{m-k}(x;y,z)+(m-1)_{k-1}(H_{m-1}-H_{m-1-k})(F_{m-k}^{(r)}(x;y)+F_{m-k}^{(s)}(x;z)),\end{aligned}\tag{4.32}$$

where $\gamma_1(x;y,z)=\gamma_0(x;y,z)=0$, and $H_0=H_{-1}=0$. The stated result in Corollary 4.3 follows immediately from (1.6), (4.9), (4.29)-(4.31). Similarly, Corollaries 4.1 and 4.22 can also be obtained from (4.29) and (4.30).

Remark 4.5. *As is well-known, the Euler polynomials $E_m(x)$ are given by*

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!}.\tag{4.33}$$

Let $p(x)$ be as in Remark 4.4. Then it is known (see[10]) that

$$p(x)=\sum_{k=0}^{m}b_kE_k(x),\quad b_k\in\mathbb{R},\tag{4.34}$$

where

$$b_k=\frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right),\ \ (0\le k\le m).\tag{4.35}$$

Now, from (4.3), (4.33), and (4.34) we derive the following expression.

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}F_k^{(r)}(x;y)F_{m-k}^{(s)}(x;z)\\&=\frac{1}{2}\sum_{k=0}^{m}\left\{\binom{m-1}{k}(\gamma_{m-k}(1;y,z)+\gamma_{m-k}(0;y,z))\right.\\&\left.\quad+\frac{1}{m}\binom{m}{k}(H_{m-1}-H_{m-1-k})\left(F_{m-k}^{(r)}(1;y)+F_{m-k}^{(r)}(y)+F_{m-k}^{(s)}(1;z)+F_{m-k}^{(s)}(z)\right)\right\}E_k(x),\end{aligned}\tag{4.36}$$

where $\gamma_1(x;y,z) = \gamma_0(x;y,z) = 0$, and $H_0 = H_{-1} = 0$.

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