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A NOTE ON DEGENERATE GAMMA FUNCTION AND DEGENERATE STIRLING NUMBER OF THE SECOND KIND

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Abstract. Recently, degenerate gamma function was introduced in [5]. In this paper, we investigate some properties of the degenerate gamma function and give some functional equations for the degenerate gamma function. In addition, we derive some identities for the degenerate Stirling number of the second kind associated with degenerate gamma function.

1. Introduction

It is well known that the gamma function is defined as

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt,\ \text{where } s\in\mathbb{C} \text{ with } Re(s)>1. \tag{1.1}$$

Thus, we note that

$$\Gamma(s+1)=s\Gamma(s),\ \Gamma(n+1)=n!,\ (n\in\mathbb{N}),\quad (\text{see } [1,5,6,7,8]). \tag{1.2}$$

The degenerate exponential function $e_\lambda(t)$ is a function of two variables $\lambda\in(0,\infty)$ and $t\in\mathbb{R}$ defined as

$$e_\lambda(t)=(1+\lambda t)^{\frac{1}{\lambda}},\quad (\text{see } [5]). \tag{1.3}$$

Note that $\lim_{\lambda\to 0} e_\lambda(t)=e^t$. From (1.1), we note that

$$\begin{aligned}\Gamma(z)&=\lim_{n\to\infty}\int_0^n\left(1-\frac{t}{n}\right)^n t^{z-1}dt\\ &=\lim_{n\to\infty} n^z\sum_{k=0}^n\binom{n}{k}(-1)^k\frac{1}{z+k}\\ &=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)},\ (z\neq 0,-1,-2,\cdots).\end{aligned} \tag{1.4}$$

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Recently, the degenerate gamma function for the complex variables with $0 < Re(s) < \frac{1}{\lambda}$ was defined as

$$\Gamma_\lambda(s) = \int_0^\infty e_\lambda^{-1}(t)t^{s-1}dt = \int_0^\infty (1+\lambda t)^{-\frac{1}{\lambda}}t^{s-1}dt, \quad \text{(see [5])}. \tag{1.5}$$

From (1.5), we have

$$\Gamma_\lambda(s+1) = \frac{s}{(1-\lambda)^{s+1}}\Gamma_{\frac{\lambda}{1-\lambda}}(s), \quad \text{(see [5])}, \tag{1.6}$$

where $\lambda \in (0,1)$ and $0 < Re(s) < \frac{1-\lambda}{\lambda}$.

For $\lambda \in \mathbb{R}$, the degenerate Stirling number of the second kind is defined by the generating function to be

$$\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k = \sum_{n=k}^\infty S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad \text{(see [3, 4, 5])}, \tag{1.7}$$

where $k \in \mathbb{N} \cup \{0\}$. Thus by (1.7), we get $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$, $(n,k \geq 0)$, where $S_2(n,k)$ are the Stirling numbers of the second kind.

In this paper, we investigate some properties of degenerate gamma function and give some formulas related to Stirling numbers.

In addition, we also give some identities for the degenerate Stirling numbers of the second kind associated with degenerate gamma function.

2. Degenerate gamma function

From (1.6), we note that

$$\begin{aligned}\Gamma_\lambda(k) &= \frac{(k-1)!}{(1-\lambda)(1-2\lambda)\cdots(1-k\lambda)} = \frac{\Gamma(k)}{(1-\lambda)(1-2\lambda)\cdots(1-k\lambda)} \\ &= \frac{1}{k}\frac{k!}{(1-\lambda)(1-2\lambda)\cdots(1-k\lambda)} = \frac{1}{k}\sum_{l=0}^k \binom{k}{l}(-1)^{k-l}\frac{l^k}{(1-\lambda l)},\end{aligned} \tag{2.1}$$

where $k \in \mathbb{N}$ and $\lambda \in (0, \frac{1}{k})$. It is not difficult to show that

$$\Gamma_{-\lambda}(k) = \frac{(k-1)!}{(1+\lambda)(1+2\lambda)\cdots(1+k\lambda)}, \text{ where } -\lambda \in (-\frac{1}{k}, 0).$$

The difference operator Δ is defined as

$$\Delta f(x) = f(x+1) - f(x).$$

From (2.1), we have

$$\begin{aligned}\Gamma_\lambda(k) &= \frac{1}{k}\sum_{m=0}^{\infty}\lambda^m\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^{k+m} \\ &= (k-1)!\sum_{m=0}^{\infty}\lambda^m\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^{k+m} \\ &= (k-1)!\sum_{m=0}^{\infty}\lambda^m\frac{1}{k!}\Delta^k 0^{k+m} = (k-1)!\sum_{m=0}^{\infty}\lambda^m S_2(k+m,k).\end{aligned} \tag{2.2}$$

Therefore, by (2.2), we obtain the following theorem

Theorem 2.1. *For $k \in \mathbb{N}$ and $\lambda \in (0, \frac{1}{k})$, we have*

$$\begin{aligned}\Gamma_\lambda(k) &= \frac{1}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{l^k}{(1-\lambda l)} = (k-1)!\sum_{m=0}^{\infty}\lambda^m S_2(k+m,k) \\ &= \frac{1}{k}\sum_{m=0}^{\infty}\lambda^m\Delta^k 0^{k+m}.\end{aligned}$$

