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SOME SUMMABLE SEQUENCE SPACES DEFINED BY THE WEIGHTED MEAN METHOD AND MODULUS FUNCTIONS

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Abstract. In this paper, we define the concept of $\left[\bar{N}_{\lambda}, p_n, f\right]-$ summability, and give the relation between $\left[\bar{N}_{\lambda}, p_n\right]-$ summability and $\left[\bar{N}_{\lambda}, p_n, f\right]-$ summability. We also study some connections between weighted statistically $\lambda-$ convergence (or $S_{\bar{N}_{\lambda}}-$ convergence) and $\left[\bar{N}_{\lambda}, p_n, f\right]-$ summability.

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1. Introduction

Let $\mathbb{N}$ and $\mathbb{C}$ be the sets of all natural numbers and complex numbers, respectively, and let ℓ_{∞} be the linear space of bounded sequences $x=(x_k)$ normed by $\|x\|=\sup_k |x_k|$, where $k \in \mathbb{N}$.

Let $p=(p_k)$ be a sequence of nonnegative numbers such that $p_0>0$ and $P_n=\sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$. Let

$$t_n(x)=\frac{1}{P_n}\sum_{k=0}^{n} p_k x_k, \qquad n=0,1,2....$$

The sequence $x=(x_k)$ is said to be $\left(\bar{N}, p_n\right)-$summable to L if $\lim_{n\to\infty} t_n(x)=L$. In this case, we write $x_k \to L\left(\bar{N}, p_n\right)$.

Moricz and Orhan [7] have defined the concept of statistical summability $\left(\bar{N}, p_n\right)$ as follows:

A sequence $x=(x_k)$ is statistically summable to L by the weighted mean method determined by the sequence (p_k) or briefly statistically summable $\left(\bar{N}, p_n\right)$ to L if $st-\lim t_n(x)=L$. In this case, we write $\bar{N}(st)-\lim x=L$. We denote by $\bar{N}(st)$ the set of all sequences which are statistically summable $\left(\bar{N}, p_n\right)$.

A sequence $x=(x_k)$ is said to be $\left[\bar{N}, p_n\right]_q-$summable $(0<q<\infty)$ to the number L if $\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n} p_k |x_k-L|^q=0$. In this case, we write $x_k \to L\left[\bar{N}, p_n\right]_q$ [9]. If $q=1$, then we write

$$\left[\bar{N}, p_n\right]=\left\{x=(x_k): \lim_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n} p_k\left|x_k-L\right|=0, \text{ for some } L\right\}$$

for the set of sequences $x=(x_k)$ which are strongly $\left(\bar{N}, p_n\right)-$summable to

L.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that
$\lambda_{n+1} \leq \lambda_n + 1, \qquad \lambda_1 = 1.$

The collection of such a sequence will be denoted by Δ. Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and
$P_{\lambda_n} := \sum_{k \in I_n} p_k \to \infty \qquad \text{and} \qquad \sigma_n := \sum_{k \in I_n} p_k x_k,$
where $I_n = [n - \lambda_n + 1, n]$. Belen and Mohiuddine [10] have defined the concepts of $(\bar{N}_\lambda, p_n)$ −summability and $[\bar{N}_\lambda, p_n]$ −summability as a generalization of $(\bar{N}, p_n)$ −summability and $[\bar{N}, p_n]$ −summability, respectively as follows:

(i) A sequence $x = (x_k)$ is said to be $(\bar{N}_\lambda, p_n)$ −summable to L if $\lim_{n\to\infty} \sigma_n = L$. In this case, we write $x_k \to L\,(\bar{N}_\lambda, p_n)$.

