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LINEAR SYMMETRY OF THE MODIFIED q-EULER POLYNOMIALS

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Abstract. In this paper we investigate certain symmetric properties of some sums involving the modified q-Euler polynomials. Using the fermionic p-adic invariant integral on $\mathbb{Z}_p$, we consider several sets of integral expressions on which S_4 acts transitively. Then we determine the isotropy subgroup of an integral expression in each set. As a result the symmetric identities of sums involving modified q-Euler polynomials follow immediately. Here the obtained isotropy subgroups are D_4, V_4 or S_3.

1. Introduction

It is L. Carlitz who initiated the study of the modified q-Euler polynomials. He used an inductive method to define the modified q-Euler polynomials ([1]). In this paper we use the p-adic q-integral to define the generating functions of the modified q-Euler polynomials. It is due to T. Kim who has established the theory of the p-adic q-integral ([5]). The p-adic q-integral reveals many essential properties of the modified q-Euler polynomials and the symmetric property is one of such properties ([4, 8, 9, 10]).

Throughout this paper p is a fixed odd prime number. We use the notations $\mathbb{Z}_p$ to express the ring of p-adic integers , $\mathbb{Q}_p$ the field of p-adic rational numbers and $\mathbb{C}_p$ the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. For $q, x \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$. We define the *$q$-analogue of a number x* to be $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$ and $f \in UD(\mathbb{Z}_p)$. T. Kim has introduced the fermionic p-adic q-integral $I_{-q}(f)$ on $\mathbb{Z}_p$(see[6, 9]).

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)du_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.1}$$

The q-Euler polynomials $\mathcal{E}_{n,q}(x)$ are defined by the generating function,

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x)\frac{t^n}{n!}, \quad ([7]). \tag{1.2}$$

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Also the modified q-Euler polynomials $E_{n,q}(x)$ are defined by the generating function,

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \quad ([2]). \tag{1.3}$$

Note that $\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n d\mu_{-q}(y)$ and $E_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n d\mu_{-1}(y)$.

For v and w odd positive integers, put $p(x,u,v,w) = \sum_{i=0}^{v-1}(-1)^i \int_{\mathbb{Z}_p} e^{[ux+vy+wi]_q t} d\mu_{-1}(y)$. Then

$$\begin{aligned} p(x,u,v,w) &= \sum_{i=0}^{v-1}(-1)^i \int_{\mathbb{Z}_p} e^{[ux+vy+wi]_q t} d\mu_{-1}(y) \\ &= \sum_{i=0}^{v-1}(-1)^i \lim_{N\to\infty} \sum_{y=0}^{p^N-1} e^{[ux+vy+wi]_q t}(-1)^y \\ &= \sum_{i=0}^{v-1}(-1)^i \lim_{N\to\infty} \sum_{y=0}^{p^N-1} \sum_{k=0}^{w-1} e^{[ux+v(k+wy)+wi]_q t}(-1)^{k+wy} \\ &= \sum_{i=0}^{v-1}\sum_{k=0}^{w-1}(-1)^{i+k} \lim_{N\to\infty} \sum_{y=0}^{p^N-1} e^{[ux+vwy+vk+wi]_q t}(-1)^y \end{aligned} \tag{1.4}$$

As the last line is invariant under the transposition (v,w), we obtain the following

$$p(x,u,v,w) = p(x,u,w,v). \tag{1.5}$$

We call (1.5) as 'basic' symmetry' in this paper.

In this paper we use 4 weights w_1, w_2, w_3, w_4 instead of u, v, w. Let $p = p(x, w_1, w_2, w_3, w_4)$ be an integral expression with a variable x and weights w_1, w_2, w_3, w_4 involved in the definition of p, see (2.1). For a permutation $\sigma \in S_4$ define

$$\sigma\left(p(x, w_1, w_2, w_3, w_4)\right) = p\left(x, w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}, w_{\sigma(4)}\right). \tag{1.6}$$

Then for $\sigma, \tau \in S_4$ (i) $id(p) = p$ and (ii) $(\sigma\tau)(p) = (\sigma(\tau(p)))$ holds. Hence this gives a group action naturally. For each group action we will choose a set of integral expressions so as to the action in (1.6) of S_4 is well-defined and transitive.

