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Indirect Influences on Directed Manifolds

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Abstract

We introduce a program aimed to building differential geometric analogues for problems arising from the theory of complex networks. We study the propagation of influences on manifolds assuming that at each point only a finite number of propagation velocities are allowed. This leads to the computation of the volume of the moduli spaces of directed paths. The proposed settings provide a fertile ground for research with potential applications in geometry, mathematical physics, differential equations, and combinatorics. The interaction between differential geometry and complex networks is a new and promising field of study.

Keywords: Path Integrals, Directed Graphs, Indirect Influences, Moduli Spaces
AMS Classification Subject: 51P05, 81Q30, 53Zxx, 28Exx, 58Z05

1 Introduction

Our aim in this work is to lay down the foundations for the study of the propagation of influences on directed manifolds. Our object of study can be approached from quite different viewpoints as indicated in the following, non-exhaustive, diagram:

$$\begin{tikzcd}
\text{Ind. Inf. on Graphs} \arrow[dr] & & \text{Geometric Control} \arrow[dl] \\
& \text{Indirect Influences on Directed Manifolds} & \\
\text{Feyman Integrals} \arrow[ur] & & \text{Directed Spaces} \arrow[ul]
\end{tikzcd}$$

Our departure point is the theory of indirect influences for weighted directed graphs, which has gradually emerged thanks to the efforts of several authors – among them Brin, Chung, Estrada, Godet, Hatano, Katz, Page, Motwani, and Winograd. Although the history of the subject is yet to be written, we regard the introduction of the Katz's index [14] as an early modern approach to the problem of understanding the propagation of influences in complex networks. Fundamental developments in the field came with the introduction of the MICMAC [11], PageRank [3], Communicability [10], and Heat Kernel [6] methods. A further method, called the PWP method, for computing the propagation of influences on networks was proposed in 2009 by the second author [7], which proceeds as follows. Assume as given a network (weighted directed graph) represented by its adjacency matrix D, also called the matrix of direct influences. Then one defines the matrix $T = T(D)$ of indirect influences, whose entry T_{ij} measures the weight of the indirect influences exerted by vertex j on vertex i. The matrix T is computed using the expression: $T = \frac{1}{e^{\lambda}-1}\sum_{n=1}^{\infty} D^n \frac{\lambda^n}{n!}$, where λ is a positive real parameter: indirect influences arise from the sum of weights of concatenations of direct influences; the weight of a concatenation of length n comes from the product of n entries of D and the factor $\frac{\lambda^n}{n!}$ which ensures convergency by attaching a rapidly decreasing weight to longer chains of direct influences. The PWP method has been applied to analyse educational programs, and to study indirect influences in international trade [8]. The stability of the method with respect to changes in D and λ is studied in [9].

We regard a differential manifold provided with a tuple of vector fields on it – we call such an object a directed manifold – as being a smooth analogue of a directed graph with numbered outgoing edges attached to each vertex. Armed with this intuition we pose the question: Is there an extension of the theory of indirect influences from the discrete to the smooth settings? We argue that the answer is in the affirmative, and that such an extension both interplays with several notions already studied in the literature, e.g. control theory [1, 19], Feynman integrals, and directed topological spaces [12], and also demands the introduction of new ideas.

The background upon which we develop our constructions is the category of directed manifolds, see Section 2, a convenient category for studying geometric control theory. Our constructions bring about a new set of problems to geometric control theory, namely, the problem of computing integrals over the moduli spaces of directed paths. We remark that strong tangency conditions are imposed in order to insure that the moduli spaces of directed paths – also called the spaces of indirect influences – split naturally into infinitely many finite dimensional pieces, each coming with a natural measure. Thus we have a notion of integration over each piece, which we extend additively to the whole moduli space, leaving the convergency of these sums to a case by case analysis. In our examples we do obtain convergent sums. These ideas are developed in Section 3, where we also introduce the wave of influences $u(p,t)$ which computes the total influence received by a point p in time t, i.e. $u(p,t)$ computes the volume of the moduli space of directed paths starting at an arbitrary point and ending up at p in time t.

Our notion of directed manifolds is strongly related to the notion of directed spaces introduced by Grandis [12], the former yielding a smooth analogue of the latter. In Section 4 we discuss invariant properties for directed manifolds and for the moduli spaces of directed paths on them. In Section 5 we study the moduli spaces of directed paths on the product and quotient of directed manifolds. In Section 6 we study the moduli spaces of directed paths arising from constant vector fields on affine spaces, which gives rise to fruitful constructions in combinatorics and probability theory [4, 20]. We left for future research the problem of generalizing our main constructions to higher dimensions, this could eventually lead to interactions with string theory.

Notation. For $n \in \mathbb{N}$, we set $[n] = \{1, ..., n\}$, $[0, n] = \{0, ..., n\}$, and let $\mathrm{P}[n]$ be the set of subsets of $[n]$. The amalgamated sum of closed subintervals of the real line $\mathbb{R}$ is given by $[a, b] \coprod_{b,c} [c, d] = [a, b + d - c]$. We let δ_{ab} be the Kronecker's delta function. We write $M \simeq N$ to indicate that M and N are homeomorphic topological spaces.

