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# Partition energy of complete product of circulant graphs and some new class of graphs

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**Abstract**

Let $G = (V, E)$ be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of $V$. The $L$-matrix with respect to a partition $P_k$ of the vertex set $V$ of graph $G$ of order $n$ is the unique square symmetric matrix $P_k(G) = [a_{ij}]$ with zero diagonal, whose entries $a_{ij}$ with $i \neq j$ are defined as follows:
(i) If $v_i$, $v_j \in V_r$, then $a_{ij} = 2$ or $-1$ according as $v_iv_j$ is an edge or not.
(ii) If $v_i \in V_r$ and $v_j \in V_s$ for $r \neq s$, then $a_{ij} = 1$ or $0$ according as $v_iv_j$ is an edge or not.
For all $V_i$ and $V_j$ in $P_k$, $i \neq j$ remove the edges between vertices of $V_i$ and $V_j$ and add the edges between the vertices of $V_i$ and $V_j$ which are not in $G$, the resulting graph is called *k-complement* of $G$ and is denoted by $\overline{(G)_k}$. For each set $V_r$ in $P_k$, remove the edges of $G$ joining the vertices within $V_r$ and add the edges of $\overline{G}$ (complement of $G$) joining the vertices of $V_r$, the graph obtained is called $k(i)$-*complement* and is denoted by $\overline{(G)_{k(i)}}$. The *k-partition energy* of a graph $G$ with respect to partition $P_k$ is denoted by $E_{P_k}(G)$ and is defined as the sum of the absolute values of $k$-partition eigenvalues of $P_k(G)$. In this paper we construct some graphs such that the graph and its 2-complement are equienergetic with respect to a given partition. We also determine partition energy of complete product of $m$ copies of a circulant graph $G$ and its subgraph, their $k$-complement and $k(i)$-complement.

KEYWORDS: $k$-partition eigenvalues, $k$-partition energy,Complete product, Equienergetic graphs, Block circulant matrix.
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# 1 Introduction

In graph theory several matrices like adjacency matrix, Laplacian matrix, distance matrix, are associated with a graphs and studied extensively for more than 30 years. Motivated by this recently E. Sampathkumar and M. A. Sriraj in [8] have introduced $L$-matrix of $G = (V, E)$ of order $n$ with respect to a partition $P_k = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V$. It is a unique square symmetric matrix $P_k(G) = [a_{ij}]$ whose entries $a_{ij}$ are defined as follows:

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i,\, v_j \in V_r, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where } v_i,\, v_j \in V_r, \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ & V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and } v_j \in V_s, \\ 0 & \text{otherwise} \end{cases}$$

This $L$-matrix determines the partition of vertex set of graph $G$ uniquely. The partition of $V$ into independent sets $V_1, V_2, \cdots, V_k$ leads to vertex coloring of graph $G$. Note that if $k$ is the chromatic number of $G$, then $k$-partition energy and color energy introduced by C.Adiga et.al in [1] are same. Further in [9], we have introduced $k$-partition energy of a graph $G$ denoted by $E_{P_k}(G)$ and defined as the sum of the absolute values of $k$-partition eigenvalues of $G$. Here $k$-partition eigenvalues of $G$ are eigenvalues of $P_k(G)$. If the vertex set of a graph $G$ of order $n$ is partitioned into $n$ sets then the partition energy coincides with the usual energy of a graph. So partition energy may be considered as a generalization of energy of a graph introduced by I. Gutman in [3]. For more details on graph energy see [4].

In this paper we construct some graphs such that the graph and its 2-complement are equienergetic with respect to a given partition. We also determine partition energy of complete product of $m$ copies of a circulant graph $G$ and its subgraph, their $k$-complement, $k(i)$-complement.

## PRELIMINARIES

In this section, we give some definitions and results which are useful to prove our main results.

**Definition 1.1.** *[6] Let $G$ be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set $V$. Then the $k$-complement of $G$ is obtained as follows: For all $V_i$ and $V_j$ in $P_k$, $i \neq j$ remove the edges between $V_i$ and $V_j$ and add the edges between the vertices of $V_i$ and $V_j$ which are not in $G$ and is denoted by $\overline{(G)_k}$.*

The matrix of $k$-complement is obtained from $L$-matrix $P_k(G)$ as follows: In $P_k(G)$ interchange 1 and 0 in the non-principal diagonal entries. The matrix thus obtained is the matrix of $\overline{G_k}$ and denoted by $P_k(\overline{(G)_k})$.

**Definition 1.2.** *[7] Let $G$ be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set $V$. Then the $k(i)$ complement of $G$ is obtained as follows: For each set $V_r$ in $P_k$, remove the edges of $G$ joining the vertices within $V_r$ and add the edges of $\overline{G}$ (complement of $G$) joining the vertices of $V_r$, and is denoted by $\overline{(G)_{k(i)}}$.*

The matrix of $k(i)$-complement is obtained by interchanging 2 and $-1$ in the matrix $P_k(G)$ and is denoted by $P_k(\overline{(G)_{k(i)}})$.