2. Linear Symmetry of the modified q-Euler polynomials with the isotropy group D_4

Let w_1, w_2, w_3, w_4 be odd positive integers and $1 \leq a, b, c, d \leq 4$ be distinct. In this section we consider the integral expression $p(x)$, where

$$p(x) = \sum_{i=0}^{w_a+w_b-1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a+w_b+w_c+w_d)x+(w_a+w_b)y+(w_c+w_d)i]_q t} d\mu_{-1}(y) \tag{2.1}$$

By the abuse of notation we denote $p(x)$ in (2.1) by $p(a+b+c+d, a+b, c+d)$. Then due to the 'basic' symmetry in (1.5), $p(a+b+c+d, a+b, c+d) = p(a+b+c+d, c+d, a+b)$. So there are only 3 generating functions,

$$f_1 = p(1{+}2{+}3{+}4, 1{+}2, 3{+}4), f_2 = p(1{+}2{+}3{+}4, 1{+}3, 2{+}4), f_3 = p(1{+}2{+}3{+}4, 1{+}4, 2{+}3). \quad (2.2)$$

Let $X = \{f_1, f_2, f_3\}$ and S_4 act on X naturally as in (1.5). This action is transitive on X as for $\tau = (2, 3, 4)$, $\tau(f_1) = f_2$ and $\tau^2(f_1) = f_3$

Choose $\sigma = (1, 3, 2, 4)$ and $\delta = (1, 2)$. Then

$$\begin{aligned} \sigma(f_1) &= \sigma(p(1+2+3+4, 1+2, 3+4)) = p(3+4+2+1, 3+4, 1+2) = f_1, \\ \delta(f_1) &= \delta(p(1+2+3+4, 1+2, 3+4)) = p(2+1+3+4, 2+1, 3+4) = f_1. \end{aligned} \quad (2.3)$$

So f_1 is fixed by the dihedral group $D_4 = \langle (1, 3, 2, 4), (1, 2)\rangle$. Moreover, D_4 is the isotropy subgroup of f_1, as $|X| \times |D_4| = |S_4|$.

Now we need to transform the q-analogue in the definition of $f_1 = p(1+2+3+4, 1+2, 3+4)$ to express it in the form of modified q-Euler polynomials.

$$\begin{aligned} &[(w_1 + w_2 + w_3 + w_4)x + (w_1 + w_2)y + (w_3 + w_4)i]_q \\ &= \frac{1 - q^{(w_1+w_2+w_3+w_4)x+(w_1+w_2)y+(w_3+w_4)i}}{1-q} \\ &= \left(\frac{1-q^{w_1+w_2}}{1-q}\right)\left(\frac{1 - q^{(w_1+w_2+w_3+w_4)x+(w_1+w_2)y+(w_3+w_4)i}}{1-q^{w_1+w_2}}\right) \\ &= \left(\frac{1-q^{w_1+w_2}}{1-q}\right)\left(\frac{1 - q^{(w_1+w_2)\left\{(1+\frac{w_3+w_4}{w_1+w_2})x+y+\frac{w_3+w_4}{w_1+w_2}i\right\}}}{1-q^{w_1+w_2}}\right) \\ &= [w_1 + w_2]_q \left[\left(1 + \frac{w_3+w_4}{w_1+w_2}\right)x + y + \frac{w_3+w_4}{w_1+w_2}i\right]_{q^{w_1+w_2}}. \end{aligned} \quad (2.4)$$

