Advanced Studies in Contemporary Mathematics
28 (2018), No. 2, pp. 167 - 176

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.2.167

Intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets

Krassimir Atanassov

Dept. of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria,
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8010, Bulgaria
e-mail: krat@bas.bg

Abstract

The basic definitions of the concept of interval-valued intuitionistic fuzzy set and of the operations, relations and operators over it are given. Some of the most important applications are described. Ideas for future development of the theory of interval-valued intuitionistic fuzzy sets are discussed.

Keywords: Interval-valued intuitionistic fuzzy set, Intuitionistic fuzzy set

AMS Classification: $11C20$

1 Introduction

Intuitionistic Fuzzy Sets (IFSs; see [1, 2, 7, 8] were introduced in 1983 as an extension of the fuzzy sets, defined by Lotfi Zadeh (1921 – 2017) in [16]. During recent years, the IFSs also were object of extensions: Intuitionistic L-Fuzzy Sets [11]; IFSs of type 2 [6], that some years ago, some authors incorrectly called relaunched as Pythagorean fuzzy sets; Temporal IFSs [5, 7, 8], Multidimensional IFSs [12, 8]; and others. But the most detailed described extension of the IFSs are Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs). They appeared in 1988, when Georgi Gargov (1947-1996) and the author read M. Gorzalczany's paper [15] on Interval-Valued Fuzzy Set (IVFS). The idea of IVIFS was announced in [3] and extended in [7, 10], where the proof that IFSs and IVIFSs are equipollent generalizations of the notion of fuzzy set, is given.

Let us have a fixed universe E and its subset A. The set

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{1}$$

is called IFS and functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ represent the *degree of membership (validity, etc.)* and *non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $A \subseteq E$.* Thus, we can also define function $\pi_A : E \to [0, 1]$ by means of

$$\pi(x) = 1 - \mu(x) - \nu(x)$$

and it corresponds to *the degree of indeterminacy (uncertainty, etc.).*

For brevity, we shall write below A instead of A^*, whenever this is possible.

The IFS A is an extension of the fuzzy set B over the fixed universe E, because it has the form

$$B = \{\langle x, \mu_B(x)\rangle \mid x \in E\},$$

where

$$0 \leq \mu_B(x) \leq 1$$

and function $\mu_B : E \to [0, 1]$ represents the *degree of membership (validity, etc.)* of element $x \in E$ to fixed set $B \subseteq E$.

Obviously, each fuzzy set can be represented in the form of an IFS:

$$B = \{\langle x, \mu_B(x), 1 - \mu_B(x)\rangle \mid x \in E\}.$$

In the last 35 years, a lot of opertions, relations and operators from different types were introduced over IFSs (see, e.g. [7, 8]). For example,

$$\begin{array}{lll}
A \subset B & \text{iff} & (\forall x \in E)((\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) > \nu_B(x)) \\
& & \qquad \vee(\mu_A(x) < \mu_B(x) \;\&\; \nu_A(x) \geq \nu_B(x)) \\
& & \qquad \vee(\mu_A(x) < \mu_B(x) \;\&\; \nu_A(x) > \nu_B(x))); \\
A \subseteq B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) \geq \nu_B(x)); \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \;\&\; \nu_A(x) = \nu_B(x)); \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A + B & = & \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle \mid x \in E\}; \\
A.B & = & \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle \mid x \in E\}; \\
A@B & = & \{\langle x, (\frac{\mu_A(x)+\mu_B(x))}{2}, \frac{(\nu_A(x)+\nu_B(x))}{2}\rangle | x \in E\}.
\end{array}$$

About 20 years ago, some of my opponents asserted that the IFSs are a trivial modification, and not extension of the fuzzy sets, because each IFS A can be represented by a pair of fuzzy sets B and C. I answered them that their idea is not new one, because 10 years before them, in 1988, Toader Buhaescu introduced it in [13, 14], but without the claim that the IFSs are a modification of fuzzy sets. In addition, I mentioned that the complex numbers are represented as pairs of real numbers, but it would be unserious to claim that the complex numbers are a trivial modification of the real numbers.

