

DEGENERATE DAEHEE POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. In this paper, we consider the degenerate Daehee numbers and polynomials of the second kind which are different from the previously introduced degenerate Daehee numbers and polynomials. We investigate some properties of these numbers and polynomials. In addition, we give some new identities and relations between the Daehee polynomials of the second kind and Carlitz's degenerate Bernoulli polynomials.

1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [1, 20]}). \quad (1.1)$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

In [3], L. Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\lambda \in \mathbb{R}). \quad (1.2)$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

The falling factorial sequence is given by

$$(x)_0 = 1, \quad (x)_n = x(x-1) \cdots (x-n+1), \quad (n \geq 1). \quad (1.3)$$

The Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0), \quad (\text{see [6, 7, 8, 13, 20]}). \quad (1.4)$$

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It is well known that the stirling numbers of the second kind are defined as

$$x^n = \sum_{l=0}^n S_2(n, l)(x)_l, \quad (n \geq 0), \quad (\text{see [6, 12, 20]}).$$

From (1.1) and (1.2), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \beta_{n, \lambda}(x) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} \\ &= \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \end{aligned} \quad (1.5)$$

Thus, by (1.5), we get

$$\lim_{\lambda \rightarrow 0} \beta_{n, \lambda}(x) = B_n(x), \quad (n \geq 0). \quad (1.6)$$

The λ -analogue of falling factorial sequence is defined by L. Carlitz as follows:

$$(x)_{0, \lambda} = 1, \quad (x)_{n, \lambda} = x(x - \lambda) \cdots (x - (n - 1)\lambda), \quad (n \geq 1), \quad (\text{see [3]}). \quad (1.7)$$

By(1.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n, \lambda}(x) \frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty} \lambda^{-m} B_m(x) \frac{(\log(1 + \lambda t))^m}{m!} \right) \left(\frac{\lambda t}{\log(1 + \lambda t)} \right) \\ &= \left(\sum_{m=0}^{\infty} \lambda^{-m} B_m(x) \sum_{k=m}^{\infty} S_1(k, m) \frac{\lambda^k t^k}{k!} \right) \left(\sum_{l=0}^{\infty} b_l \frac{\lambda^l t^l}{l!} \right) \\ &= \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^k \lambda^{k-m} B_m(x) S_1(k, m) \right) \frac{t^k}{k!} \right) \left(\sum_{l=0}^{\infty} b_l \frac{\lambda^l t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \lambda^{n-m} B_m(x) S_1(k, m) b_{n-k} \right) \frac{t^n}{n!}, \end{aligned} \quad (1.8)$$

Here $b_n, (n \geq 0)$, are the Bernoulli numbers of the second kind given by the generating function

$$\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}, \quad (\text{see [1, 20]}). \quad (1.9)$$

From (1.8), we note that

$$\beta_{n, \lambda}(x) = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \lambda^{n-m} B_m(x) S_1(k, m) b_{n-k}. \quad (1.10)$$

The Daehee polynomials are defined by

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad (\text{see [10, 21 - 23]}). \quad (1.11)$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

From (1.1) and (1.11), we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} D_m(x) \frac{1}{m!} (e^t - 1)^m \\ &= \sum_{m=0}^{\infty} D_m(x) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n S_2(n, m) D_m(x) \right) \frac{t^n}{n!}. \end{aligned} \quad (1.12)$$

Comparing the coefficients on both sides of (1.12), we have

$$B_n(x) = \sum_{m=0}^n S_2(n, m) D_m(x), \quad (n \geq 0), \quad (\text{see [4, 6, 12]}). \quad (1.13)$$

The degenerate Daehee numbers are given by the generating function

$$\frac{\lambda \log(1 + \frac{1}{\lambda} \log(1 + \lambda t))}{\log(1 + \lambda t)} = \sum_{n=0}^{\infty} \tilde{D}_{n,\lambda} \frac{t^n}{n!}, \quad (\text{see [14]}). \quad (1.14)$$

Recently, many authors have studied the Daehee numbers and polynomials and the degenerate Daehee numbers and polynomials (see [2-19,21-23]).