2 Basic Definitions

We let diman be the category of directed manifolds. A directed manifold is a tuple $(M, v_1, ..., v_k)$ where M is a smooth manifold, and $v_1, ..., v_k$ are smooth vector fields on M, with $k \geq 1$. A morphism $(f, \alpha) : (M, v_1, ..., v_k) \longrightarrow (N, w_1, ..., w_l)$ in diman is a pair (f, α) where $f : M \longrightarrow N$ is a smooth map, $\alpha : [k] \longrightarrow [l]$ is a map, and the following identity holds $df(v_i) = w_{\alpha(i)}$, for $i \in [k]$. Let $(g, \beta) : (N, w_1, ..., w_l) \longrightarrow (K, z_1, ..., z_r)$ be another morphism. The composition morphism $(g, \beta) \circ (f, \alpha)$ is given by $(g, \beta) \circ (f, \alpha) = (g \circ f, \beta \circ \alpha)$. Indeed $d(gf)(v_i) = dg(df(v_i)) = dg(w_{\alpha(i)}) = z_{\beta(\alpha(i))} = z_{\beta\circ\alpha(i)}$.

One can think of a directed manifold $(M, v_1, ..., v_k)$ as being a smooth analogue of a finite directed graph with up to k outgoing numbered edges at each vertex. Points in the manifold M are thought as vertices in the smooth graph. The tangent vectors $v_i(p) \in T_pM$ are thought as infinitesimal edges starting at p. The out-degree of a vertex $p \in M$ is the number of non-zero infinitesimal edges starting at p, i.e. the cardinality of the set $\{i \in [k] \mid v_i(p) \neq 0\}$. An actual edge from p to q is a smooth path $\varphi : [0, t] \longrightarrow M$ with $\varphi(0) = p$, $\varphi(t) = q$, and such that the tangent vector at each point of φ is an infinitesimal edge, i.e. $\dot{\varphi} = v_i(\varphi)$ for some $i \in [k]$, or more explicitly $\dot{\varphi}(s) = v_i(\varphi(s))$ for all $s \in [0, t]$. We say that p exerts a direct influence, in time $t > 0$, on the vertex q through the path φ. Note that φ is determined by p and the index i of vector field v_i, thus we are entitled to use the notation $\varphi(s) = \varphi_i(p, s)$, where φ_i is the flow generated by v_i.

Definition 1. Let $(M, v_1, ..., v_k)$ be a directed manifold and $p, q \in M$. The set of one-direction paths, also called direct influences, $D_{p,q}(t)$ from p to q developed in time $t > 0$ is given by

$$D_{p,q}(t) = \{i \in [k] \mid \varphi_i(p, t) = q\}, \quad \text{and} \quad D_{p,q}(0) = \begin{cases} \{p\} & \text{if } p = q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Next we introduce the notion of indirect influences which arise from the concatenation of direct influences. Our focus is on finding a convenient parametrization for the space of all such concatenations.

Definition 2. Let $(M, v_1, ..., v_k)$ be a directed manifold and $p, q \in M$. A directed path, also called a indirect influence, from p to q displayed in time $t > 0$ through $n \geq 0$ changes of directions is given by a pair (c, s) with the following properties:

- $c = (c_0, c_1, ..., c_n)$ is a $(n+1)$-tuple with $c_i \in [k]$ and such that $c_i \neq c_{i+1}$. We say that c defines the pattern (of directions) of the directed path (c, s). We let $D(n, k)$ be the set of all such tuples and $l(c) = n+1$ be the length of c. There are $k(k-1)^n$ different patterns in $D(n, k)$. We often regard a pattern c as a map $c : [0, n] \longrightarrow [k]$.

- $s = (s_0, ..., s_n)$ is a $(n+1)$-tuple with $s_i \in \mathbb{R}_{\geq 0}$ and such that $s_0 + \cdots + s_n = t$. We say that s defines the time distribution of the directed path (c, s), and let Δ_n^t be the n-simplex of all such tuples.

- The pair (c, s) determines a $(n+2)$-tuple of points $(p_0, \ldots, p_{n+1}) \in M^{n+2}$ given by $p_0 = p$ and $p_i = \varphi_{c_{i-1}}(p_{i-1}, s_{i-1})$ for $1 \leq i \leq n+1$, where $\varphi_{c_{i-1}}$ is the flow generated by the vector field $v_{c_{i-1}}$. We denote the last point p_{n+1} by $\varphi_c(p, s)$.

- The pair (c, s) must be such that $\varphi_c(p, s) = q$.

- Directed paths in time $t = 0$ are the same as one-direction paths in time $t = 0$.

The geometric meaning of directed paths is made clear through the following construction. A pair (c, s) as above determines a piece-wise smooth path

$$\varphi_{c,s} : [0, s_0 + \cdots + s_n] \quad \simeq \quad [0, s_0] \bigsqcup_{s_0, 0} \cdots \bigsqcup_{s_{n-1}, 0} [0, s_n] \quad \longrightarrow \quad M$$

such that the restriction of $\varphi_{c,s}$ to the interval $[0, s_i]$, for $0 \leq i \leq n$, is given by $\varphi_{c,s}|_{[0,s_i]}(r) = \varphi_{c_i}(p_i, r)$ for all $r \in [0, s_i]$. Indirect influences are exerted through such directed paths. Figure 1 shows the directed path associated to a pair $(c_0, c_1, c_2, c_3, s_0, s_1, s_2, s_3)$.