In the present paper, we discuss the way for representing the IVIFSs by IFSs following Buhaescu's idea, but, of course, without to claim that the IVIFSs are "a trivial modification" of the IFSs. We will discuss some properties of this represenation.

2 On representability of interval valued intuitionistic fuzzy sets by intuitionistic fuzzy sets

First, we mention that in [4] the second geometrical interpretation of the IFSs is given (see Fig. 1). The set of the points of the interpretation triangle from Fig. 1 can be written as

$$L = \{\langle p, q\rangle \mid p, q, p + q \in [0, 1]\}.$$

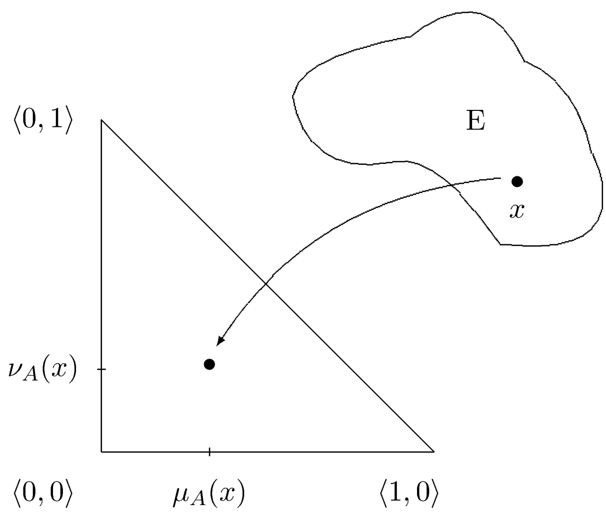

Fig. 1.

An IVIFS A over E is an object of the form:

$$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\},$$

where $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and for all $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \leq 1.$$

Obviously, this definition is analogous to the definition of IFS.

IVIFSs have geometrical interpretations similar to, but more complex than these of the IFSs. For example, the analogue of the geometrical interpretation from Figs. 1 is shown on Fig. 2.

The author is not aware of a geometrical interpretation of an Interval-Valued Fuzzy Set in the sense of Figures 1 and 2. Now, we will do this.

Obviously, each IVFS A can be represented by an IVIFS as

$$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\}$$
$$= \{\langle x, M_A(x), [1 - \sup M_A(x), 1 - \inf N_A(x)\rangle \mid x \in E\}.$$

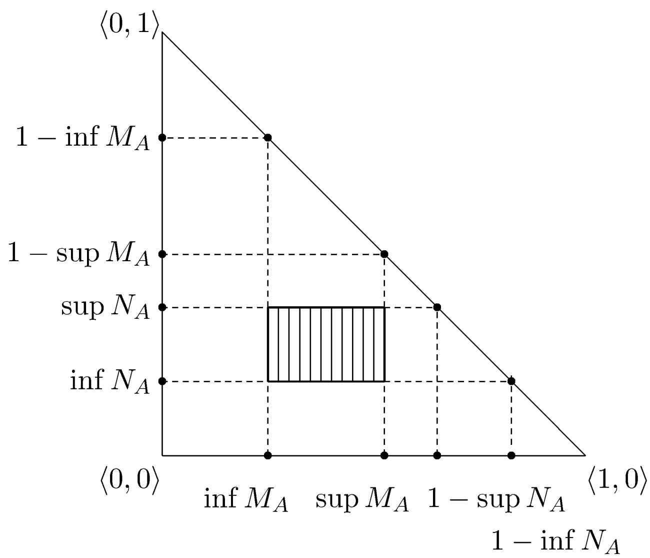

Fig. 2

The geometrical interpretation of the IVFS A is shown on Fig. 3. It has the form of a section lying on the triangles hypothenuse.

Second, let for the arbitrary element $x \in E$ with degrees a and b (i.e., $a, b, a+b \in [0,1]$), for the triple $\langle x, a, b\rangle$ we call that it is *nesting* in the IVIFS A and denote

$$\langle x, a, b\rangle \varepsilon A \text{ if and only if (iff) } a \in M_A(x), b \in N_A(x).$$

We will call that the IFS

$$C = \{\langle x, \mu_C(x), \nu_C(x)\rangle \mid x \in E\}$$

is *nesting* in the IVIFS A iff for each $x \in E$, the triple $\langle x, \mu_C(x), \nu_C(x)\rangle \varepsilon A$.