In this paper, we consider the degenerate Daehee numbers and polynomials of the second kind which are different from the previous introduced degenerate Daehee numbers and polynomials. We investigate some properties of these numbers and polynomials. In addition, we give some new identities and relations between the Daehee polynomials of the second kind and Carlitz's degenerate Bernoulli polynomials.

2. Degenerate Daehee polynomials

For $\lambda \in \mathbb{R}$, we consider the degenerate Daehee polynomials of the second kind given by the generating function

$$\frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1} (1+\lambda \log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}. \quad (2.1)$$

When $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the degenerate Daehee numbers of the second kind.

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} D_{n,\lambda}(x) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1} (1+\lambda \log(1+t))^{\frac{x}{\lambda}} \\ &= \frac{\log(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \end{aligned} \quad (2.2)$$

Thus, by (2.2), we get

$$\lim_{\lambda \rightarrow 0} D_{n,\lambda}(x) = D_n(x), \quad (n \geq 0).$$

From (1.2), we note that

$$\begin{aligned} &\frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1} (1+\lambda \log(1+t))^{\frac{x}{\lambda}} \\ &= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \frac{1}{m!} (\log(1+t))^m \\ &= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \beta_{m,\lambda}(x) S_1(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.3)$$

Therefore, by (2.1) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_{n,\lambda}(x) = \sum_{m=0}^n \beta_{m,\lambda}(x) S_1(n, m).$$

We observe that

$$\begin{aligned}
 & \frac{\log(1+t)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1} (1+\lambda\log(1+t))^{\frac{x}{\lambda}} \\
 &= \frac{\log(1+t)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1} \sum_{l=0}^{\infty} \left(\frac{x}{\lambda}\right)_l \lambda^l \frac{1}{l!} (\log(1+t))^\lambda \\
 &= \left(\sum_{m=0}^{\infty} D_{m,\lambda} \frac{t^m}{m!}\right) \left(\sum_{l=0}^{\infty} (x)_{l,\lambda} \sum_{k=l}^{\infty} S_1(k,l) \frac{t^k}{k!}\right) \\
 &= \left(\sum_{m=0}^{\infty} D_{m,\lambda} \frac{t^m}{m!}\right) \left(\sum_{k=0}^{\infty} \left(\sum_{l=0}^k (x)_{l,\lambda} S_1(k,l)\right) \frac{t^k}{k!}\right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} (x)_{l,\lambda} S_1(k,l) D_{n-k,\lambda}\right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.4}$$

Therefore, by (2.1) and (2.4), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$D_{n,\lambda}(x) = \sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} (x)_{l,\lambda} S_1(k,l) D_{n-k,\lambda}.$$

By replacing t by $e^t - 1$ in (2.1), we get

$$\begin{aligned}
 \sum_{m=0}^{\infty} D_{m,\lambda}(x) \frac{1}{m!} (e^t - 1)^m &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} \\
 &= \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}.
 \end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned}
 \sum_{m=0}^{\infty} D_{m,\lambda}(x) \frac{1}{m!} (e^t - 1)^m &= \sum_{m=0}^{\infty} D_{m,\lambda}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n D_{m,\lambda}(x) S_2(n,m)\right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$\beta_{n,\lambda}(x) = \sum_{m=0}^n D_{m,\lambda}(x) S_2(n,m).$$

By (2.1), we get

$$\begin{aligned}
\log(1+t) &= \sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^n}{n!} \left((1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1 \right) \\
&= \frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} (1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - \sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (D_{n,\lambda}(1) - D_{n,\lambda}) \frac{t^n}{n!}.
\end{aligned} \tag{2.7}$$