(ii) A sequence $x = (x_k)$ is said to be strongly $(\bar{N}_\lambda, p_n)$ −summable to L if
$\lim_{n\to\infty} \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} p_k |x_k - L| = 0.$

In this case, we write $x_k \to L\,[\bar{N}_\lambda, p_n]$ and $[\bar{N}_\lambda, p_n]$ denotes the set of all strongly $(\bar{N}_\lambda, p_n)$ −summable sequences, i.e.,

$$[\bar{N}_\lambda, p_n] = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} p_k |x_k - L| = 0, \text{ for some } L \right\}.$$

If $\lambda_n = n$ for all $n \in \mathbb{N}$, $(\bar{N}_\lambda, p_n)$ −summability and $[\bar{N}_\lambda, p_n]$ −summability are reduced to $(\bar{N}, p_n)$ −summability and $[\bar{N}, p_n]$ −summability which are introduced in [9], respectively.

The notion of a modulus function was introduced by Nakano [6]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that
(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$.
Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ from condition (ii), and so

$$f(x) = f\left(nx\frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right)$$

Hence, for $n \in \mathbb{N}$

$\frac{1}{n} f(x) \leq f\left(\frac{x}{n}\right)$.

Ruckle [12] used the idea of a modulus function f to construct a class of $FK-$ spaces

$L(f) = \left\{ x = (x_k) : \sum\limits_{k=1}^{\infty} f(|x_k|) < \infty \right\}$.

The space $L(f)$ is closely related to the space ℓ_1 which is a $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.
Furthermore, modulus function has been discussed in [5], [13], [14], [15], [16], [17], and many others.

In this paper, we extend the strongly $\left(\bar{N}_\lambda, p_n\right)-$summable sequences which are defined by Belen and Mohiuddine [10], and give the relation between strongly $\left(\bar{N}_\lambda, p_n\right)-$summability and strongly $\left(\bar{N}_\lambda, f, p_n\right)-$summability with respect to a modulus. We also study some connections between weighted $\lambda-$statistically convergence and $\left[\bar{N}_\lambda, p_n, f\right]-$summability.

2. Some sequence spaces and inclusion theorems

We now introduce some new sequence spaces by using a modulus function f and investigate some inclusion relations.

Definition 1. *Let f be a modulus. We define the spaces*

$$\left[\bar{N}_\lambda, p_n, f\right] = \left\{ x = (x_k) : \lim_n \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} f(p_k |x_k - L|) = 0, \text{ for some } L \right\},$$

$$\left[\bar{N}_\lambda, p_n, f\right]_0 = \left\{ x = (x_k) : \lim_n \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} f(p_k |x_k|) = 0 \right\}.$$

If we take $f(x) = x$, then we have $\left[\bar{N}_\lambda, p_n, f\right] = \left[\bar{N}_\lambda, p_n\right]$ and $\left[\bar{N}_\lambda, p_n, f\right]_0 = \left[\bar{N}_\lambda, p_n\right]_0$, where

$$\left[\bar{N}_\lambda, p_n\right]_0 = \left\{ x = (x_k) : \lim_n \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} p_k |x_k| = 0 \right\}.$$

Now, we begin with the following theorem.

Theorem 2.1. *The spaces $\left[\bar{N}_\lambda, p_n, f\right]$ and $\left[\bar{N}_\lambda, p_n, f\right]_0$ are linear spaces.*

Proof. We consider only $\left[\bar{N}_\lambda, p_n, f\right]$. The proof for the other space will follow similarly. Suppose that $x_i \to L$ and $y_j \to L'$ in $\left[\bar{N}_\lambda, p_n, f\right]$ and α, β are in $\mathbb{C}$. Then there exists integers M_α and M_β such that $|\alpha| \leq M_\alpha$ and $|\beta| \leq M_\beta$. Therefore, we have

$$\frac{1}{P_{\lambda_n}} \sum_{k \in I_n} f(p_k |\alpha x_k + \beta y_k - (\alpha L + \beta L')|)$$

$$\leq M_\alpha \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} f(p_k |x_k - L|) + M_\beta \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} f(p_k |y_k - L'|).$$

This implies that $\alpha x + \beta y \to \alpha L + \beta L'$ in $\left[\bar{N}_\lambda, p_n, f\right]$, and so $\left[\bar{N}_\lambda, p_n, f\right]$ is a linear space. This completes the proof. □

Proposition 2.2. *([14]) Let f be any modulus. Then $\lim_{t\to\infty}\frac{f(t)}{t}=\beta$ exists.*

Proposition 2.3. *([16]) Let f be a modulus and let $0<\delta<1$. Then for each $x\geq\delta$ we have $f(x)\leq 2f(1)\,\delta^{-1}x$.*

Next theorems give the relation between $\left[\bar{N}_\lambda,p_n\right]$ and $\left[\bar{N}_\lambda,p_n,f\right]$ spaces.