For $k \in \mathbb{N}$ and $\lambda \in (0, \frac{1}{k})$, we observe that

$$\begin{aligned}\left(\frac{d}{d\lambda}\right)^r\Gamma_\lambda(k) &= \left(\frac{d}{d\lambda}\right)^r\left(\frac{\Gamma(k)}{(1-\lambda)(1-2\lambda)\cdots(1-k\lambda)}\right) \\ &= \frac{1}{k}\left(\frac{d}{d\lambda}\right)^r\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{l^k}{(1-\lambda l)} \\ &= \frac{1}{k}r!\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{l^{k+r}}{(1-\lambda l)^{r+1}}.\end{aligned} \tag{2.3}$$

Thus, by (2.3), we get

$$\left(\frac{d}{d\lambda}\right)^r\Gamma_\lambda(k) = \frac{r!}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{l^{k+r}}{(1-\lambda l)^{r+1}}. \tag{2.4}$$

Now, we define the following notation:

$$\Gamma_0^{(r)}(k) = \left(\frac{d}{d\lambda}\right)^r\Gamma_\lambda(k)\bigg|_{\lambda=0}, \ (r \in \mathbb{N}). \tag{2.5}$$

Then, by (2.4) and (2.5), we get

$$\begin{aligned}\Gamma_0^{(r)}(k) &= \frac{r!}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^{k+r}\\ &= \frac{r!k!}{k}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^{k+r}\\ &= \frac{(k+r-1)!}{\binom{k+r-1}{r}}\frac{1}{k!}\Delta^k 0^{k+r} = \frac{(k+r-1)!}{\binom{k+r-1}{r}}S_2(k+r,k).\end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.2. *For $r,k\in\mathbb{N}$ and $\lambda\in(0,\frac{1}{k})$, we have*

$$\Gamma_0^{(r)}(k) = \frac{(k+r-1)!}{\binom{k+r-1}{r}}S_2(k+r,k).$$

For $k\in\mathbb{N}$ and $\lambda\in(0,\frac{1}{k})$, we note that

$$\begin{aligned}\Gamma_\lambda(k)\Gamma_{-\lambda}(k) &= \frac{\Big((k-1)!\Big)^2}{(1-\lambda^2)(1-(2\lambda)^2)\cdots(1-(k\lambda)^2)}\\ &= (-1)^k\frac{\Big((k-1)!\Big)^2}{\lambda^{2k}1^2 2^2\cdots k^2}\times\frac{1}{\big(1-\frac{1}{\lambda^2}\big)\big(1-(\frac{1}{2\lambda})^2\big)\cdots\big(1-(\frac{1}{k\lambda})^2\big)}\\ &= (-1)^k\frac{\lambda^{-2k}}{k^2}\times\frac{1}{\big(1-(\frac{\frac{1}{\lambda}}{1})^2\big)\big(1-(\frac{\frac{1}{\lambda}}{2})^2\big)\cdots\big(1-(\frac{\frac{1}{\lambda}}{k})^2\big)}\\ &= \frac{(-\lambda^{-2})^k}{k^2}\frac{\prod_{n=1}^{\infty}\Big(1-(\frac{\frac{1}{\lambda}}{n+k})^2\Big)}{\prod_{n=1}^{\infty}\Big(1-(\frac{\frac{1}{\lambda}}{n})^2\Big)}\\ &= \frac{(-\lambda^{-2})^k}{k^2}\frac{\frac{\pi}{\lambda}}{\sin\frac{\pi}{\lambda}}\prod_{n=1}^{\infty}\left(1-\Big(\frac{1}{\lambda(n+k)}\Big)^2\right)\end{aligned} \tag{2.7}$$

From (2.7), we have

$$\begin{aligned}\Gamma_\lambda(k)\Gamma_{-\lambda}(k) &= \frac{(-1)^k}{k^2}\lambda^{-2k-1}\frac{\pi}{\sin\frac{\pi}{\lambda}}\prod_{n=1}^{\infty}\left(1-\Big(\frac{1}{\lambda(n+k)}\Big)^2\right)\\ &= \frac{(-1)^k}{k^2}\lambda^{-2k-1}\Gamma(\frac{1}{\lambda})\Gamma(1-\frac{1}{\lambda})\prod_{n=1}^{\infty}\left(1-\Big(\frac{1}{\lambda(n+k)}\Big)^2\right).\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For $r, k \in \mathbb{N}$ wwith $\lambda \in (0, \frac{1}{k})$, we have*

$$\Gamma_\lambda(k)\Gamma_{-\lambda}(k) = \frac{(-1)^k}{k^2}\lambda^{-2k-1}\frac{\pi}{\sin\frac{\pi}{\lambda}}\prod_{n=1}^{\infty}\left(1-\Big(\frac{1}{\lambda(n+k)}\Big)^2\right)$$