**Definition 1.3.** *The complete product of two graphs $G_1$ and $G_2$ is obtained by joining every vertex of $G_1$ to every vertex of the other graph $G_2$ and denoted by $G_1 \nabla G_2$.*

**Definition 1.4.** *Two graphs of same order are said to be equienergetic if $E(G_1) = E(G_2)$.*

**Definition 1.5.** *[2] Let $A_1, A_2, \cdots, A_m$ be square matrices of order $n$. A block circulant matrix of type $(m, n)$ (of order $mn$) is an $mn \times mn$ matrix of the form*

$$bcirc(A_1, A_2, \cdots, A_m) = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ A_m & A_1 & \dots & A_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \dots & A_1 \end{pmatrix}.$$

**Definition 1.6.** *[2] Let $A$ be of type $(m, n)$. If this matrix is circulant and if each block is circulant, then $A$ is called a block circulant with circulant blocks and we say that it is of the class $BCCB_{m,n}$.*

**Definition 1.7.** *[2] Let $A$ and $B$ be matrices of order $m \times n$ and $p \times q$ respectively. Then the Kronecker product(Tensor or Direct product of $A$ and $B$)*
*is the $mp \times nq$ matrix defined by*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Now we state the following results which are used for computation of spectrum of block circulant matrices.

**Lemma 1.8.** *[2] Let*

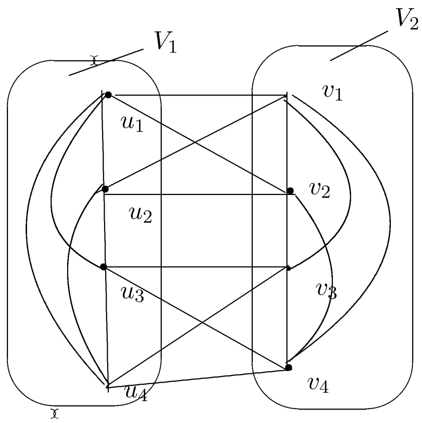
$$A = \left[\begin{array}{c|c} A_0 & A_1 \\ \hline A_1 & A_0 \end{array}\right]$$

*be a symmetric $2 \times 2$ block matrix. Then the spectrum of $A$ is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.*

**Theorem 1.9.** *[2] All matrices in $BCCB_{m,n}$ are simultaneously diagonalizable by the unitary matrix $F_m \otimes F_n$. Hence they commute. If the eigenvalues of the circulant blocks are given by $\wedge_{k+1}, k = 0, 1, 2, \cdots, m-1$, the diagonal matrix of the eigenvalues of the BCCB matrix is given by $\sum_{k=0}^{m-1} \Omega_m^k \otimes \wedge_{k+1}$ where $\wedge_{k+1} = diag(\lambda_1^{(k+1)}, \lambda_2^{(k+1)}, \cdots, \lambda_n^{(k+1)})$, $\Omega_m^k = diag(1^k, w^k, \cdots, w^{(m-1)k})$ and $\omega = exp(\frac{2\pi i}{m})$.*

## 2 Construction of some equienergetic graphs with respect to a given partition

In this section, we construct some new graphs such that 2-partition energy of the graph and its 2-complement are same. For $n \geq 2$ and $n \equiv 0(mod\ d)$, we consider two copies of $K_n$ with vertex sets, $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ respectively. Now we construct a $(n-1+d)$-regular graph $G = (V, E)$, with $V = V_1 \cup V_2$, and the edge set $E = \{u_iu_j, v_iv_j, u_{t+i}v_{t+j} : \ \forall\ i, j \in \{1, 2, \ldots, d\}\}$, where $t = 0, d, 2d, \ldots, (l-1)d$.




In the above graph we have $n = 4$ and $d = 2$, $P_2 = \{V_1, V_2\}$. Its matrix $P_2(G)$ is

$$\left(\begin{array}{cccc|cccc}
0 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\
2 & 0 & 2 & 2 & 1 & 1 & 0 & 0 \\
2 & 2 & 0 & 2 & 0 & 0 & 1 & 1 \\
2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \\
1 & 1 & 0 & 0 & 2 & 0 & 2 & 2 \\
0 & 0 & 1 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 2 & 2 & 2 & 0
\end{array}\right).$$

In the following theorem we find the 2-partition energy of the above constructed graph $G$, its 2-complement, $2(i)$-complement and discuss the conditions under which the graphs are equienergetic with respect to the given partition.

**Theorem 2.1.** *Let $G = (V, E)$ be the graph constructed as above and $P_2 = \{V_1, V_2\}$ be a partition of $V$ where $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Then*
*$(i)$ $G$ and $\overline{(G)_2}$ are equienergetic with respect to $P_2$. Moreover*
$E_{P_2}(G) = E_{P_2}(\overline{(G)_2}) = 10n - 4\frac{n}{d} - 2d - 4$, *where* $d \neq 1$,

*(ii)* $E_{P_2}(\overline{(G)_{2(i)}}) = 6n - 2\frac{n}{d} - 2d - 2$, *where* $d \neq 1$.

