Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 2. pp. 309 – 317
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.2.309

SOME EXPLICIT FORMULAS OF DEGENERATE STIRLING NUMBERS ASSOCIATED WITH THE DEGENERATE SPECIAL NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we study the degenerate Stirling numbers arising from the generating functions and we give some identities and formulas of these numbers which are related to the degenerate special numbers and polynomials.

1. Introduction

For $\lambda \in \mathbb{R}$, the degenerate Bernoulli polynomials are defined by L. Carlitz which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [1, 2])}. \tag{1.1}$$

Note that $\lim_{\lambda\to 0} \beta_{n,\lambda}(x) = B_n(x)$, where $B_n(x)$ are ordinary Bernoulli polynomials.

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

For $r \in \mathbb{N}$, the higher-order degenerate Bernoulli polynomials are also defined by the generating function to be

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [1, 2])}. \tag{1.2}$$

When $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Bernoulli numbers.

It is well known that the Stirling number of the second kind is defined by the generating function to be

$$\frac{1}{k!}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [3, 5, 6, 7, 8])}. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 11B68; 11S80.
Key words and phrases. Explicit formula, Degenerate Stirling numbers.

In the viewpoint of inversion formula of (1.3), the Stirling number of the first kind is given by

$$\frac{1}{k!}\big(\log(1+t)\big)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\quad \text{(see [3,8])}. \tag{1.4}$$

Let us consider the following degenerate exponential function.

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\ e_\lambda(t)=e_\lambda^1(t)=(1+\lambda t)^{\frac{1}{\lambda}},\quad \text{(see [9])}. \tag{1.5}$$

From (1.5), we note that the degenerate Stirling number of the second kind is defined by the generating function to be

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad \text{(see [6])}, \tag{1.6}$$

where $k\in\mathbb{N}=\{1,2,3,\cdots\}$.

Let $\log_\lambda t$ be the inverse function of $e_\lambda(t)$. Then, by (1.5), we easily get

$$\log_\lambda t=\frac{1}{\lambda}\big(t^\lambda-1\big),\ (\lambda\in\mathbb{R}),\quad \text{(see [7])}. \tag{1.7}$$

From (1.7), we note that

$$e_\lambda\big(\log_\lambda(t)\big)=\log_\lambda\big(e_\lambda(t)\big)=t, \tag{1.8}$$

and

$$\log_\lambda(t+1)=\frac{1}{\lambda}\sum_{n=1}^{\infty}(\lambda)_n\frac{t^n}{n!}, \tag{1.9}$$

where $(\lambda)_0=1$, $(\lambda)_n=\lambda(\lambda-1)\cdots(\lambda-n+1)$, $(n\geq 1)$.

By (1.5) and (1.9), we easily get

$$\lim_{\lambda\to 0}e_\lambda(t)=\lim_{\lambda\to 0}(1+\lambda t)^{\frac{1}{\lambda}}=e^t,\ \lim_{\lambda\to 0}\log_\lambda(1+t)=\log(1+t). \tag{1.10}$$

Now, we define the degenerate Stirling number of the first kind which is given by the generating function to be

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!},\quad \text{(see [7])}, \tag{1.11}$$

where $k\in\mathbb{N}=\{1,2,3,\cdots\}$.

For $r\in\mathbb{N}$, the higher-order Cauchy polynomials are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^x=\sum_{n=0}^{\infty}C_n^{(r)}(x)\frac{t^n}{n!},\quad \text{(see [5,11])}. \tag{1.12}$$

When $x=0$, $C_n^{(r)}=C_n^{(r)}(0)$ are called the Cauchy numbers of order r.

As is known, the Daehee polynomials of order $r(\in \mathbb{N})$ are given by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [4, 5, 10, 11])}. \tag{1.13}$$

When $x = 0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Dahee numbers of order r.

In this paper, we consider the degenerate Stirling numbers and degenerate Cauchy and Daehee polynomials which are given by the generating functions and we give some explicit formulas of the degenerate Stirling numbers associated with the degenerate special numbers and polynomials.

2. Some explicit identities of the degenerate Stirling numbers

Now, we define the higher-order Daehee polynomials which are given by the generating function to be

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} d_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $d_{n,\lambda}^{(r)} = d_{n,\lambda}^{(r)}(0)$ are called the degenerate Daehee numbers of order $r \in \mathbb{N}$.

From (1.5), (1.8) and (1.13), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} d_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{\log_\lambda(1+t)}{e_\lambda\big(\log_\lambda(1+t)\big)-1}\right)^r e_\lambda^x\big(\log(1+t)\big) \\
&= \sum_{k=0}^{\infty} \beta_{k,\lambda}^{(r)}(x)\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k \\
&= \sum_{k=0}^{\infty} \beta_{k,\lambda}^{(r)}(x)\left(\sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \beta_{k,\lambda}^{(r)}(x)S_{1,\lambda}(n,k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.2}$$

Therefore, by comparing the coefficients on the both sides of (2.2), we obtain the following theorem.