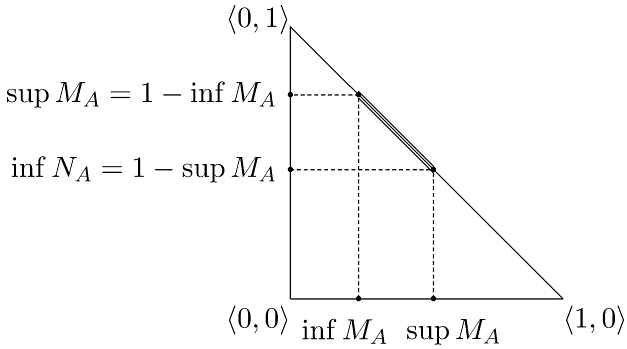

Fig. 3

Now, let us define for the fixed above IVIFS A,

$$\underline{A} = \{\langle x, \inf M_A(x), \sup N_A(x)\rangle \mid x \in E\},$$

$$\overline{A} = \{\langle x, \sup M_A(x), \inf N_A(x)\rangle \mid x \in E\}.$$

Obviously, for each IVIFS A,

$$\underline{A} \subseteq \overline{A}.$$

Moreover, sets $\underline{A}$ and $\overline{A}$ are IFSs and they are determined bijectively from A.

Theorem 1. Let A be an IVIFS and B be an IFS.

$$\underline{A} \subseteq B \subseteq \overline{A} \text{ iff } B \text{ is nesting in } A.$$

Proof. Let for the IVIFS A and IFS B, the inclusion $\underline{A} \subseteq B \subseteq \overline{A}$ be valid. Therefore for each $x \in E$:

$$\inf M_A(x) \leq \mu_B(x) \leq \sup M_A(x)$$

and

$$\inf N_A(x) \leq \nu_B(x) \leq \sup N_A(x).$$

Hence for each $x \in E$: $\mu_B(x) \in M_A(x)$ and $\nu_B(x) \in N_A(x)$, i.e. IFS B is nesting in A.

The opposite check of the present Theorem and of the next theorems are similar.

We call the two IFSs $\underline{A}$ and $\overline{A}$ *lower* and *upper* boundary IFSs of the IVIFS A.

Theorem 2. For every two IFSs B and C over universe E, such that for each $x \in E$

$$\sup M_B(x) + \sup N_C \leq 1$$

and

$$C \subseteq B,$$

there exists a unique IVIFS A with upper and lower boundary IFSs $\overline{A} = B$ and $\underline{A} = C$.

From both theorems it follows that each IVIFS can be represented by a pair of two IFSs.

3 Some propertiees of the lower and upper boundary IFSs of a given IVIFS

First, we introduce some operations and relations over two IVIFSs A and B:

$$\begin{array}{lll}
A \subset_{\Box,\inf} B & \text{iff} & (\forall x \in E)(\inf M_A(x) \leq \inf M_B(x)), \\
A \subset_{\Box,\sup} B & \text{iff} & (\forall x \in E)(\sup M_A(x) \leq \sup M_B(x)), \\
A \subset_{\Diamond,\inf} B & \text{iff} & (\forall x \in E)(\inf N_A(x) \geq \inf N_B(x)), \\
A \subset_{\Diamond,\sup} B & \text{iff} & (\forall x \in E)(\sup N_A(x) \geq \sup N_B(x)), \\
A \subset_{\Box} B & \text{iff} & A \subset_{\Box,\inf} B \ \& \ A \subset_{\Box,\sup} B, \\
A \subset_{\Diamond} B & \text{iff} & A \subset_{\Diamond,\inf} B \ \& \ A \subset_{\Diamond,\sup} B, \\
A \subseteq B & \text{iff} & A \subset_{\Box} B \ \& \ B \subset_{\Diamond} A, \\
A = B & \text{iff} & A \subseteq B \ \& \ B \subseteq A,
\end{array}$$