*Proof.* (i) The matrix of the graph $G$ stated above with $P_2 = \{V_1, V_2\}$ is of the form

$$P_2(G) = \left(\begin{array}{c|c} A & B \\ B & A \end{array}\right).$$

Here $A$ and $B$ are block circulant matrices of order $n$ with $l$ blocks of order $d$. In $A$, the blocks corresponding to principal diagonal are such that, all its principal diagonal entries are 0's, non principal diagonal entries are 2 and all other blocks have the entries 2. In $B$, the blocks corresponding to principal diagonal are of order $d$ and all its entries are 1's and entries in the remaining blocks are 0's. By Lemma 1.8, the spectrum of $P_2(G)$ is the union of the spectra of $A + B$ and $A - B$. Since $A$ and $B$ are block circulants with circulant blocks, it follows that $A+B$ and $A-B$ are also block circulants with circulant blocks. From Theorem 1.9 we can write the diagonal form of $A + B$ as

$$\sum_{k=0}^{l-1} \Omega_l^k \otimes \wedge_{k+1},$$

where $\wedge_{k+1} = diag(\lambda_1^{(k+1)}, \lambda_2^{(k+1)}, \cdots, \lambda_d^{(k+1)})$ represents the diagonal matrix of eigenvalues of $A_{k+1}$ and $\Omega_l^k = diag(1^k, w^k, \cdots, w^{(l-1)k})$.

Here $A+B$ is of the form $\begin{pmatrix} A_1 & A_2 & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & A_1 \end{pmatrix}_{n\times n}$ and $A-B$ is of the form $\begin{pmatrix} B_1 & A_2 & \cdots & A_2 \\ A_2 & B_1 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & B_1 \end{pmatrix}_{n\times n}$,

where $A_1 = \begin{pmatrix} 1 & 3 & \dots & 3 \\ 3 & 1 & \dots & 3 \\ \vdots & \vdots & \ddots & \vdots \\ 3 & 3 & \dots & 1 \end{pmatrix}_{d\times d}$ $A_2 = \begin{pmatrix} 2 & 2 & \dots & 2 \\ 2 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 2 \end{pmatrix}_{d\times d}$ and $B_1 = \begin{pmatrix} -1 & 1 & \dots & 1 \\ 1 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -1 \end{pmatrix}_{d\times d}$.

Thus, we get the diagonal form of $A + B$ as

$$\begin{pmatrix} \wedge_1 + (l-1)\wedge_2 & 0 & \dots & 0 \\ 0 & \wedge_1 - \wedge_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \wedge_1 - \wedge_2 \end{pmatrix}_{n\times n}.$$

where $\wedge_1 = diag(3d-2, -2, -2, \dots, -2)$ and $\wedge_2 = diag(2d, 0, 0, \dots, 0)$ represent the matrices of eigenvalues of $A_1$ and $A_2$ respectively.

Hence, eigenvalues of $A+B$ are

$$\begin{cases} 2(n-1)+d & once \\ d-2 & l-1 \quad times \\ -2 & n-l \quad times. \end{cases}$$

Similarly the diagonal form of $A-B$ is

$$\begin{pmatrix} \wedge_1^1+(l-1)\wedge_2 & 0 & \dots & 0 \\ 0 & \wedge_1^1-\wedge_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \wedge_1^1-\wedge_2 \end{pmatrix}_{n\times n},$$

where $\wedge_1^1 = diag(d-2,-2,-2,\dots,-2)$ and $\wedge_2 = diag(2d,0,0,\dots,0)$ are matrices of eigenvalues of $B_1$ and $A_2$ respectively.
Thus, eigenvalues of $A-B$ are

$$\begin{cases} 2(n-1)-d & once \\ -d-2 & l-1 \quad times \\ -2 & n-l \quad times. \end{cases}$$

Hence, 2-partition eigenvalues of $G$ are

$$\begin{cases} 2(n-1)+d & once \\ 2(n-1)-d & once \\ d-2 & l-1 \quad times \\ -d-2 & l-1 \quad times \\ -2 & 2n-2l \quad times. \end{cases}$$

Therefore,

$E_{P_2}(G) = \mid 2(n-1)+d \mid + \mid 2(n-1)-d \mid +(l-1) \mid (d-2) \mid +(2n-2l) \mid (-2) \mid +(l-1) \mid (-2-d) \mid .$

$E_{P_2}(G) = 10n-4l-2d-4 = 10n-4\frac{n}{d}-2d-4 \quad for \quad d \neq 1.$
Further, if $d=1$, $d=2$ and $d=n$, then $E_{P_2}(G) = 8(n-1)$.