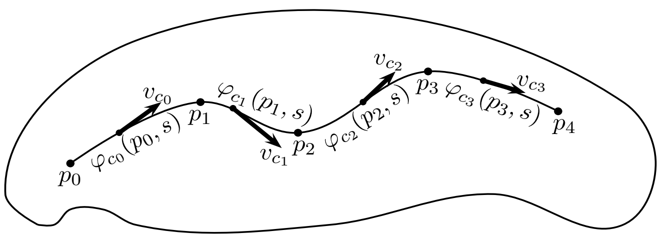

Figure 1. Directed path associated to a pair $(c_0, c_1, c_2, c_3, s_0, s_1, s_2, s_3)$.

We assume that the flows generated by the vector fields v_j are globally defined by smooth maps $\varphi_j(\ ,\ ) : M \times \mathbb{R} \longrightarrow M$ yielding a one-parameter group of diffeomorphisms of M. A pattern $c \in D(n,k)$ defines an iterated flow given by the smooth map $\varphi_c : M \times \mathbb{R}^{n+1} \longrightarrow M$ defined by recursion on the length of c as $\varphi_c(p, s_0, ..., s_n) = \varphi_{c_n}(\varphi_{c|_{[0,n-1]}}(p, s_0, ..., s_{n-1}), s_n)$. Fixing a time distribution $(s_0, ..., s_n)$ we obtain the diffeomorphism $\varphi_c(\ , s_0, ..., s_n) : M \longrightarrow M$. These construction justify the notation $\varphi_c(p, s)$ for the point $p_{n+1}(c, s)$ introduced in Definition 4.

We regard the n-simplex Δ_n^t introduced in Definition 4 as a smooth manifold with corners. There are at least three different approaches to differential geometry on manifolds with corners. First we can apply differential geometric notions on the interior of Δ_n^t. Second it is possible to introduce differential geometric notions on Δ_n^t by considering objects that are smooth on an open neighborhood of Δ_n^t in $\mathbb{R}^{n+1}$. A third and more intrinsic approach for doing differential geometry on Δ_n^t relies on deeper results in the theory of manifolds with corners. Although this more comprehensive approach is certainly desirable, for simplicity, we will not further consider it.

Proposition 3. For a pattern $c \in D(n,k)$, the map $\varphi_c : M \times \Delta_n^t \longrightarrow M$ sending a pair $(p, s) \in M \times \Delta_n^t$ to the point $\varphi_c(p, s) \in M$ is a smooth map and a diffeomorphism for a fixed time distribution $s \in \Delta_n^t$.

Next we introduce the main objects of study in this work, namely, the moduli spaces of directed paths on directed manifolds, also called the spaces of indirect influences.

Definition 4. Let $(M, v_1, ..., v_k)$ be a directed manifold and $p, q \in M$. The moduli space $\Gamma_{p,q}(t)$ of directed paths from p to q developed in time $t > 0$ is given by $\Gamma_{p,q}(t) = \Big\{(c,s)\ \Big|\ \varphi_c(p,s) = q\Big\} =$

$$\coprod_{n=0}^{\infty} \coprod_{c \in D(n,k)} \{s \in \Delta_n^t \mid \varphi_c(p,s) = q\} \ = \ \coprod_{n=0}^{\infty} \coprod_{c \in D(n,k)} \Gamma_{p,q}^c(t).$$

We also set

$$\Gamma_{p,q}(0) \ = \ \Gamma_{p,q}^{\emptyset}(0) \ = \ \begin{cases} \{p\} & \text{if } p = q, \\ \emptyset & \text{otherwise,} \end{cases}$$

Figure 2 shows a schematic picture of a component $\Gamma_{p,q}^c(t)$ of the moduli space of indirect influences.

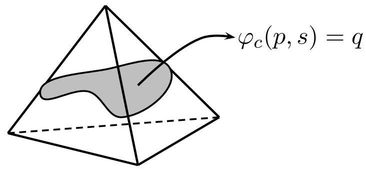

Figure 2. Moduli space of directed paths $\Gamma^c_{p,q}(t)$.

Theorem 5. The moduli spaces of directed paths on a directed manifold form a topological category.

Proof. Given a directed manifold $(M, v_1, ..., v_k)$ we let $\Gamma = \Gamma(M, v_1, ..., v_k)$ be the category of directed paths on M. The objects of Γ are the points of M. Given $p, q \in M$, the space of morphisms in Γ from p to q is given by $\Gamma_{p,q} = \coprod_{n\in\mathbb{N}} \coprod_{c\in D(n,k)} \Gamma^c_{p,q}$ where $\Gamma^c_{p,q} = \{(s,t) \in \mathbb{R}^{n+2}_{\geq 0} \mid s \in \Delta^t_n, \ \varphi_c(p,s) = q\}$. In order to define continuous composition maps $\circ : \Gamma_{p,q} \times \Gamma_{q,r} \longrightarrow \Gamma_{p,r}$, it is enough to define componentwise composition maps $\circ : \Gamma^c_{p,q} \times \Gamma^d_{q,r} \longrightarrow \Gamma^{c*d}_{p,r}$ for given patterns c and d with $n = l(c)$ and $m = l(d)$. We consider two cases:

- If $c_n \neq d_0$, then $c * d = (c, d)$ and $(s_0, ..., s_n) \circ (u_0, ..., u_m) = (s_0, ..., s_n, u_0, ..., u_m)$.
- If $c_n = d_0$, then $c * d = (c_0, ..., c_n) * (d_0, ..., d_m) = (c_0, ..., c_n, d_1, ..., d_m)$ and

$$(s_0, ..., s_n) \circ (u_0, ..., u_m) = (s_0, ..., s_n + u_0, ..., u_m).$$

These compositions are well-defined continuous maps satisfying the associative property. The unique $t = 0$ directed path from $p \in M$ to itself gives the identity morphism for each object $p \in \Gamma$. $\square$