Hence

$$\begin{aligned} f_1 &= p(1 + 2 + 3 + 4, 1 + 2, 3 + 4) \\ &= \sum_{i=0}^{w_1+w_2-1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_1+w_2+w_3+w_4)x+(w_1+w_2)y+(w_3+w_4)i]_q t} d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{w_1+w_2-1} (-1)^i [w_1+w_2]_q^n E_{n,q^{w_1+w_2}}\left(\left(1+\frac{w_3+w_4}{w_1+w_2}\right)x + \frac{w_3+w_4}{w_1+w_2}i\right)\frac{t^n}{n!}. \end{aligned} \quad (2.5)$$

Theorem 2.1. *For $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $n \geq 0$, the following sum of modified q-Euler polynomials*

$$\sum_{i=0}^{w_1+w_2-1} (-1)^i [w_1+w_2]_q^n E_{n,q^{w_1+w_2}}\left(\left(1+\frac{w_3+w_4}{w_1+w_2}\right)x + \frac{w_3+w_4}{w_1+w_2}i\right) \quad (2.6)$$

is invariant under any permutation in the dihedral group $D_4 = \langle (1,3,2,4),(1,2)\rangle$. *Moreover,* $D_4 = \langle (1,3,2,4),(1,2)\rangle$ *is the isotropy subgroup of the integral expression* f_1 *when* S_4 *acts on* $\{f_1, f_2, f_3\}$ *in* (2.2).

3. Linear symmetry of the modified q-Euler polynomials with the isotropy group V_4 or S_3

Let w_1, w_2, w_3, w_4 be odd positive integers and $1 \leq a, b, c, d \leq 4$ be distinct. In this section we investigate the integral expressions $p(x)$ as below.

$$p(x) = \sum_{i=0}^{w_c-1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a+w_b)x+w_cy+w_di]_q t} d\mu_{-1}(y) \tag{3.1}$$

By the abuse of notation we denote $p(x)$ in (3.1) by $p(a+b,c,d)$. Then due to the 'basic' symmetry in (1.5), $q(a+b,c,d) = q(a+b,d,c)$. So there are 6 integral expressions,

$$\begin{aligned} g_1 &= q(1+2,3,4), g_2 = q(1+3,2,4), g_3 = q(1+4,2,3), \\ g_4 &= q(2+3,1,4), g_5 = q(2+4,1,3), g_6 = q(3+4,1,2). \end{aligned} \tag{3.2}$$

Let $X = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ and let S_4 act on X naturally as in (1.6). For $\sigma = (1,2,3,4)$ $\sigma(g_1) = g_4$, $\sigma^2(g_1) = g_6$ and $\sigma^3(g_1) = g_3$. Also $(2,3)(g_1) = g_2$, $(1,4)(g_1) = g_5$. Hence S_4 acts on Y transitively.

It is obvious that $(1,2)g_1 = g_1$. Due to the 'basic' symmetry in (1.5) $(3,4)g_1 = g_1$. Hence the Klein 4 group $V_4 = \langle (1,2),(3,4)\rangle$ is the isotropy group of g_1 under the action of S_4 on X.

To express the modified q-Euler polynomials generated by g_1 we prepare the q-analogue as follows.

$$\begin{aligned} &[(w_1+w_2)x + w_3y + w_4i]_q \\ &= \frac{1-q^{(w_1+w_2)x+w_3y+w_4i}}{1-q} \\ &= \left(\frac{1-q^{w_3}}{1-q}\right)\left(\frac{1-q^{(w_1+w_2)x+w_3y+w_4i}}{1-q^{w_3}}\right) \\ &= \left(\frac{1-q^{w_3}}{1-q}\right)\left(\frac{1-q^{w_3\left\{\frac{w_1+w_2}{w_3}x+y+\frac{w_4}{w_3}i\right\}}}{1-q^{w_3}}\right) \\ &= [w_3]_q \left[\frac{w_1+w_2}{w_3}x + y + \frac{w_4}{w_3}i\right]_{q^{w_3}}. \end{aligned} \tag{3.3}$$