On the other hand

$$\log(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n. \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$D_{n,\lambda}(1) - D_{n,\lambda} = \begin{cases} 0, & \text{if } n = 0, \\ (-1)^{n-1} (n-1)!, & \text{if } n \geq 1. \end{cases}$$

From (2.1), we have

$$\begin{aligned}
&\frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} (1 + \lambda \log(1+t))^{\frac{x}{\lambda}} \\
&= \frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{d}{\lambda}} - 1} \sum_{a=0}^{d-1} (1 + \lambda \log(1+t))^{\frac{a+x}{\lambda}} \\
&= \frac{1}{d} \left(\frac{d \log(1+t)}{(1 + \frac{\lambda}{d} (d \log(1+t))^{\frac{d}{\lambda}} - 1)} \right) \sum_{a=0}^{d-1} (1 + \frac{\lambda}{d} (d \log(1+t))^{\frac{d}{\lambda}})^{\frac{d}{\lambda} (\frac{a+x}{d})} \\
&= \frac{1}{d} \sum_{a=0}^{d-1} \sum_{m=0}^{\infty} \beta_{m, \frac{\lambda}{d}} \left(\frac{a+x}{d} \right) \frac{1}{m!} (d \log(1+t))^m \\
&= \frac{1}{d} \sum_{a=0}^{d-1} \sum_{m=0}^{\infty} \beta_{m, \frac{\lambda}{d}} \left(\frac{a+x}{d} \right) \sum_{n=m}^{\infty} d^m S_1(n, m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^n \sum_{a=0}^{d-1} d^{m-1} \beta_{m, \frac{\lambda}{d}} \left(\frac{a+x}{d} \right) S_1(n, m) \right\} \frac{t^n}{n!},
\end{aligned} \tag{2.9}$$

where $d \in \mathbb{N}$ and $n \geq 0$.

By (2.1) and (2.9), we get the following theorem.

Theorem 2.5. For $d \in \mathbb{N}$ and $n \geq 0$, we have

$$D_{n,\lambda}(x) = \sum_{m=0}^n d^{m-1} S_1(n, m) \sum_{a=0}^{d-1} \beta_{m, \frac{\lambda}{d}} \left(\frac{a+x}{d} \right).$$

On the one hand, we have

$$\begin{aligned} & \log(1+t) \sum_{l=0}^{n-1} (1 + \lambda \log(1+t))^{\frac{l}{\lambda}} \\ &= \frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} (1 + \lambda \log(1+t))^{\frac{n}{\lambda}} - \frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \\ &= \sum_{m=0}^{\infty} \{D_{m,\lambda}(n) - D_{m,\lambda}\} \frac{t^m}{m!} = t \sum_{m=0}^{\infty} \left\{ \frac{D_{m+1,\lambda}(n) - D_{m+1,\lambda}}{m+1} \right\} \frac{t^m}{m!}. \end{aligned} \quad (2.10)$$

On the other hand, we have

$$\begin{aligned} & \log(1+t) \sum_{l=0}^{n-1} (1 + \lambda \log(1+t))^{\frac{l}{\lambda}} = t \left(\frac{\log(1+t)}{t} \right) \sum_{l=0}^{n-1} (1 + \lambda \log(1+t))^{\frac{l}{\lambda}} \\ &= t \left(\sum_{p=0}^{\infty} D_p \frac{t^p}{p!} \right) \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^j (l)_{k,\lambda} S_1(j, k) \right) \frac{t^j}{j!} \right) \\ &= t \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} (l)_{k,\lambda} S_1(j, k) D_{m-j} \right) \frac{t^m}{m!}. \end{aligned} \quad (2.11)$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.6. For $n \in \mathbb{N}$, $m \geq 0$, we have

$$\frac{1}{m+1} (D_{m+1,\lambda}(n) - D_{m+1,\lambda}) = \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} (l)_{k,\lambda} S_1(j, k) D_{m-j}.$$

For $r \in \mathbb{N}$, we define the higher-order degenerate Daehee polynomials of the second kind given by the generating function

$$\left(\frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^r (1 + \lambda \log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} D_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}. \quad (2.12)$$

When $x = 0$, $D_{n,\lambda}^{(r)} = D_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Daehee numbers of the second kind.