Remark 6. The moduli spaces $\Gamma_{p,q}(t)$ can be extended from points to arbitrary subsets of M as follows. Given $A, B \subseteq M$ we define the moduli space of directed paths from A to B as $\Gamma_{A,B}(t) = \left\{(c,s) \,\middle|\, p \in A, \ \varphi_c(p,s) \in B\right\} = \coprod_{n=0}^{\infty} \coprod_{c\in D(n,k)} \{s \in \Delta^t_n \mid p \in A, \ \varphi_c(p,s) \in B\} = \coprod_{n=0}^{\infty} \coprod_{c\in D(n,k)} \Gamma^c_{A,B}(t)$. Looking at embedded oriented submanifolds of M and following techniques from Chas and Sullivan's string topology [5], this construction gives rise to a transversal category.

We introduce a few distinguished subsets of M useful for understanding the propagation of influences on M. These sets are usually called the reachable sets in geometric control theory, and are natural generalizations of the corresponding graph theoretical notions. They also play a prominent role in general relativity [17]. For $A \subseteq M$ we set: 1) $\Gamma_A(t) = \{q \in M \mid \Gamma_{A,q}(t) \neq \emptyset\}$ is the set of points in M influenced by A in time t. 2)

$\Gamma_{A,\leq}(t) = \{q \in M \mid \text{there is } 0 \leq s \leq t, \text{ such that } \Gamma_{A,q}(s) \neq \emptyset\}$ is the set of points in M influenced by A in time less or equal to t. 3) $\Gamma_A = \{q \in M \mid \Gamma_{A,q}(t) \neq \emptyset \text{ for some } t \geq 0\}$ is the set of points in M that are influenced by A. 4) $\Gamma^-_A(t) = \{q \in M \mid \Gamma_{q,A}(t) \neq \emptyset\}$ is the set of points in M that influence A in time t, i.e. the set of points on which A depends on time t. 5) $\Gamma^-_{A,\leq}(t) = \{q \in M \mid \text{there is } 0 \leq s \leq t, \text{ such that } \Gamma_{q,A}(s) \neq \emptyset\}$ is the set of points in M that influence A in time less or equal to t. 6) $\Gamma^-_A = \{q \in M \mid \Gamma_{q,A}(t) \neq \emptyset \text{ for some } t \geq 0\}$ is the set of points in M that influence A. 7) $\mathrm{F}_A(t) = \partial\Gamma_{A,\leq}(t)$ and $\mathrm{F}^-_A(t) = \partial\Gamma^-_{A,\leq}(t)$ are called, respectively, the front of influence and the front of dependence of A in time t.

Note that a directed manifold M is naturally a pre-poset by setting $p \leq q$ if and only if $q \in \Gamma_p$. The associated poset is the quotient space $M_\sim$, where the equivalence relation $\sim$ on M is given by $p \sim q$ if and only if $q \in \Gamma_p$ and $p \in \Gamma_q$. The space $M_\sim$ tell us how M splits into components of co-influences, i.e. the path connected components of M through directed paths.

3 Measuring the Moduli Spaces of Directed Paths

In order to measure directed paths on M we assume from now on that an orientation on M has been chosen. To gauge the amount of indirect influences exerted, in time t, by a point $p \in M$ on a point $q \in M$ we need to define measures on the moduli spaces $\Gamma_{p,q}(t)$ of directed paths. From Definition 4 we see that $\Gamma_{p,q}(t)$ is a disjoint union of pieces, one for each pattern $c \in D(n,k)$, of the form $\Gamma^c_{p,q}(t) = \{s \in \Delta^t_n \mid \varphi_c(p,s) = q\}$. So, our problem reduces to imposing measures on the pieces $\Gamma^c_{p,q}(t)$. The n-simplex Δ^t_n is a smooth manifold with corners, and comes equipped with a Riemannian metric and its associated volume form. Indeed using Cartesian coordinates $l_i = s_0 + \cdots + s_{i-1}$, we have that $\Delta^t_n = \Big\{(l_1,...,l_n) \in \mathbb{R}^n \mid 0 \leq l_1 \leq l_2 \leq ...... \leq l_n \leq t\Big\}$. Thus $\Delta^t_n \subseteq \mathbb{R}^n$ inherits a Riemannian metric, an orientation, and the corresponding volume form $dl_1 \wedge \cdots \wedge dl_n$. With this measure $\mathrm{vol}(\Delta^t_n) = \frac{t^n}{n!}$.

Definition 7. A directed manifold $(M, v_1, ..., v_k)$ has smooth spaces of directed paths if for any pattern $c \in D(n,k)$ and points $p, q \in M$ the space of indirect influences $\Gamma^c_{p,q}(t)$ is a smooth embedded sub-manifold of Δ^t_n.