In particular,

$$\frac{\Gamma_\lambda(k)\Gamma_{-\lambda}(k)}{\Gamma(\frac{1}{\lambda})\Gamma(1-\frac{1}{\lambda})} = \frac{(-1)^k}{k^2}\lambda^{-2k-1}\prod_{n=1}^{\infty}\left(1-\Big(\frac{1}{\lambda(n+k)}\Big)^2\right).$$

From (1.3), we note that

$$(1+\lambda t)^{\frac{x}{\lambda}} = \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 + 1\right)^x = \sum_{k=0}^{\infty}(x)_k\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k, \tag{2.8}$$

where $(x)_0 = 1$, $(x)_k = x(x-1)\cdots(x-k+1)$, $(k \geq 1)$.
On the other hand

$$\begin{aligned}(1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k\right)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\right)(x)_k,\end{aligned} \tag{2.9}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.
From (2.8) and (2.9), we have

$$\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k, \tag{2.10}$$

and

$$(x)_{n,\lambda} = \sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k, \ \ (n \geq 0). \tag{2.11}$$

By (2.11), we easily get

$$S_{2,\lambda}(n+1,k) = S_{2,\lambda}(n,k-1) + kS_{2,\lambda}(n,k) - n\lambda S_{2,\lambda}(n,k), \tag{2.12}$$

where $n, k \geq 0$.

From (2.12), we can derive the following equations.

$$\begin{aligned}S_{2,\lambda}(n+1,n) &= nS_{2,\lambda}(n,n)+S_{2,\lambda}(n,n-1)-n\lambda S_{2,\lambda}(n,n)\\ &= n(1-\lambda)S_{2,\lambda}(n,n)+S_{2,\lambda}(n,n-1)\\ &= n(1-\lambda)+S_{2,\lambda}(n,n-1)=n(1-\lambda)+(n-1)(1-\lambda)+S_{2,\lambda}(n-1,n-2)\\ &= \cdots\\ &= n(1-\lambda)+(n-1)(1-\lambda)+\cdots+(1-\lambda)+S_{2,\lambda}(1,0)\\ &= \frac{n(n+1)}{2}(1-\lambda)=\binom{n+1}{2}(1-\lambda).\end{aligned} \tag{2.13}$$

Thus, by (2.13), we get

$$S_{2,\lambda}(n+1,n)=\binom{n+1}{2}(1-\lambda),\ (n\in\mathbb{N}). \tag{2.14}$$

Now, we observe that

$$\begin{aligned}\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}}-1\Big)^k=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(1+\lambda t)^{\frac{l}{\lambda}}\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{1}{k!}\Delta^k(0)_{n,\lambda}\frac{t^n}{n!}.\end{aligned} \tag{2.15}$$

By (2.15), we get

$$\begin{aligned}\frac{1}{k!}\Delta^k(0)_{n,\lambda} &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}\\ &= \begin{cases}S_{2,\lambda}(n,k), & \text{if } n\ge k,\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{2.16}$$

From (2.16), we have

$$\frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}(l)_{n+1,\lambda}=S_{2,\lambda}(n+1,n)=\binom{n+1}{2}(1-\lambda). \tag{2.17}$$

So, by (2.17), we get

$$\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}(l)_{n+1,\lambda}=n!\binom{n+1}{2}(1-\lambda)=\frac{n(n+1)!}{2}(1-\lambda). \tag{2.18}$$

From the definition of $(x)_{n,\lambda}$, we have

$$(l)_{n+1,\lambda} = l(l-\lambda)(l-2\lambda)\cdots(l-n\lambda) = l^{n+1}\Big(1-\frac{\lambda}{l}\Big)\Big(1-\frac{2\lambda}{l}\Big)\cdots\Big(1-\frac{n\lambda}{l}\Big)$$
$$= l^{n+1}\frac{(n-1)!}{\Gamma_{\frac{\lambda}{l}}(n)}, \text{ where } \frac{\lambda}{l} \in (0, \frac{1}{n}). \tag{2.19}$$

By (2.18), we get

$$\sum_{l=0}^{n}\binom{n}{l}(-1)^l l^{n+1}\frac{(n-1)!}{\Gamma_{\frac{\lambda}{l}}(n)} = \frac{1}{2}(-1)^n n(n+1)!(1-\lambda). \tag{2.20}$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{N}$ and $\lambda \in (0, \frac{1}{n})$, we have*

$$\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l l^{n+1}}{\Gamma_{\frac{\lambda}{n}}(n)} = \frac{1}{2}(-1)^n n^2(n+1)!(1-\lambda).$$

Remark. Note that

$$\frac{1}{2}(-1)^n n^2(n+1) = \sum_{l=0}^{n}\binom{n}{l}(-1)^l l^{n+1}.$$

Thus, we have

$$\frac{1}{2}n(n+1) = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}l^{n+1}$$
$$= S_2(n, n+1).$$

Remark. The research problem of degenerate gamma function for $\frac{1}{\lambda}$ is remained in this paper.

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