Theorem 2.1. *For $r \in \mathbb{N}$ and $n \geq 0$, we have*

$$d_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \beta_{k,\lambda}^{(r)}(x)S_{1,\lambda}(n,k).$$

In particular,

$$d_{n,\lambda}^{(r)}=\sum_{k=0}^{n}\beta_{k,\lambda}^{(r)}S_{1,\lambda}(n,k).$$

From (2.1), we note that

$$\begin{aligned}\sum_{k=0}^{\infty}d_{k,\lambda}^{(r)}(x)\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^{k}&=\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!}.\end{aligned}\tag{2.3}$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^{\infty}d_{k,\lambda}^{(r)}(x)\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^{k}&=\sum_{k=0}^{\infty}d_{k,\lambda}^{(r)}(x)\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^{n}}{n}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}d_{k,\lambda}^{(r)}(x)S_{2,\lambda}(n,k)\right)\frac{t^{n}}{n!}.\end{aligned}\tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.2. *For $r\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, we have*

$$\beta_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}d_{k,\lambda}^{(r)}(x)S_{2,\lambda}(n,k).$$

In particular,

$$\beta_{n,\lambda}^{(r)}=\sum_{k=0}^{n}d_{k,\lambda}^{(r)}S_{2,\lambda}(n,k).$$

In the viewpoint of (1.12), we can consider the degenerate Cauchy polynomials of order r which are given by the generating function to be

$$\left(\frac{t}{\log_{\lambda}(1+t)}\right)^{r}(1+t)^{x}=\sum_{n=0}^{\infty}C_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!}.\tag{2.5}$$

When $x = 0$, $C_{n,\lambda}^{(r)} = C_{n,\lambda}^{(r)}(0)$ are called the degenerate Cauchy numbers of order r. Now, we observe that

$$\begin{aligned}(1+t)^{x+y} &= \left(\frac{\log_\lambda(1+t)}{t}\right)^r (1+t)^x \left(\frac{t}{\log_\lambda(1+t)}\right)^r (1+t)^y \\ &= \left(\sum_{l=0}^{\infty} d_{l,\lambda}^{(r)}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} C_{m,\lambda}^{(r)}(y)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} d_{l,\lambda}^{(r)}(x) C_{n-l,\lambda}^{(r)}(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n}\binom{x}{l}\binom{y}{n-l} = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l} d_{l,\lambda}^{(r)}(x) C_{n-l,\lambda}^{(r)}(y).$$

In particular,

$$\sum_{l=0}^{n}\binom{n}{l}^2 = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l} d_{l,\lambda}^{(r)}(n) C_{n-l,\lambda}^{(r)}(n).$$

By (1.8), and (2.5), we get

$$\begin{aligned}\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}\right)^r &= \left(\frac{e_\lambda(t)-1}{t}\right)^r = \sum_{n=0}^{\infty} C_{n,\lambda}^{(r)}\frac{1}{n!}\big(e_\lambda(t)-1\big)^n \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}^{(r)}\sum_{k=n}^{\infty} S_{2,\lambda}(k,n)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} C_{n,\lambda}^{(r)} S_{2,\lambda}(k,n)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned}\left(\frac{e_\lambda(t)-1}{t}\right)^r &= \frac{r!}{t^r}\frac{1}{r!}\big(e_\lambda(t)-1\big)^r = \frac{r!}{t^r}\frac{1}{r!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^r \\ &= \frac{r!}{t^r}\sum_{k=r}^{\infty} S_{2,\lambda}(k,r)\frac{t^k}{k!} = \sum_{k=0}^{\infty}\frac{S_{2,\lambda}(k+r,r)}{\binom{k+r}{k}}\frac{t^k}{k!}.\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.4. *For $r, k \geq 0$, we have*

$$\frac{1}{\binom{k+r}{r}} S_{2,\lambda}(k+r,r) = \sum_{n=0}^{k} C_{n,\lambda}^{(r)} S_{2,\lambda}(k,n).$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,\lambda}^{(r)} \frac{t^n}{n!} &= \frac{1}{t^r}\big(\log_\lambda(1+t)\big)^r = \frac{r!}{t^r}\frac{1}{r!}\big(\log_\lambda(1+t)\big)^r \\ &= r!\sum_{n=0}^{\infty} S_{1,\lambda}(n+r,r)\frac{t^n}{(n+r)!} = \sum_{n=0}^{\infty}\frac{S_{1,\lambda}(n+r,r)}{\binom{n+r}{r}}\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

Therefore, by comparing the coefficients on the both sides of (2.9), we obtain the following theorem.

Theorem 2.5. *For $n, r \geq 0$, we have*

$$d_{n,\lambda}^{(r)} = \frac{1}{\binom{n+r}{r}} S_{1,\lambda}(n+r,r).$$

For $n, k \geq 0$, it is known that the associated Stirling number is defined by the generating function to be

$$\frac{1}{k!}\big(\log(1+t)-t\big)^k = \sum_{n=2k}^{\infty}(-1)^{n-k}S^*(n,k)\frac{t^n}{n!}, \quad \text{(see [3])}. \tag{2.10}$$

Now, we define the associated degenerate Stirling numbers which are given by the generating function to be

$$\frac{1}{k!}\big(\log_\lambda(1+t)-t\big)^k = \sum_{n=2k}^{\infty}(-1)^{n-k}S_\lambda^*(n,k)\frac{t^n}{n!}, \ (k \geq 0). \tag{2.11}$$

Note that $\lim_{\lambda\to 0} S_\lambda^*(n,k) = S^*(n,k)$, $(n,k \geq 0)$.