For our next result we use the implicit function theorem for manifolds [13, 21]. Let $f : N \longrightarrow M$ be a smooth map between differential manifolds and fix $q \in M$. If for each $p \in f^{-1}(q)$ the linear map $d_pf : T_pN \longrightarrow T_qM$ has maximal rank, that is $\mathrm{rank}(d_pf) = \min\{\dim(N), \dim(M)\}$, then $f^{-1}(q)$ is a smooth sub-manifold of N. If $\mathrm{rank}(d_pf) = \dim(N)$, then d_pf is injective, f is an immersion, and $f^{-1}(q)$ is a set of isolated points. If $\mathrm{rank}(d_pf) = \dim(M)$, then d_pf is surjective, f is a submersion, and $f^{-1}(q)$ is a sub-manifold of N of dimension $\dim(N) - \dim(M)$.

Theorem 8. Let $(M, v_1, ..., v_k)$ be a directed manifold. Fix a pattern $c \in D(n,k)$ with $n \geq 1$, and a point $p \in M$. If for any $(s_0, ..., s_n)$ in the open part of $\Gamma_{p,q}(t)$ there are $\min(n, \dim(M))$ linearly independent vectors among the vectors given for $i \in [0, n-1]$ by

$$d^M \varphi_{(c_{i+1},...,c_n)}\Big|_{(s_{i+1},...,s_n)} \big[v_{c_i}(\varphi_{c_0,...,c_i}(s_0, ..., s_i))\big] \; - \; v_{c_n}(\varphi_c(s_0, ..., s_n)) \;\; \in \; T_{\varphi_c(p,s_0,...,s_n)},$$

then $\Gamma^c_{p,q}(t)$ is a smooth sub-manifold of Δ^t_n.

Proof. Fix $c \in D(n,k)$ with $n \geq 1$, and consider $\varphi_c : M \times \mathbb{R}^{n+1} \longrightarrow M$ the iterated flow associated to c. The differential of φ_c naturally split as $d\varphi_c = d^M \varphi_c + d^{\mathbb{R}^{n+1}} \varphi_c$. Consider the map $\phi : \Delta^t_n \longrightarrow M$ given by $\phi(s) = \phi(s_0, ..., s_{n-1}) = \varphi_c(p, s_0, ..., s_{n-1}, t - s_0 - \cdots - s_{n-1})$, where we are using the identification $\Delta^t_n = \left\{ s = (s_0, ..., s_{n-1}) \in \mathbb{R}^n_{\geq 0} \;\middle|\; |s| = s_0 + \cdots + s_{n-1} \leq t \right\}$. In order to guarantee that $\Gamma^c_{p,q}(t) = \phi^{-1}(p)$ is a smooth sub-manifold of Δ^t_n we impose the condition that $d_s\phi$ has maximal rank for $s \in \phi^{-1}(p)$. Next we compute for $i \in [0, n-1]$ the vectors $\frac{\partial \phi}{\partial s_i}(s) = d_s\phi(\frac{\partial}{\partial s_i}) \in T_{\phi(s)}M$. Using the identity $\frac{\partial}{\partial s_n}(\varphi_{c_0,\cdots,c_n})(p, s_0, \cdots, s_n) = v_{c_n}(\varphi_{c_1,\cdots,c_n}(p, s_0, \cdots, s_n))$, one can show that $\frac{\partial \phi}{\partial s_i}(s)$ is given by $d^M \varphi_{c_{i+1},...,c_n}\Big|_{(s_{i+1},...,s_{n-1},s_n)} \big[v_{c_i}(\varphi_{c_0,...,c_i}(s_0, ..., s_i))\big] - v_{c_n}(\varphi_{c_0,...,c_n}(s_0, ..., s_n))$, where we recall that $s_n = t - |s|$, $d^M \varphi_{c_{i+1},...,c_n} = d\varphi_{c_n}(\;\;, t - |s|) \circ \cdots \circ d^M \varphi_{c_{i+1}}(\;\;, s_{i+1})$, and $\varphi_{c_0,...,c_i}(s_0, ..., s_i) = \varphi_{c_i}[\varphi_{c_0,...,c_{i-1}}(s_0, ..., s_{i-1}), s_i]$ for $i \geq 1$. Thus the rank of $d_s\phi$ is maximal at each point $s \in \phi^{-1}(q)$ if and only if there are exactly $\min(n, \dim(M))$ linearly independent vectors among the vectors $\frac{\partial \phi}{\partial s_i}(s)$ given by the expression above. We have shown the desired result. $\square$

Corollary 9. Under the hypothesis of Theorem 8, the interior of the moduli space $\Gamma^c_{p,q}(t)$ is an oriented Riemannian sub-manifold of Δ^t_n.

Proof. We use oriented differential intersection theory as developed by Guillemin [13]. Since $\Gamma^c_{p,q}(t)$ is a smooth sub-manifold of Δ^t_n it acquires by restriction a Riemannian metric. The orientation on $\Gamma^c_{p,q}(t)$ arises as follows. For $s \in \Gamma^c_{p,q}(t)$ write $T_s\Delta^t_n \simeq N_s\Gamma^c_{p,q}(t) \oplus T_s\Gamma^c_{p,q}(t)$, where $N_s\Gamma^c_{p,q}(t) \simeq T_s\Delta^t_n / T_s\Gamma^c_{p,q}(t)$ is the normal bundle of $\Gamma^c_{p,q}(t)$. Note that $d_s\phi(T_s\Delta^t_n) = T_{\phi(s)}M$ and thus $d_s\phi : N_s\Gamma^c_{p,q}(t) \longrightarrow T_{\phi(s)}M$ is an isomorphism. Since $T_s\Delta^t_n$ is oriented, and $N_s\Gamma^c_{p,q}(t)$ acquires an orientation from the isomorphism above, then $T_s\Gamma^c_{p,q}(t)$ naturally acquires an orientation. $\square$

For a directed manifold with a smooth moduli space of directed paths each piece $\Gamma^c_{p,q}(t) \subseteq \Delta^t_n$ acquires from Δ^t_n a Riemannian metric. If in addition we assume that each piece $\Gamma^c_{p,q}(t)$ is given an orientation, then $\Gamma^c_{p,q}(t)$ acquires a volume form denoted by dl_c. As we have just shown this is the situation arising from the conditions of Theorem 8. We are ready to highlight a few functions on the moduli spaces of directed paths, for a fix a time $t > 0$, that one would like to integrate against these measures.