From (2.12), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} D_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \left(\frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda \log(1+t))^{\frac{x}{\lambda}} \\ &= \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \end{aligned}$$

where $D_n^{(r)}(x)$ are called the higher-order Daehee polynomials.

As is well known, the higher-order degenerate Bernoulli polynomials are considered by L. Carlitz as follows:

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}. \quad (2.13)$$

Note that $\lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x)$, ($n \geq 0$), where $B_n^{(r)}(x)$ are the higher-order Bernoulli polynomials.

From (2.12), we note that

$$\begin{aligned} &\left(\frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda \log(1+t))^{\frac{x}{\lambda}} \\ &= \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(r)}(x) \frac{1}{m!} (\log(1+t))^m = \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(r)}(x) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \beta_{m,\lambda}^{(r)} S_1(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.14)$$

Thus, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$D_{n,\lambda}^{(r)}(x) = \sum_{m=0}^n \beta_{m,\lambda}^{(r)} S_1(n, m).$$

By replacing t by $e^t - 1$ in (2.12), we get

$$\begin{aligned}
 \sum_{m=0}^{\infty} D_{m,\lambda}^{(r)}(x) \frac{1}{m!} (e^t - 1)^m &= \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right) (1 + \lambda t)^{\frac{x}{\lambda}} \\
 &= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}.
 \end{aligned} \tag{2.15}$$

On the other hand,

$$\begin{aligned}
 \sum_{m=0}^{\infty} D_{m,\lambda}^{(r)}(x) \frac{1}{m!} (e^t - 1)^m &= \sum_{m=0}^{\infty} D_{m,\lambda}^{(r)}(x) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n D_{m,\lambda}^{(r)}(x) S_2(n, m) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.16}$$

Therefore, by (2.15) and (2.16), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(r)}(x) = \sum_{m=0}^n D_{m,\lambda}^{(r)}(x) S_2(n, m).$$

For $r, k \in \mathbb{N}$, with $r > k$, by (2.12), we get

$$\begin{aligned}
 &\left(\frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^r (1 + \lambda \log(1+t))^{\frac{x}{\lambda}} \\
 &= \left(\frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^{r-k} \left(\frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^k (1 + \lambda \log(1+t))^{\frac{x}{\lambda}} \\
 &= \left(\sum_{l=0}^{\infty} D_{l,\lambda}^{(r-k)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} D_{m,\lambda}^{(k)}(x) \frac{t^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} D_{l,\lambda}^{(r-k)} D_{n-l,\lambda}^{(k)}(x) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.17}$$

Therefore, by (2.12) and (2.17), we obtain the following theorem.

Theorem 2.9. *For $r, k \in \mathbb{N}$, with $r > k$, we have*

$$D_{n,\lambda}^{(r)}(x) = \sum_{l=0}^n \binom{n}{l} D_{l,\lambda}^{(r-k)} D_{n-l,\lambda}^{(k)}(x), \quad (n \geq 0).$$

It is well known that

$$\left(\frac{t}{\log(1+t)}\right)^k (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-k+1)}(x) \frac{t^n}{n!}, \quad (k \in \mathbb{Z}), \quad (2.18)$$

where $B_n^{(\alpha)}(x)$ are called the higher-order Bernoulli polynomials which are given by the generating function

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}.$$

Thus, by (2.18), we get

$$\begin{aligned} & \left(\frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda \log(1+t))^{\frac{x}{\lambda}} \\ &= \left(\frac{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1}{\frac{1}{\lambda} \log(1+\lambda \log(1+t))}\right)^{-r} (1+\lambda \log(1+t))^{\frac{x}{\lambda}} \left(\frac{\lambda \log(1+t)}{\log(1+\lambda \log(1+t))}\right)^r \\ &= \left(\sum_{m=0}^{\infty} B_m^{(m+r+1)}(x+1) \frac{1}{m!} \left((1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1\right)^m\right) \\ & \quad \times \left(\frac{\lambda \log(1+t)}{\log(1+\lambda \log(1+t))}\right)^r. \end{aligned} \quad (2.19)$$