Also the matrix of $\overline{(G)_2}$ is

$$P_2\overline{(G)_2} = \left(\begin{array}{c|c} A & C \\ \hline C & A \end{array}\right).$$

In $C$, all the blocks corresponding to principal diagonal are null matrices and remaining blocks have the entries 1. Proceeding as above, the eigenvalues of $A+C$ are

$$\begin{cases} 3n-d-2 & once \\ -d-2 & l-1 \quad times \\ -2 & n-l \quad times. \end{cases}$$

Similarly the eigenvalues of $A-C$ are

$$\begin{cases} n-2+d & once \\ d-2 & l-1 \quad times \\ -2 & n-l \quad times. \end{cases}$$

Hence, the 2-partition eigenvalues of $P_2\overline{(G)_2}$ are

$$\begin{cases} 3n-2-d & once \\ n-2+d & once \\ d-2 & l-1 \quad times \\ -d-2 & l-1 \quad times \\ -2 & 2n-2l \quad times. \end{cases}$$

$E_{P_2}(\overline{(G)_2}) = \mid 3n{-}2{-}d \mid + \mid n{-}2{+}d \mid +(l{-}1) \mid (d{-}2) \mid +(2n{-}2l) \mid (-2) \mid +(l{-}1) \mid (-2{-}d) \mid .$
$E_{P_2}(\overline{(G)_2}) = 10n-4l-2d-4 = 10n-4\frac{n}{d}-2d-4. \ \ for \ \ d \neq 1$
For $d=1$ and $d=n$, $E_{P_2}(\overline{(G)_2}) = 8(n-1)$.
Thus $G$ and $\overline{(G)_2}$ are equienergetic with respect to the given partition $P_2$.

(iii) The matrix of $\overline{(G)_{2(i)}}$ is

$$P_2\overline{(G)_{2(i)}} = \left(\begin{array}{c|c} D & B \\ \hline B & D \end{array}\right),$$

where $D$ is obtained from $A$ by replacing 2 by -1. With similar discussion, the eigenvalues of $D+B$ are

$$\begin{cases} d-n+1 & once \\ d+1 & l-1 \quad times \\ 1 & n-l \quad times \end{cases}$$

and the eigenvalues of $D-B$ are

$$\begin{cases} 1-d-n & once \\ 1-d & l-1 \quad times \\ 1 & n-l \quad times. \end{cases}$$

Thus, the 2-partition eigenvalues of $\overline{(G)_{2(i)}}$ are

$$\begin{cases} d+1-n & once \\ 1-d-n & once \\ 1+d & l-1 \quad times \\ 1-d & l-1 \quad times \\ 1 & 2n-2l \quad times. \end{cases}$$

$\therefore E_{P_2}(\overline{(G)_{2(i)}}) = \mid d{+}1{-}n \mid + \mid 1{-}d{-}n \mid +(l{-}1) \mid (1{+}d) \mid +(2n{-}2l) \mid 1 \mid +(l{-}1) \mid (1{-}d) \mid .$

For $d = 1$,

$$E_{P_2}(\overline{(G)}_{2(i)}) = 4(n-1),$$

for $d = n$,

$$E_{P_2}(\overline{(G)}_{2(i)}) = 4n-2,$$

and

$$E_{P_2}(\overline{(G)_{2(i)}}) = 6n - 2\frac{n}{d} - 2d - 2 \quad for \quad d \neq 1.$$

□

## 3 Partition energy of complete product of $m$-copies of a circulant graph

In this section we obtain partition energies of complete product of $m$-copies of a circulant graph, its subgraph obtained by deleting some edges, their $k$-complement and $k(i)$-complement.

**Theorem 3.1.** *[9] If $G$ is a $r$-regular graph with $n$ vertices and $3r-n+1$, $\lambda_2, \lambda_3, \ldots, \lambda_n$ are eigenvalues of $P_1(G)$, then 1-partition eigenvalues of its $1(i)$-complement $\overline{(G)_{1(i)}}$ are $2n-3r-2, -\lambda_2-1, -\lambda_3-1, \ldots, -\lambda_n-1$.*

**Theorem 3.2.** *Let $G_i = (V_i, E_i)$, where $i = 1, 2, \ldots, m$ be the $i^{th}$ copy of an $r$-regular circulant graph $G = (V, E)$ of order $n$ and $S$ denote the complete product of $G_1, G_2, \ldots, G_m$. Then*

$(i) E_{P_m}(S) = mE_{P_1}(G) - m \mid 3r-n+1 \mid + \mid 3r-n+1+n(m-1) \mid + (m-1) \mid 3r-2n+1 \mid$.

$(ii)$ $E_{P_m}(\overline{(S)_m}) = mE_{P_1}(G)$.

$(iii) E_{P_m}\overline{(S)_{m(i)}} = \mid 2n-3r-2+m(n-1) \mid + (m-1) \mid n-3r-2 \mid - m \mid 2n-3r-2 \mid + m(E_{P_1}\overline{(G)_{1(i)}})$,

*where $P_m = \{V_1, V_2, V_3, \ldots, V_m\}$ is the partition of the vertex set of $S$.*

*Proof.* (i) The matrix of $S$ with respect to $P_m = \{V_1, V_2, V_3, \ldots, V_m\}$ is

$$P_m(S) = \begin{pmatrix} A_1 & A_2 & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & A_1 \end{pmatrix}_{mn \times mn},$$

where $A_1 = P_1(G)$ and $A_2 = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}_{n \times n}$.

Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the 1-partition eigenvalues of $G$. Since $G$ is $r$-regular,

$\lambda_0 = 3r - n + 1$. We know that $\wedge_1 = diag(3r - n + 1, \lambda_1, \ldots, \lambda_{n-1})$ and $\wedge_2 = diag(n, 0, 0, \ldots, 0)$ are the matrices of eigenvalues of $A_1$ and $A_2$ respectively. Then from Theorem 1.9 the diagonal form of $P_m(S)$ is

$$\begin{pmatrix} \wedge_1 + (m-1)\wedge_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \wedge_1 - \wedge_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \wedge_1 - \wedge_2 \end{pmatrix}_{mn \times mn}.$$

Here $\wedge_1 + (m-1)\wedge_2 = diag(3r - n + 1 + (m-1)n, \lambda_1, \ldots, \lambda_{n-1})$ and $\wedge_1 - \wedge_2 = diag(3r - 2n + 1, \lambda_1, \ldots, \lambda_{n-1})$.
Hence, $m$-partition eigenvalues of $S$ are

$$\begin{cases} 3r - n + 1 + (m-1)n & once \\ 3r - 2n + 1 & m - 1 \quad times \\ \lambda_1 & m \quad times \\ \lambda_2 & m \quad times \\ \vdots & \\ \lambda_{n-1} & m \quad times. \end{cases}$$

Thus,
$E_{P_m}(S) = mE_{P_1}(G) - m \mid 3r - n + 1 \mid + \mid 3r - n + 1 + n(m-1) \mid + (m-1) \mid 3r - 2n + 1 \mid .$

(ii) The matrix of $\overline{(S)_m}$ is

$$P_m(\overline{(S)_m}) = \begin{pmatrix} A_1 & B_2 & \cdots & B_2 \\ B_2 & A_1 & \cdots & B_2 \\ \vdots & \vdots & \ddots & \vdots \\ B_2 & B_2 & \cdots & A_1 \end{pmatrix}_{mn \times mn},$$

where $A_1 = P_1(G)$ and $B_2$ is a null matrix of order $n$. We know that $\wedge_1 = diag(3r - n + 1, \lambda_1, \ldots, \lambda_{n-1})$ is the matrix of eigenvalues of $A_1$ and $\wedge_2^1 = diag(0, 0, \ldots, 0)$ is the matrix of eigenvalues of $B_2$. Then, from Theorem 1.9 the diagonal form of $\overline{S_m}$ is

$$\begin{pmatrix} \wedge_1 + (m-1)\wedge_2^1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \wedge_1 - \wedge_2^1 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \wedge_1 - \wedge_2^1 \end{pmatrix}_{mn \times mn}.$$

Hence, $m$-partition eigenvalues of $\overline{(S)_m}$ are

$$\begin{cases} 3r - n + 1 & m \quad times \\ \lambda_1 & m \quad times \\ \lambda_2 & m \quad times \\ \vdots & \\ \lambda_{n-1} & m \quad times. \end{cases}$$

Thus,

$$E_{P_m}(\overline{(S)_m}) = mE_{P_1}(G).$$

(iii) The matrix of $\overline{(S)_{m(i)}}$ is

$$P_m(\overline{(S)_{m(i)}}) = \begin{pmatrix} B_1 & A_2 & \cdots & A_2 \\ A_2 & B_1 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & B_1 \end{pmatrix}_{mn\times mn},$$

where $B_1 = P_1\overline{(G)_{1(i)}}$.
We know that the matrix of eigenvalues of $A_2$ is $\wedge_2 = diag(n, 0, 0, \ldots, 0)$ and of $B_1$ is $\wedge_1^1 = diag(2n-3r-2, -\lambda_1 - 1, -\lambda_2 - 1, \ldots, -\lambda_{n-1} - 1)$ as stated in Theorem 3.1. Proceeding as in (i), we get the $m$-partition eigenvalues of $\overline{S_{m(i)}}$ as

$$\begin{cases} 2n-3r-2+n(m-1) & once \\ n-3r-2 & m-1 \quad times \\ -\lambda_1 - 1 & m \quad times \\ -\lambda_2 - 1 & m \quad times \\ \vdots & \\ -\lambda_{n-1} - 1 & m \quad times. \end{cases}$$

Thus, $E_{P_m}\overline{(S)_{m(i)}} = \mid 2n-3r-2+n(m-1) \mid + (m-1) \mid n-3r-2 \mid + m\sum_{j=1}^{n-1} \mid -\lambda_j - 1 \mid$.

Hence,

$$E_{P_m}\overline{(S)_{m(i)}} = \mid 2n-3r-2+n(m-1) \mid + (m-1) \mid n-3r-2 \mid - m \mid 2n-3r-2 \mid + m(E_{P_1}\overline{(G)_{1(i)}}).$$

□

As a consequence of the above theorem we have the following corollaries.

**Corollary 3.3.** *If $G = K_n$ where $n \geq 2$, then*
*(i)* $E_{P_m}\overline{(S)} = mE_{P_1}(K_n)$.
*(ii)*$E_{P_m}(\overline{(S)_m}) = mE_{P_1}(K_n)$.
*(iii)*$E_{P_m}\overline{(S)_{m(i)}} = 4mn - 4n - 2m + 2$.