1. Volume of Moduli Space of Directed Paths. Each component $\Gamma^c_{p,q}(t)$ of the space of indirect influences is compact and thus of bounded volume. We define the volume or total measure of $\Gamma_{p,q}(t)$, leaving convergency issues to be discussed on a case by case basis, as follows:

$$\mathrm{vol}(\Gamma_{p,q}(t)) \;=\; \int_{\Gamma_{p,q}(t)} 1dl \;=\; \sum_{n=1}^{\infty} \sum_{c\in D(n,k)} \int_{\Gamma^c_{p,q}(t)} 1\; dl_c \;=\; \sum_{n=1}^{\infty} \sum_{c\in C(n,k)} \mathrm{vol}(\Gamma^c_{p,q}(t)).$$

2. Functions on directed paths coming from differential 1-forms on M. Let A be a differential 1-form on M. We formally write

$$\int_{\Gamma_{p,q}(t)} \widehat{A}\; dl \;=\; \sum_{n=1}^{\infty} \sum_{c\in D(n,k)} \int_{\Gamma^c_{p,q}(t)} \widehat{A}\; dl_c,$$

where the map $\widehat{A} : \Gamma^c_{p,q}(t) \longrightarrow \mathbb{R}$ is given by $A(c,s) = \int_0^t \varphi^*_{c,s}A = \sum_{i=0}^{l(c)} \int_0^{s_i} \varphi_{c,s}|^*_{[0,s_i]}A,$ with $\varphi_{c,s} : [0, s_0 + \cdots + s_n] \longrightarrow M$ the directed path associated to $(c,s) \in \Gamma_{p,q}(t)$.

3. Functions on directed paths from Riemannian metrics on M. Let g be a Riemannian metric on M. We formally write

$$\int_{\Gamma_{p,q}(t)} e^{-l_g}\; dl \;=\; \sum_{n=1}^{\infty} \sum_{c\in D(n,k)} \int_{\Gamma^c_{p,q}(t)} e^{-l_g}\; dl_c,$$

where $e^{-l_g} : \Gamma^c_{p,q}(t) \longrightarrow \mathbb{R}$ is the map given by $\epsilon^{-l_g}(c,s) \;=\; e^{-l_g(\varphi_{c,s})}$ and $l_g(\varphi_{c,s})$ is the length of the path $\varphi_{c,s}$, i.e.:

$$l_g(\varphi_{c,s}) \;=\; \sum_{i=0}^{l(c)} l_g(\varphi_{c,s}|_{[0,s_i]}) \;=\; \sum_{i=0}^{l(c)} \int_0^{s_i} g\Big(v_{c_i}(\varphi_{c_i}(p_i,u))\;,\; v_{c_i}(\varphi_{c_i}(p_i,u))\Big)du.$$

4. Functions on direct paths from functions on M. Given a smooth map $f : M \longrightarrow \mathbb{R}$ we formally write

$$\int_{\Gamma_{p,q}(t)} \widehat{f}\; dl \;=\; \sum_{n=1}^{\infty} \sum_{c\in D(n,k)} \int_{\Gamma^c_{p,q}(t)} f(p_0)\cdots f(p_n)\; dl_c$$

with $p_0 = p$ and $p_{i+1} = \varphi_{c_i}(p_i, s_i)$ for $0 \leq i \leq n$.

5. Functions on directed paths from Lagrangian functions on TM. Let $L : TM \longrightarrow \mathbb{R}$ be a Lagrangian map. In the applications L is usually built from a Riemannian metric g on

M and a potential map $U : M \longrightarrow \mathbb{R}$ as $L(p,v) = g(v,v) - U(p)$. Given a Lagrangian L we consider the following analogue of the Feynman integrals:

$$\int_{\Gamma_{p,q}(t)} e^{\frac{i}{\hbar}S}\ dl \ = \ \sum_{n=1}^{\infty} \sum_{c \in D(n,k)} \int_{\Gamma^c_{p,q}(t)} e^{\frac{i}{\hbar}S}\ dl_c,$$

where we set $e^{\frac{i}{\hbar}S}(c,s) = e^{\frac{i}{\hbar}S(c,s)}$, and the action map S is given by

$$S(c,s) \ = \ \int_0^t L(\varphi_{c,s}(u), \dot{\varphi}_{c,s}(u))\ du \ = \ \sum_{i=0}^{l(c)} \int_0^{s_i} L(\varphi_{c,s}|_{[0,s_i]}(u), \dot{\varphi}_{c,s}|_{[0,s_i]}(u))\ du.$$

We have shown how to construct and integrate functions on the moduli spaces of directed paths on directed manifolds, let us pick one such a function and call it h. Integrating over the moduli spaces of directed paths we obtain the kernel for the propagation of influences $k : M \times M \times \mathbb{R} \longrightarrow \mathbb{R}$ which is given by $k(p,q,t) = \int_{\Gamma_{p,q}(t)} h\ dl$.