$$\begin{array}{lll}
\neg A & = & \{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\}, \\
A \cap B & = & \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))], \\
 & & [\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}, \\
A \cup B & = & \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x) \sup M_B(x))], \\
 & & [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\} \\
A + B & = & \{\langle x, [\inf M_A(x) + \inf M_B(x) - \inf M_A(x). \inf M_B(x), \\
 & & \sup M_A(x) + \sup M_B(x) - \sup M_A(x). \sup M_B(x)], \\
 & & [\inf N_A(x). \inf N_B(x), \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\} \\
A.B & = & \{\langle x, [\inf M_A(x). \inf M_B(x), \sup M_A(x). \sup M_B(x)], \\
 & & [\inf N_A(x) + \inf N_B(x) - \inf N_A(x). \inf N_B(x), \\
 & & \sup N_A(x) + \sup N_B(x) - \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\} \\
A@B & = & \{\langle x, [(\inf M_A(x) + \inf M_B(x))/2, (\sup M_A(x) + \sup M_B(x))/2], \\
 & & [(\inf N_A(x) + \inf N_B(x))/2, (\sup N_A(x)\rangle + \sup N_B(x))/2] \mid x \in E\}.
\end{array}$$

Theorem 3. For every two IVIFSs A and B the folowing equalities are valid:

$$\underline{A \cap B} = \underline{A} \cap \underline{B},$$

$$\underline{A \cup B} = \underline{A} \cup \underline{B},$$

$$\underline{A + B} = \underline{A} + \underline{B},$$

$$\underline{A.B} = \underline{A}.\underline{B},$$

$$\underline{A@B} = \underline{A}@\underline{B},$$

$$\overline{A \cap B} = \overline{A} \cap \overline{B},$$

$$\overline{A \cup B} = \overline{A} \cup \overline{B},$$

$$\overline{A + B} = \overline{A} + \overline{B},$$

$$\overline{A.B} = \overline{A}.\overline{B},$$

$$\overline{A@B} = \overline{A}@\overline{B},$$

$$\underline{\neg A} = \neg \overline{A},$$

$$\overline{\neg A} = \neg \underline{A}.$$

Second, we introduce the first two (classical) intuitionistic fuzzy modal operators over the IVIFS A (see [7]):

$$\Box A = \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle \mid x \in E\},$$

$$\Diamond A = \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle \mid x \in E\}.$$

Theorem 4. For each IVIFS A, the inequalities:

$$\underline{\Box A} \supseteq \Box \underline{A},$$

$$\overline{\Box A} \supseteq \Box \overline{A},$$

$$\underline{\Diamond A} \subseteq \Diamond \underline{A},$$

$$\overline{\Diamond A} \subseteq \Diamond \overline{A}$$

hold.

Third, we introduce the standard intuitionistic fuzzy topological operators over the IVIFS A (see [7]):

$$C(A) = \{\langle x, [K'_{\inf}, K'_{\sup}], [L'_{\inf}, L'_{\sup}]\rangle \mid x \in E\},$$

$$I(A) = \{\langle x, [K''_{\inf}, K''_{\sup}], [L''_{\inf}, L''_{\sup}]\rangle \mid x \in E\},$$

where:

$$
\begin{aligned}
K'_{\inf} &= \sup_{x\in E} \inf M_A(x),\\
K'_{\sup} &= \sup_{x\in E} \sup M_A(x),\\
L'_{\inf} &= \inf_{x\in E} \inf N_A(x),\\
L'_{\sup} &= \inf_{x\in E} \sup N_A(x),\\
K''_{\inf} &= \inf_{x\in E} \inf M_A(x),\\
K''_{\sup} &= \inf_{x\in E} \sup M_A(x),\\
L''_{\inf} &= \sup_{x\in E} \inf N_A(x),\\
L''_{\sup} &= \sup_{x\in E} \sup N_A(x).
\end{aligned}
$$

Theorem 5. For every IVIFS A the equalities:

$$
\begin{aligned}
C(\underline{A}) &= \underline{C(A)},\\
C(\overline{A}) &= \overline{C(A)},\\
I(\underline{A}) &= \underline{I(A)},\\
I(\overline{A}) &= \overline{I(A)}
\end{aligned}
$$

hold.