From (2.11), we have

$$\begin{aligned}\left(\frac{\log_\lambda(1+t)}{t}\right)^r &= \left(1+\frac{1}{t}\big(\log_\lambda(1+t)-t\big)\right)^r \\ &= \sum_{m=0}^{r}\binom{r}{m}\frac{1}{t^m}\big(\log_\lambda(1+t)-t\big)^m \\ &= \sum_{m=0}^{r}\binom{r}{m}\sum_{n=m}^{\infty}S_\lambda^*(n+m,m)\frac{m!n!}{(n+m)!}(-1)^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left((-1)^n\sum_{m=0}^{n}\frac{\binom{r}{m}}{\binom{n+m}{n}}S_\lambda^*(n+m,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

On the other hand,

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r = \sum_{n=0}^{\infty} d_{n,\lambda}^{(r)}\frac{t^n}{n!}. \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$d_{n,\lambda}^{(r)} = (-1)^n \sum_{m=0}^{n} \frac{\binom{r}{m}}{\binom{n+m}{n}} S_\lambda^*(n+m,m).$$

Now, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n,\lambda}^{(r)} \frac{t^n}{n!} &= \left(\frac{t}{\log_\lambda(1+t)}\right)^r = \left(1 + \frac{1}{\log_\lambda(1+t)}\big(t - \log_\lambda(1+t)\big)\right)^r \\
&= \sum_{m=0}^{r} \binom{r}{m} \left(\frac{t}{\log_\lambda(1+t)}\right)^m \frac{1}{t^m}\big(t - \log_\lambda(1+t)\big)^m \\
&= \sum_{m=0}^{r} \binom{r}{m} \left(\frac{t}{\log_\lambda(1+t)}\right)^m (-1)^m \frac{m!}{t^m} \frac{1}{m!}\big(\log_\lambda(1+t) - t\big)^m \\
&= \sum_{m=0}^{r} \binom{r}{m} (-1)^m \frac{m!}{t^m} \left(\sum_{k=2m}^{\infty} S_\lambda^*(k,m)(-1)^{k-m} \frac{t^k}{k!}\right) \left(\frac{t}{\log_\lambda(1+t)}\right)^m \\
&= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} \binom{r}{m} \frac{S_\lambda^*(k+m,m)}{\binom{k+m}{k}} (-1)^{k-m} \frac{t^k}{k!}\right) \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{(m)} \frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{m=0}^{k} \frac{\binom{r}{m}\binom{n}{k}}{\binom{k+m}{k}} (-1)^{k-m} S_\lambda^*(k+m,m) C_{n-k,\lambda}^{(m)}\right) \frac{t^n}{n!}.
\end{aligned} \tag{2.14}$$

Therefore, by comparing the coefficients on the both sides of (2.14), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$C_{n,\lambda}^{(r)} = \sum_{k=0}^{n} \sum_{m=0}^{k} \frac{\binom{r}{m}\binom{n}{k}}{\binom{k+m}{k}} (-1)^{k-m} S_\lambda^*(k+m,m) C_{n-k,\lambda}^{(m)}.$$

It is easy to show that

$$\sum_{n=k}^{\infty} S_{1,\lambda}(n,k) S_{2,\lambda}(m,k) = \delta_{k,m}, \;\; (k, m \geq 0),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

From (1.1), we note that

$$\begin{aligned}\sum_{k=0}^{\infty}\beta_{k,\lambda}\frac{t^k}{k!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} = \frac{\log_\lambda(1+(1+\lambda t)^{\frac{1}{\lambda}}-1)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\\ &= \left(\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)\frac{1}{\lambda}\sum_{n=1}^{\infty}(\lambda)_n\frac{1}{n!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^n\\ &= \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_{n+1}}{n+1}\frac{1}{n!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^n\\ &= \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_{n+1}}{n+1}\sum_{k=n}^{\infty}S_{2,\lambda}(k,n)\frac{t^k}{k!}\\ &= \sum_{k=0}^{\infty}\left(\frac{1}{\lambda}\sum_{n=0}^{k}\frac{(\lambda)_{n+1}}{n+1}S_{2,\lambda}(k,n)\right)\frac{t^k}{k!}.\end{aligned}\tag{2.15}$$

Therefore, by comparing on the both sides of (2.15), we obtain the following equation:

For $k \geq 0$, we have

$$\beta_{k,\lambda} = \frac{1}{\lambda}\sum_{n=0}^{k}\frac{(\lambda)_{n+1}}{n+1}S_{2,\lambda}(k,n), \quad \text{(see [7])}.$$

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