Definition 10. Let $(M, v_1, ..., v_k)$ be an oriented directed manifold with smooth moduli space of directed paths. M is given a Riemannian metric, and thus it acquires a volume form. Let $f : M \longrightarrow \mathbb{R}$ be a map representing the density of influences originated at time $t = 0$. Let g be a map on directed paths, and consider its associated kernel of influences $k = k_g$. The wave of influences $u : M \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ is the map given by

$$u(q,t) \ = \ \int_{p \in \Gamma^-_q(t)} k(p,q,t) f(p)\ dp,$$

where we assume that $\Gamma^-_q(t)$ is a compact oriented smooth sub-manifold of M; thus it acquires by restriction a Riemannian metric, and comes with a volume form dp.

4 Invariance, Involution, and Limit Properties

Let $(M, v_1, ..., v_k)$ be a directed manifold and $f : M \longrightarrow N$ be a diffeomorphism. Then we obtain the directed manifold $(N, f_*v_1, ..., f_*v_k)$ where the push-forward vector fields f_*v_i are given for $q \in N$ by $f_*v_i(q) = d_pf(v_i(p))$, with $p = f^{-1}(q)$. With this notation we have the following result.

Theorem 11. Let $(M, v_1, ..., v_k)$ be a directed manifold and $f : M \longrightarrow N$ be a diffeomorphism. For $p, q \in M$ the identity map gives a natural homeomorphism $\Gamma^M_{p,q}(t) \simeq \Gamma^N_{f(p),f(q)}(t)$. Moreover, if $(M, v_1, ..., v_k)$ has a smooth moduli space of directed paths, M and N are oriented manifolds, and f is an orientation preserving diffeomorphism, then the identification above is an identity between Riemannian manifolds, and thus $\mathrm{vol}(\Gamma^M_{p,q}(t)) = \mathrm{vol}(\Gamma^N_{f(p),f(q)}(t))$.

Proof. We show that $s \in \Gamma^{M,c}_{p,q}(t)$ if and only if $s \in \Gamma^{N,c}_{f(p),f(q)}(t)$. By construction we have that $f(\varphi_{v_i}(p,t)) = \varphi_{f_*(v_i)}(f(p),t)$, and thus by induction on the length of c we have that

$f(\varphi_{v,c,s}(p,t)) = \varphi_{f_*v,c,s}(f(p),t)$. Therefore the equations $\varphi_{v,c}(p,s) = q$ and $\varphi_{f_*v,c}(f(p),s) = f(q)$ are equivalent. To show the second statement, note that the identity map $\Gamma^{M,c}_{p,q}(t) \longrightarrow \Gamma^{N,c}_{f(p),f(q)}(t)$ preserves orientation. Indeed since the identity map preserves the splittings $T_s\Delta^t_n \simeq N_s\Gamma^{M,c}_{p,q}(t) \oplus T_s\Gamma^{M,c}_{p,q}(t)$ and $T_s\Delta^t_n \simeq N_s\Gamma^{N,c}_{f(p),f(q)}(t) \oplus T_s\Gamma^{N,c}_{f(p),f(q)}(t)$, we just have to show that $N_s\Gamma^{M,c}_{p,q}(t)$ and $N_s\Gamma^{N,c}_{f(p),f(q)}(t)$ are given compatible orientations. This follows by construction, see the proof of Theorem 8, as the square

$$\begin{tikzcd} N_s\Gamma^{c}_{p,q}(t) \arrow[r, "d_s\phi_{v,c}"] \arrow[d, "1"'] & T_{\phi_{v,c}(s)}M \arrow[d, "df"] \\ N_s\Gamma^{N,c}_{f(p),f(q)}(t) \arrow[r, "d_s\phi_{f_\star v,c}"] & T_{\phi_{f_\star v,c}(s)}N \end{tikzcd}$$

is a commutative diagram of orientation preserving isomorphisms, see Corollary 9.

□

Next result tell us how the moduli spaces of directed paths depend on the ordering on vector fields.

Proposition 12. Let $(M,v) = (M, v_1, ..., v_k)$ be a directed manifold and $\alpha : [k] \longrightarrow [k]$ be a permutation. For the directed manifold $(M, v\alpha) = (M, v_{\alpha 1}, ..., v_{\alpha k})$ we have that $\Gamma^{v}_{p,q}(t) \simeq \Gamma^{v\alpha}_{p,q}(t)$. Moreover, if (M,v) is oriented and has a smooth moduli space of directed paths, then so does $(M, v\alpha)$ and we have that $\mathrm{vol}(\Gamma^{v}_{p,q}(t)) = \mathrm{vol}(\Gamma^{v\alpha}_{p,q}(t))$.