**Corollary 3.4.** *If $G = C_n$, then*
*(i)* $E_{P_m}\overline{(S)} = mE_{P_1}(C_n) + \mid 7 - 2n + mn \mid + (m-1) \mid 7 - 2n \mid - m \mid 7 - n \mid$.
*(ii)*$E_{P_m}(\overline{(S)_m}) = mE_{P_1}(C_n)$.
*(iii)* $E_{P_m}\overline{(S)_{m(i)}} = \mid mn + n - 8 \mid + (m-1) \mid n - 8 \mid - m \mid 2n - 8 \mid + mE_{P_1}\overline{(C_n)_{1(i)}}$.

The above Corollaries 3.3 and 3.4 have been proved in [5].

**Observation 3.5.** *If we know* 1*-partition eigenvalues of circulant graph* $G$*, then we can find* $m$*-partition eigenvalues and* $m$*-partition energy of* $S$*,* $\overline{S_m}$*,* $\overline{S_{m(i)}}$*, where* $S$ *is defined as in Theorem 3.2.*

**Observation 3.6.** *The graphs* $S$ *and* $\overline{(S)_m}$ *in Corollary 3.3 are equienergetic with respect to* $P_m$*.*

**Theorem 3.7.** *Let* $G_i = (V_i, E_i)$*, where* $i = 1, 2, \ldots, m$ *be the* $i^{th}$ *copy of an* $r$*-regular circulant graph* $G = (V, E)$ *of order* $n$ *and* $S$ *denote the complete product of* $G_1, G_2, \cdots, G_m$*. If* $H = (V', E')$ *is the subgraph of* $S$ *obtained by removing the edges* $v_{is}v_{js}$ *for* $1 \le i < j \le m$ *and* $s = 1, 2, \ldots, n$*, then*
(*i*) $E_{P_m}(H) = \mid 3r - n + 1 + (m-1)(n-1) \mid + (m-1) \mid 3r - 2n + 2 \mid + \sum_{j=1}^{n-1} \mid \lambda_j - (m-1) \mid + (m-1)\sum_{j=1}^{n-1} \mid \lambda_j + 1 \mid.$
(*ii*) $E_{P_m}\overline{(H)_m} = \mid 3r - n + 1 + (m-1) \mid + (m-1) \mid 3r - n \mid + \sum_{j=1}^{n-1} \mid \lambda_j + (m-1) \mid + (m-1)\sum_{j=1}^{n-1} \mid \lambda_j - 1 \mid.$
(*iii*) $E_{P_m}\overline{(H)_{m(i)}} = |2n - 3r - 2 + (n-1)(m-1)| + (m-1)|n - 3r - 1| + \sum_{j=1}^{n-1} \mid -\lambda_j - m \mid + (m-1)\sum_{j=1}^{n-1} \mid -\lambda_j \mid,$
*where* $(3r - n + 1, \lambda_1, \ldots, \lambda_{n-1})$ *represent the* 1*-partition eigenvalues of* $G$ *and* $P_m = \{V_1, V_2, V_3, \ldots, V_m\}$*.*

*Proof.* (*i*) The matrix of $H$ with respect to $P_m = \{V_1, V_2, \ldots, V_m\}$ is of the form

$$P_m(H) = \begin{pmatrix} A_1 & A_2 & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & A_1 \end{pmatrix}_{mn \times mn},$$

where $A_1 = P_1(G)$ and $A_2 = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}_{n \times n}$.

Clearly, $\wedge_1 = diag(3r - n + 1, \lambda_1, \ldots, \lambda_{n-1})$ and $\wedge_2 = diag(n-1, -1, -1, \ldots, -1)$ represents the matrix of eigenvalues of $A_1$ and $A_2$ respectively. Then proceeding as in (*i*) of Theorem 3.2, we get the $m$-partition eigenvalues of $H$ as follows:

$$\begin{cases} 3r - n + 1 + (m-1)(n-1) & once \\ 3r - 2n + 2 & m-1 \quad times \\ \lambda_j - (m-1) & for \quad j = 1, 2, \ldots, n-1 \\ \lambda_j + 1 & for \quad j = 1, 2, \ldots, n-1, \quad m-1 \quad times. \end{cases}$$

Hence,
$E_{P_m}(H) =| 3r-n+1+(m-1)(n-1) | +(m-1) | 3r-2n+2 | + \sum_{j=1}^{n-1} | \lambda_j-(m-1) | + (m-1)\sum_{j=1}^{n-1} | \lambda_j+1 |.$

$(ii)$ The matrix of $m$-complement of $H$ is of the form

$$P_m\overline{(H)_m} = \begin{pmatrix} A_1 & B_2 & \cdots & B_2 \\ B_2 & A_1 & \cdots & B_2 \\ \vdots & \vdots & \ddots & \vdots \\ B_2 & B_2 & \cdots & A_1 \end{pmatrix}_{mn\times mn},$$

where $A_1 = P_1(G)$ and $B_2 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}_{n\times n}$.