Theorem 2.4. *Let f be a modulus. Then $\left[\bar{N}_\lambda,p_n\right]\subset\left[\bar{N}_\lambda,p_n,f\right]$.*

Proof. Let $x\in\left[\bar{N}_\lambda,p_n\right]$. Then, we have

$$s_n=\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}p_k\left|x_k-L\right|\to 0 \qquad \text{as} \qquad n\to\infty, \qquad \text{for some } L.$$

Write $K_{P_{\lambda_n}}(\varepsilon)=\{k\leq P_{\lambda_n}:p_k\left|x_k-L\right|\geq\varepsilon\}$. Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $f(t)<\varepsilon$ for every t with $0\leq t\leq\delta$. We can write

$$\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}f\left(p_k\left|x_k-L\right|\right)=\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{P_{\lambda_n}}(\varepsilon)}}f\left(p_k\left|x_k-L\right|\right)+\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\notin K_{P_{\lambda_n}}(\varepsilon)}}f\left(p_k\left|x_k-L\right|\right)$$
$$\leq\frac{1}{P_{\lambda_n}}\left(P_{\lambda_n}\varepsilon\right)+2f(1)\,\delta^{-1}s_n,$$

by Proposition 2.3 Therefore $x\in\left[\bar{N}_\lambda,p_n,f\right]$. □

Theorem 2.5. *Let f be a modulus. If $\lim_{t\to\infty}\frac{f(t)}{t}=\beta>0$, then $\left[\bar{N}_\lambda,p_n,f\right]=\left[\bar{N}_\lambda,p_n\right]$.*

Proof. By Theorem 2.4, we need only show that $\left[\bar{N}_\lambda,p_n,f\right]\subset\left[\bar{N}_\lambda,p_n\right]$. Let $\beta>0$ and $x\in\left[\bar{N}_\lambda,p_n,f\right]$. Since $\beta>0$, we have $f(t)\geq\beta t$ for all $t\geq 0$. Hence, we have

$$\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}f\left(p_k\left|x_k-L\right|\right)\geq\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}\beta p_k\left|x_k-L\right|=\beta\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}p_k\left|x_k-L\right|$$

Therefore, we have $x\in\left[\bar{N}_\lambda,p_n\right]$. This completes the proof. □

In Theorem 2.5, the condition $\beta>0$ cannot be omitted. For this, consider the following example.

Example 1. *Let $f(x)=\ln(1+x)$. Then $\beta=0$. Let $p_k=1$ for all $k\in\mathbb{N}$ and define x_k to be P_{λ_n} at the $(n-\lambda_n+1)$th term in I_n for every $n\geq 1$ and $x_k=0$ otherwise. Note that $x\notin\ell_\infty$. Then, we have*

$$\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}f\left(p_k\left|x_k\right|\right)=\frac{\ln(1+\lambda_n)}{\lambda_n}\to 0, \qquad \text{as} \qquad n\to\infty$$

and so $x\in\left[\bar{N}_\lambda,p_n,f\right]$, but

$$\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}p_k\left|x_k\right|=\frac{\lambda_n}{\lambda_n}\to 1, \qquad \text{as} \qquad n\to\infty$$

and so $x\notin\left[\bar{N}_\lambda,p_n\right]$.