From (2.18), we note that

$$\begin{aligned} \left(\frac{\lambda \log(1+t)}{\log(1+\lambda \log(1+t))}\right)^r &= \sum_{j=0}^{\infty} B_j^{(j-r+1)}(1) \frac{1}{j!} \lambda^j (\log(1+t))^j \\ &= \sum_{l=0}^{\infty} \left(\sum_{j=0}^l B_j^{(j-r+1)}(1) \lambda^j S_1(l, j)\right) \frac{t^l}{l!}. \end{aligned} \quad (2.20)$$

As is known, the degenerate Stirling numbers of the second kind are defined by the generating function

$$\frac{1}{m!} ((1+\lambda t)^{\frac{1}{\lambda}} - 1)^m = \sum_{n=m}^{\infty} S_{2,\lambda}(n, m) \frac{t^n}{n!}, \quad (2.21)$$

where $m \in \mathbb{N}$ with $m \geq 0$, (see [7]).

Note that $\lim_{\lambda \rightarrow 0} S_{2,\lambda}(n, m) = S_2(n, m)$, ($n, m \geq 0$). Also, we note that

$$\begin{aligned}
 & \sum_{m=0}^{\infty} B_m^{(m+r+1)}(x+1) \frac{1}{m!} \left((1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1 \right)^m \\
 &= \sum_{m=0}^{\infty} B_m^{(m+r+1)}(x+1) \sum_{k=m}^{\infty} S_{2,\lambda}(k, m) \frac{1}{k!} (\log(1+t))^k \\
 &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^k B_m^{(m+r+1)}(x+1) S_{2,\lambda}(k, m) \right) \sum_{p=k}^{\infty} S_1(p, k) \frac{t^p}{p!} \\
 &= \sum_{p=0}^{\infty} \left\{ \sum_{k=0}^p \sum_{m=0}^k B_m^{(m+r+1)}(x+1) S_{2,\lambda}(k, m) S_1(p, k) \right\} \frac{t^p}{p!}.
 \end{aligned} \tag{2.22}$$

From (2.19), (2.20), and (2.22), we have

$$\begin{aligned}
 & \left(\frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^r (1 + \lambda \log(1+t))^{\frac{x}{\lambda}} \\
 &= \left(\sum_{p=0}^{\infty} \left\{ \sum_{k=0}^p \sum_{m=0}^k B_m^{(m+r+1)}(x+1) S_{2,\lambda}(k, m) S_1(p, k) \right\} \frac{t^p}{p!} \right) \\
 & \quad \times \left(\sum_{l=0}^{\infty} \left(\sum_{j=0}^l B_j^{(j-r+1)}(1) \lambda^j S_1(l, j) \right) \frac{t^l}{l!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{p=0}^n \sum_{k=0}^p \sum_{m=0}^k \sum_{j=0}^{n-p} \binom{n}{p} B_m^{(m+r+1)}(x+1) B_j^{(j-r+1)}(1) \lambda^j S_{2,\lambda}(k, m) S_1(p, k) \right. \\
 & \quad \left. \times S_1(n-p, j) \right) \frac{t^n}{n!}
 \end{aligned} \tag{2.23}$$

Therefore, by (2.12) and (2.23), we get the following result.