Clearly, $\wedge_1 = diag(3r-n+1, \lambda_1, \dots, \lambda_{n-1})$ and $\wedge_2^1 = diag(1, 1, 1, \dots, 1)$ represent the matrix of eigenvalues of $A_1$ and $B_2$ respectively. With arguments similar to those in Theorem 3.2, we get the $m$-partition eigenvalues of $\overline{(H)_m}$ as follows:

$$\begin{cases} 3r-n+1+(m-1) & once \\ 3r-n & m-1 \quad times \\ \lambda_j+(m-1) & for \quad j=1,2,\dots,n-1 \\ \lambda_j-1 & for \quad j=1,2,\dots,n-1, \quad m-1 \quad times. \end{cases}$$

Thus,
$E_{P_m}\overline{(H)_m} =| 3r-n+1+(m-1) | +(m-1) | 3r-n | + \sum_{j=1}^{n-1} | \lambda_j+(m-1) | + (m-1)\sum_{j=1}^{n-1} | \lambda_j-1 |.$

$(iii)$ The matrix of $m(i)$-complement of $H$ is of the form

$$P_m\overline{(H)_{m(i)}} = \begin{pmatrix} B_1 & A_2 & \cdots & A_2 \\ A_2 & B_1 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & B_1 \end{pmatrix}_{mn\times mn},$$

where $B_1 = P_1\overline{(G)_{1(i)}}$ and $A_2$ as in (i). From Theorem 2.2, $\wedge_1^1 = diag(2n-3r-2, -\lambda_1-1, \dots, -\lambda_{n-1}-1)$ and $\wedge_2 = diag(n-1, -1, -1, \cdots, -1)$ represents the matrix of eigenvalues of $B_1$ and $A_2$ respectively. With arguments similar to those in Theorem 3.2, we get the $m$-partition eigenvalues of $\overline{(H)_{m(i)}}$ as follows:

$$\begin{cases} 2n-3r-2+(n-1)(m-1) & once \\ n-3r-1 & m-1 \quad times \\ -\lambda_j-m & for \quad j=1,2,\dots,n-1 \\ -\lambda_j & for \quad j=1,2,\dots,n-1, \quad m-1 \quad times. \end{cases}$$

Hence,

$$E_{P_m}\overline{(H)_{m(i)}} = \mid 2n-3r-2+(n-1)(m-1) \mid + (m-1) \mid n-3r-1 \mid + \sum_{j=1}^{n-1} \mid -\lambda_j - m \mid + (m-1)\sum_{j=1}^{n-1} \mid -\lambda_j \mid .$$

□

As a consequence of the above theorem, we have the following corollary.

**Corollary 3.8.** *If $G = K_n$ where $n \geq 2$, then*
*(i)* $E_{P_m}\overline{(H)} = 4m(n-1) = mE_{P_1}(K_n)$.
*(ii)* $E_{P_m}\overline{(H)_m} = 6(m-1)(n-1)$, *for* $m > 2$ *and*
$E_{P_m}\overline{(H)_m} = 8(n-1) = 2E_{P_1}(K_n)$ *for* $m$=2.
*(iii)* $E_{P_m}\overline{(H)_{m(i)}} = 2(n-1)(3m-4)$, *for* $m > 2$ *and*
$E_{P_m}\overline{(H)_{m(i)}} = 4(n-1) = 2E_{P_1}\overline{(K_n)_{1(i)}}$, *for* $m = 2$.

**Observation 3.9.** *The graphs $S$ considered in Corollary 3.3 (i) and $H$ in Corollary 3.8 (i) are equienergetic with respect to the given partition $P_m$.*

**Observation 3.10.** *For $m = 2$, the graphs $H$ and $\overline{H_m}$ are equienergetic.*

**Theorem 3.11.** *[9] Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two $r_1$, $r_2$ regular graphs of order $n_1$ and $n_2$. Suppose $V = V_1 \cup V_2$ is the 2-partition of $G_1 \nabla G_2 = (V, E)$. Then*

$$E_{P_2}[G_1 \nabla G_2] = E_{P_1}[G_1] + E_{P_1}[G_2] - \mid 3r_1 - n_1 + 1 \mid - \mid 3r_2 - n_2 + 1 \mid + \mid \alpha_1 \mid + \mid \alpha_2 \mid,$$

*where*

$$\alpha_1 = \frac{3(r_1+r_2) - (n_1+n_2) + 2 + \sqrt{9(r_1-r_2)^2 + (n_1+n_2)^2 - 6(n_1-n_2)(r_1-r_2)}}{2},$$

$$\alpha_2 = \frac{3(r_1+r_2) - (n_1+n_2) + 2 - \sqrt{9(r_1-r_2)^2 + (n_1+n_2)^2 - 6(n_1-n_2)(r_1-r_2)}}{2}.$$

**Theorem 3.12.** *Let $G_i = (V_i, E_i)$, where $i = 1, 2, \ldots, m$ be the $i^{th}$ copy of an $r$-regular circulant graph $G = (V, E)$ of order $n$ and $S$ denote the complete product of $G_1, G_2, \cdots, G_m$, then*

*(i)* $E_{P_2}(S) = (m-2) \mid 3r - 3n + 1 \mid -m \mid 3r - n + 1 \mid + mE_{P_1}(G) + \mid \alpha_1 \mid + \mid \alpha_2 \mid$,

*where* $\alpha_1 = \dfrac{6r - 6n + 2mn + 2 + \sqrt{3(2ln - mn)^2 + (mn)^2}}{2}$,

$\alpha_2 = \dfrac{6r - 6n + 2mn + 2 - \sqrt{3(2ln - mn)^2 + (mn)^2}}{2}$,
$P_2 = \{U_1, U_2\}$, $U_1 = V_1 \cup V_2 \cup \cdots \cup V_l$, *and* $U_2 = V_{l+1} \cup V_{l+2} \cup \cdots \cup V_m$.