Finally, following [7], we introduce for all IVIFS A the four operators:

$$
\begin{aligned}
*_1 A = &\quad \{\langle x, \inf M_A(x), \inf N_A(x)\rangle \mid x \in E\},\\
*_2 A = &\quad \{\langle x, \inf M_A(x), \sup N_A(x)\rangle \mid x \in E\},\\
*_3 A = &\quad \{\langle x, \sup M_A(x), \inf N_A(x)\rangle \mid x \in E\},\\
*_4 A = &\quad \{\langle x, \sup M_A(x), \sup N_A(x)\rangle \mid x \in E\}.
\end{aligned}
$$

Therefore, the operators $\underline{A}$ and $\overline{A}$, defined in the present paper for a given IVIFS A, coincide with operators $*_2$ and $*_3$.

4 Conclusion

In near future, the author plans to extend the research from the present paper for the case of the extended intuitionistic fuzzy modal operators, of the intuitionistic fuzzy topological operators and of the intuitionistic fuzzy level operators. All results will be included in a book.

In [9], it is shown that the IFSs are a suitable tool for evaluation of Data mining processes and objects. In the near future, we plan to discuss the possibilities to use IVIFSs as a similar tool.

Acknowledgements

The author is thankful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. DN-02-10/2016.

References

[1] Atanassov, K. Intuitionistic fuzzy sets, VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci. - Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: Int. J. Bioautomation, 2016, 20(S1), S1-S6.

[2] Atanassov, K. Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 20, 1986, No. 1, 87-96.

[3] Atanassov K. Review and new results on intuitionistic fuzzy sets. Preprint IM-MFAIS-1-88, Sofia, 1988.

[4] Atanassov K. Geometrical interpretation of the elements of the intuitionistic fuzzy objects, Preprint IM-MFAIS-1-89, Sofia, 1989. Reprinted: Int. J. Bioautomation, 2016, 20(S1), S27-S42.

[5] Atanassov, K., Temporal intuitionistic fuzzy sets. Comptes Rendus de l'Academie bulgare des Sciences, Tome 44, 1991, No. 7, 5-7.

[6] Atanassov K. A second type of intuitionistic fuzzy sets, BUSEFAL, Vol. 56, 1993, 66-70.

[7] Atanassov, K. Intuitionistic Fuzzy Sets. Springer, Heidelberg, 1999.

[8] Atanassov, K. On Intuitionistic Fuzzy Sets Theory, Springer, Berlin, 2012.

[9] Atanassov, K. Intuitionistic fuzzy logics as tools for evaluation of Data Mining processes, Knowledge-Based Systems, Vol. 80, 2015, 122-130.

[10] Atanassov, K., G. Gargov. Interval valued intuitionistic fuzzy sets. Fuzzy Sets and Systems, Vol. 31, 1989, No. 3, 343-349.

[11] Atanassov K., S. Stoeva. Intuitionistic L-fuzzy sets, in R. Trappl (Ed.), Cybernetics and Systems Research 2 (Elsevier Sci. Publ., Amsterdam, 1984), 539-540.

[12] Atanassov, K., E. Szmidt, J. Kacprzyk, On intuitionistic fuzzy multi-dimensional sets. Issues in Intuitionistic Fuzzy Sets and Generalized Nets, Vol. 7, 2008, 1-6.

[13] Buhaescu, T. On the convexity of intuitionistic fuzzy sets. Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1988, 137-144.

[14] Buhaescu, T. Some observations on intuitionistic fuzzy rerelations. Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1989, 111-118.

[15] Gorzalczany M. Interval-valued fuzzy fuzzy inference method - some basic properties. Fuzzy Sets and Systems, Vol. 31, 1989, No. 2, 243-251.

[16] Zadeh, L. Fuzzy sets. Information and Control, Vol. 8, 1965, 338-353.