Theorem 2.10. *For $n \geq 0$, we have*

$$\begin{aligned}
 D_{n,\lambda}^{(r)}(x) &= \sum_{p=0}^n \sum_{k=0}^p \sum_{m=0}^k \sum_{j=0}^{n-p} \binom{n}{p} B_m^{(m+r+1)}(x+1) B_j^{(j-r+1)}(1) \lambda^j S_{2,\lambda}(k, m) \\
 & \quad \times S_1(p, k) S_1(n-p, j).
 \end{aligned}$$

From (2.12), we note that

$$\begin{aligned}
& \left(\frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda \log(1+t))^{\frac{x}{\lambda}} \\
&= \left(\sum_{l=0}^{\infty} D_{l,\lambda}^{(r)} \frac{t^l}{l!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^k (x)_{m,\lambda} S_1(k,m) \right) \frac{t^k}{k!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} (x)_{m,\lambda} S_1(k,m) D_{n-k,\lambda}^{(r)} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.24}$$

Thus, by (2.12) and (2.24), we get the next theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$D_{n,\lambda}^{(r)}(x) = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} (x)_{m,\lambda} S_1(k,m) D_{n-k,\lambda}^{(r)}.$$

Now, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,\lambda}^{(r)}(x+y) \frac{t^n}{n!} &= \left(\frac{\log(1+t)}{(1+\lambda \log(1+t))^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda \log(1+t))^{\frac{x+y}{\lambda}} \\
&= \left(\sum_{l=0}^{\infty} D_{l,\lambda}^{(r)}(x) \frac{t^l}{l!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^k (y)_{m,\lambda} S_1(k,m) \right) \frac{t^k}{k!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} D_{n-k}^{(r)}(x) (y)_{m,\lambda} S_1(k,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.25}$$

Thus, by (2.25), we get

$$D_{n,\lambda}^{(r)}(x+y) = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} D_{n-k}^{(r)}(x) (y)_{m,\lambda} S_1(k,m). \tag{2.26}$$

From (2.18), we note that

$$\begin{aligned}
 & \left(\frac{(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1}{\log(1 + t)} \right)^r = \left(\frac{t}{\log(1 + t)} \right)^r \frac{r!}{t^r} \frac{1}{r!} \left((1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1 \right)^r \\
 & = \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{t^m}{m!} \right) \frac{r!}{t^r} \left(\sum_{l=r}^{\infty} S_{2,\lambda}(l, r) \frac{1}{l!} (\log(1 + t))^l \right) \\
 & = \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{t^m}{m!} \right) \frac{r!}{t^r} \left(\sum_{l=0}^{\infty} S_{2,\lambda}(l + r, r) \frac{1}{(l + r)!} (\log(1 + t))^{l+r} \right) \\
 & = \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{t^m}{m!} \right) \frac{r!}{t^r} \left(\sum_{k=r}^{\infty} \left(\sum_{l=0}^{k-r} S_{2,\lambda}(l + r, r) S_1(k, l + r) \right) \frac{t^k}{k!} \right) \\
 & = \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{t^m}{m!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{l=0}^k \frac{S_{2,\lambda}(l + r, r) S_1(k + r, l + r)}{\binom{k+r}{k}} \right) \frac{t^k}{k!} \right) \\
 & = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^n \sum_{l=0}^k \frac{\binom{n}{k}}{\binom{k+r}{k}} S_{2,\lambda}(l + r, r) S_1(k + r, l + r) B_{n-k}^{(n-k-r+1)}(1) \right\} \frac{t^n}{n!}.
 \end{aligned} \tag{2.27}$$

On the other hand,

$$\left(\frac{(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1}{\log(1 + t)} \right)^r = \sum_{n=0}^{\infty} D_{n,\lambda}^{(-r)} \frac{t^n}{n!}. \tag{2.28}$$

Therefore, by (2.27) and (2.28), we obtain the following theorem.

Theorem 2.12. For $n \geq 0$, $r \in \mathbb{N}$, we have

$$D_{n,\lambda}^{(-r)} = \sum_{k=0}^n \sum_{l=0}^k \frac{\binom{n}{k}}{\binom{k+r}{k}} S_{2,\lambda}(l + r, r) S_1(k + r, l + r) B_{n-k}^{(n-k-r+1)}(1).$$

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