Hence

$$
\begin{aligned}
g_1 &= p(1+2,3,4) \\
&= \sum_{i=0}^{w_3-1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_1+w_2)x+w_3y+w_4i]_q t} d\mu_{-1}(y) \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{w_3-1} (-1)^i [w_3]_q^n E_{n,q^{w_3}} \left(\frac{w_1+w_2}{w_3} x + \frac{w_4}{w_3} i \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.4}
$$

Theorem 3.1. *For $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $n \geq 0$, the following sum of modified q-Euler polynomials*

$$
\sum_{i=0}^{w_3-1} (-1)^i [w_3]_q^n E_{n,q^{w_3}} \left(\frac{w_1+w_2}{w_3} x + \frac{w_4}{w_3} i \right) \tag{3.5}
$$

is invariant under any permutation in the Klein 4 group $V_4 = \langle (1,2),(3,4) \rangle$. Moreover, $V_4 = \langle (1,2),(3,4) \rangle$ is the isotropy subgroup of the integral expression g_1 when S_4 acts on $\{g_1, \cdots, g_6\}$ in (3.2).

We add 2 more types of integral expressions. Though it is trivial to compute the isotropy groups, they are the integral expressions with the isotropy subgroups having more than 3 elements.

Let $q(x)$ be the integral expressions as follows.

$$
q(x) = \sum_{i=0}^{w_a+w_b-1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a+w_b+w_c+w_d)x+(w_a+w_b)y+(w_a+w_b)i]_q t} d\mu_{-1}(y). \tag{3.6}
$$

Let's denote $q(x)$ in (3.6) by $q(a+b+c+d, a+b, a+b)$. Then there are 6 generating functions.

$$
\begin{aligned}
&h_1 = q(1+2+3+4, 1+2, 1+2), h_2 = q(1+2+3+4, 1+3, 1+3), \\
&h_3 = q(1+2+3+4, 1+4, 1+4), h_4 = q(1+2+3+4, 2+3, 2+3), \\
&h_2 = q(1+2+3+4, 2+4, 2+4), h_3 = q(1+2+3+4, 3+4, 3+4).
\end{aligned} \tag{3.7}
$$

It is obvious that 2 transpositions $(1,2)$ and $(3,4)$ fix h_1. So the Klein 4 group $V_4 = \langle (1,2),(3,4) \rangle$ is the isotropy subgroup of the integral expression h_1 when S_4 acts on $\{h_1, \cdots, h_6\}$ in (3.7). The integral expression h_1 induces the modified q-Euler polynomials as follows.

$$
\sum_{i=0}^{w_1+w_2-1} (-1)^i [w_1+w_2]_q^n E_{n,q^{w_1+w_2}} \left(\left(1 + \frac{w_3+w_4}{w_1+w_2} \right) x + i \right). \tag{3.8}
$$

Consider the integral expression $\hat{q}(x)$,

$$
\hat{q}(x) = \sum_{i=0}^{w_a-1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a+w_b+w_c+w_d)x+w_a y+w_a i]_q t} d\mu_{-1}(y). \tag{3.9}
$$

Let's denote $\hat{q}(x)$ in (3.9) by $\hat{q}(a+b+c+d, a, a)$. Then there are 4 integral expressions.

$$\hat{h}_a = q(1+2+3+4, a, a), a = 1, 2, 3, 4. \tag{3.10}$$

It is obvious that the symmetric group S_3 of all permutations on $\{2, 3, 4\}$ fixes $\hat{h}_1$. So the symmetric group S_3 is the isotropy subgroup of the integral expression $\hat{h}_1$. The integral expression $\hat{h}_1$ induces the modified q-Euler polynomials as follows.

$$\sum_{i=0}^{w_1-1} (-1)^i \, [w_1]_q^n \, E_{n,q^{w_1}} \left(\left(1 + \frac{w_2 + w_3 + w_4}{w_1} \right) x + i \right). \tag{3.11}$$

Finally we ask a problem whether D_n or A_n $(n \geq 5)$ is an isotropy subgroup of modified q-Euler polynomials when S_n acts naturally on a prescribed set of integral expressions involving n weights.

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