$(ii)$ $E_{P_2}\overline{(S)_2} = E_{P_1}(G_1) + E_{P_1}(G_2)$.

$(iii)$ $E_{P_2}\overline{(S)_{2(i)}} = (m-1) \mid 3n-3r-2 \mid -m \mid 2n-3r-2 \mid + \mid 3n-3r-mn-2 \mid + mE_{P_1}\overline{(G)_{1(i)}}$.

*Proof.* $(i)$ The matrix of $S$ with respect to $P_2 = \{U_1, U_2\}$ is $P_2(S) = P_2(G_1 \nabla G_2)$ where $G_1$ represents the complete product of $l$ copies of $G$ and $G_2$ represents the complete product of $m-l$ copies of $G$. Clearly, $G_1$ has $nl$ vertices and is $(r+(l-1)n)$-regular and $G_2$ has $(m-l)n$ vertices and is $(r+(m-l-1)n)$-regular. Then from Theorem 3.11 we get,

$$E_{P_2}(G_1 \nabla G_2) = E_{P_1}(G_1) + E_{P_1}(G_2) - \mid 3r-3n+2ln+1 \mid - \mid 3r-3n+2mn-2ln+1 \mid + \mid \alpha_1 \mid + \mid \alpha_2 \mid, \tag{3.1}$$

where

$$\alpha_1 = \frac{6r-6n+2mn+2+\sqrt{3(2ln-mn)^2+(mn)^2}}{2},$$
$$\alpha_2 = \frac{6r-6n+2mn+2-\sqrt{3(2ln-mn)^2+(mn)^2}}{2}.$$

With simplification similar to that in Theorem 3.2, we get
$E_{P_1}(G_1) = \mid 3r-3n+2ln+1 \mid + (l-1) \mid 3r-3n+1 \mid -l \mid 3r-n+1 \mid + lE_{P_1}(G)$
and
$E_{P_1}(G_2) = \mid 3r-3n+2mn-2ln+1 \mid + (m-l-1) \mid 3r-3n+1 \mid - (m-l) \mid 3r-n+1 \mid + (m-l)E_{P_1}(G)$.

Substituting this in equation (3.1) we get,

$$E_{P_2}(S) = E_{P_2}(G_1 \nabla G_2) = (m-2) \mid 3r-3n+1 \mid -m \mid 3r-n+1 \mid + mE_{P_1}(G) + \mid \alpha_1 \mid + \mid \alpha_2 \mid .$$

$(ii)$ The matrix of 2-complement of $S$ is $P_2\overline{(S)_2} = P_2\overline{(G_1 \nabla G_2)_2} = P_2(G_1 \oplus G_2)$.

Clearly, the 2-partition eigenvalues of $\overline{(S)_2}$ are 1-partition eigenvalues of $G_1$ and $G_2$. Thus,

$$E_{P_2}\overline{(S)_2} = E_{P_1}(G_1) + E_{P_1}(G_2).$$

$(iii)$ The matrix of $2(i)$-complement of $S$ is $P_2\overline{(S)_{2(i)}} = P_2\overline{(G_1 \nabla G_2)_{2(i)}}$.

Here $\overline{(G_1)_{1(i)}}$ has $nl$ vertices and is $(n-r-1)$-regular and $\overline{(G_2)_{1(i)}}$ has $(m-l)n$ vertices and is $(n-r-1)$ - regular. Then from Theorem 3.11 we get,

$$E_{P_2}\overline{(G_1 \nabla G_2)_{2(i)}} = E_{P_1}\overline{(G_1)_{1(i)}} + E_{P_1}\overline{(G_2)_{1(i)}} - \mid 3n-3r-ln-2 \mid - \mid 3n-3r-mn+ln-2 \mid + \mid \alpha_1 \mid + \mid \alpha_2 \mid . \tag{3.2}$$

It can be easily observed that

$E_{P_1}\overline{(G_1)_{1(i)}} =\mid 3n-3r-ln-2 \mid +(l-1) \mid 3n-3r-2 \mid -l \mid 2n-3r-2 \mid +lE_{P_1}\overline{(G)_{1(i)}}$,

$E_{P_1}\overline{(G_2)_{1(i)}} =\mid 3n-3r-mn+ln-2 \mid +(m-l-1) \mid 3n-3r-2 \mid -(m-l) \mid 2n-3r-2 \mid + (m-l)E_{P_1}\overline{(G)_{1(i)}}$,

$\alpha_1 = 3n-3r-2$ and $\alpha_2 = 3n-3r-mn-2$.

Substituting this in equation (3.2) we get,

$$E_{P_2}\overline{(S)_{2(i)}} = (m-1) \mid 3n-3r-2 \mid -m \mid 2n-3r-2 \mid + \mid 3n-3r-mn-2 \mid + mE_{P_1}\overline{(G)_{1(i)}}.$$ □

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