3. Some results on generalized weighted statistical convergence

The concept of statistical convergence for sequences of real numbers was introduced by Fast [1] in 1951, and since then several generalizations and applications of this notion have been investigated by many authors such as Schoenberg [2], Fridy [3], Connor [4, 5], etc.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $st - \lim x = L$. [1]

Mursaleen et al. [9] has modified the definition of weighted statistical convergence due to Karakaya and Chishti [8] and through an example, they showed that the definition must be as follows:
Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$.

A sequence $x = (x_k)$ is weighted statistically convergent (or $S_{\bar{N}}-$convergent) to L if for every $\varepsilon > 0$,
$$\lim_n \frac{1}{P_n} |\{k \leq P_n : p_k |x_k - L| \geq \varepsilon\}| = 0.$$

Recently, Belen and Mohiuddine [10] have introduced the definition of generalized weighted statistical convergence following the line of Mursaleen [11].

Let $\lambda \in \Delta$ and $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and
$$P_{\lambda_n} := \sum_{k \in I_n} p_k \to \infty$$
where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be weighted $\lambda-$statistically convergent (or $S_{\bar{N}_\lambda}-$convergent) to L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{P_{\lambda_n}} |\{k \leq P_{\lambda_n} : p_k |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S_{\bar{N}_\lambda} - \lim x = L$ and $S_{\bar{N}_\lambda}$ denotes the set of all weighted statistically $\lambda-$convergent sequences. [10]
Note that if $\lambda_n = n$ for all $n \in \mathbb{N}$, then $S_{\bar{N}_\lambda}-$convergence coincides with $S_{\bar{N}}-$convergence.

In this section, we establish inclusion relations between $\left[\bar{N}_\lambda, p_n, f\right]$ and $S_{\bar{N}_\lambda}$.

Theorem 3.1. *Let f be any modulus. Then $\left[\bar{N}_\lambda, p_n, f\right] \subset S_{\bar{N}_\lambda}$.*

Proof. Let $x \in \left[\bar{N}_\lambda, p_n, f\right]$, $\varepsilon > 0$ and $K_{P_{\lambda_n}}(\varepsilon) = \{k \leq P_{\lambda_n} : p_k |x_k - L| \geq \varepsilon\}$. Then, we have

$$\begin{aligned} \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} f\left(p_k |x_k - L|\right) &\geq \frac{1}{P_{\lambda_n}} \sum_{\substack{k \in I_n \\ k \in K_{P_{\lambda_n}}(\varepsilon)}} f\left(p_k |x_k - L|\right) \\ &\geq \frac{1}{P_{\lambda_n}} f(\varepsilon) \left|\{k \leq P_{\lambda_n} : p_k |x_k - L| \geq \varepsilon\}\right| \end{aligned}$$

which yields that $x \in S_{\bar{N}_\lambda}$. This completes the proof. □

Theorem 3.2. *Let f be any modulus and $p = (p_k)$ be bounded, then we have $S_{\bar{N}_\lambda} \cap \ell_\infty = \left[\bar{N}_\lambda, p_n, f\right] \cap \ell_\infty$.*

Proof. Let $\varepsilon > 0$ be given and M be a constant. Then we have this result in view of above Theorem 3.1 and the following inequality:

$$\begin{aligned} \frac{1}{P_{\lambda_n}} \sum_{k \in I_n} f\left(p_k |x_k - L|\right) &= \frac{1}{P_{\lambda_n}} \sum_{\substack{k \in I_n \\ k \in K_{P_{\lambda_n}}(\varepsilon)}} f\left(p_k |x_k - L|\right) + \frac{1}{P_{\lambda_n}} \sum_{\substack{k \in I_n \\ k \notin K_{P_{\lambda_n}}(\varepsilon)}} f\left(p_k |x_k - L|\right) \\ &\leq \frac{M}{P_{\lambda_n}} \left|\{k \leq P_{\lambda_n} : p_k |x_k - L| \geq \varepsilon\}\right| + f(\varepsilon). \end{aligned}$$

□

4. Concluding Remark

Under what condition(s) the equality $S_{\bar{N}_\lambda} = \left[\bar{N}_\lambda, p_n, f\right]$ holds?

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