Proof. Associated to the permutation α we have the map $\alpha_* : \Gamma^{v}_{p,q}(t) \longrightarrow \Gamma^{v\alpha}_{p,q}(t)$ given by $\alpha_*(c,s) = (\alpha^{-1}c, s)$. It follows that α_* is an homeomorphism as its restriction map $\alpha_* : \Gamma^{v,c}_{p,q}(t) \longrightarrow \Gamma^{v\alpha,\alpha^{-1}c}_{p,q}(t)$ is just the identity map and is a well-defined homeomorphism since $\varphi_{v\alpha,\alpha^{-1}c}(p,s) = \varphi_{v,\alpha\alpha^{-1}c}(p,s) = \varphi_{v,c}(p,s) = q$. In the case of smooth moduli spaces of directed paths, the map above is clearly orientation preserving, since it is just the identity map, and we have a commutative diagram of orientation-preserving isomorphisms

$$\begin{tikzcd} N_s\Gamma^{v,c}_{p,q}(t) \arrow[r, "d_s\phi_{v,c}"] \arrow[d, "1"'] & T_{\phi_{v,c}(s)}M \arrow[d, "1"] \\ N_s\Gamma^{v\alpha,\alpha^{-1}c}_{p,q}(t) \arrow[r, "d_s\phi_{v\alpha,\alpha^{-1}c}"] & T_{\phi_{v\alpha,\alpha^{-1}c}(s)}M \end{tikzcd}$$

□

From Theorem 11 and Proposition 12 we see that the invariant study of directed paths on a directed oriented manifold M relies on the study, for $k \geq 1$, of the quotient spaces $\chi(M)^k/\mathrm{Diff}_+(M) \times S_k$, where $\chi(M)$ is the space of vector fields on M, S_k the group of permutations of $[k]$, and $\mathrm{Diff}(M)_+$ is the group of orientation preserving diffeomorphism of

M, i.e. the study of equivalence classes of tuples of vector fields under diffeomorphisms and permutations. Next we define the direction reversion functor $-$: diman $\longrightarrow$ diman. It sends a directed manifold $(M, v_1, ..., v_k)$ to its reversed directed manifold $(M, -v_1, ..., -v_k)$.

Proposition 13. Let $(M, v_1, ..., v_k)$ be a directed manifold and $(M, -v_1, ..., -v_k)$ its reversed directed manifold. Given $A, B \subseteq M$ we have canonical homeomorphism $\Gamma_{v,A,B}(t) \simeq \Gamma_{-v,B,A}(t)$. Therefore the respective reachable sets are related by $\Gamma_{-v,A}(t) \simeq \Gamma^{-}_{v,A}(t)$, $\Gamma_{-v,A,\leq}(t) \simeq \Gamma^{-}_{v,A,\leq}(t)$, $\Gamma_{-v,A} \simeq \Gamma^{-}_{v,A}$, $\mathrm{F}_{-v,A}(t) \simeq \partial\Gamma^{-}_{A,\leq}(t)$. If $(M, v_1, ..., v_k)$ is oriented and has a smooth moduli space of directed paths, then so does $(M, -v_1, ..., -v_k)$ and the maps above are actually diffeomorphisms which may or may not preserve orientation.

Proof. We define a map $\overline{(\ )} : \Gamma_{v,A,B}(t) \longrightarrow \Gamma_{-v,B,A}(t)$ as follows $\overline{(c,s)} = \overline{(c_0, ..., c_n, s_0, ..., s_n)} = (\overline{c}, \overline{s}) = (c_n, ..., c_0, s_n, ..., s_0)$. This map is an homeomorphism since the map $\overline{(\ )} : D(n,k) \longrightarrow D(n,k)$ is bijective, and the map $\overline{(\ )} : \Gamma^{c}_{v,A,B}(t) \longrightarrow \Gamma^{\overline{c}}_{-v,B,A}(t)$ is an homeomorphism as the equations $\varphi_{v,c}(p,s) = q$ and $\varphi_{-v,\overline{c}}(q,\overline{s}) = p$ are equivalent. □

5 Indirect Influences on Product/Quotient Manifolds

Let $(M, v_1, ..., v_k)$ and $(N, u_1, ..., u_l)$ be directed manifolds. The natural isomorphism $T(M \times N) \simeq \pi^*_M TM \oplus \pi^*_N TN$, allows us to consider $(M \times N, v_1, ..., v_k, u_1, ..., u_l)$ as a directed manifold, where one should more formally write $(v_i, 0)$ instead of v_i, and $(0, u_j)$ instead of u_j. Recall that diman is the category of directed manifolds, and that we are allowing in diman manifolds with connected components of different dimensions.

Proposition 14. The product defined above gives diman the structure of a monoidal category.

Fix $A \subseteq [n]$. We say that a map $c : A \longrightarrow [k]$ is a pattern if $c(i) \neq c(i+1)$ for all contiguous elements $i,\ i+1 \in A$. Thus a pattern for the product manifold $M \times N$ is given by a map $c : [n] \longrightarrow [k+l] \simeq [k] \sqcup [l]$ such that its restrictions $c|_{c^{-1}[k]} : c^{-1}[k] \longrightarrow [k]$ and $c|_{c^{-1}[l]} : c^{-1}[l] \longrightarrow [l]$ are patterns on $c^{-1}[k]$ and $c^{-1}[l]$, respectively.

Proposition 15. Let $(p_1, p_2),\ (q_1, q_2) \in M \times N$, and let $c : [n] \longrightarrow [k] \sqcup [l]$ be a pattern. We have a canonical homeomorphism $\Gamma^{M \times N,\ c}_{(p_1,p_2),(q_1,q_2)} \simeq \Gamma^{N,\ c|_{c^{-1}[k]}}_{p_1,\ q_1} \times \Gamma^{N,\ c|_{c^{-1}[l]}}_{p_2,\ q_2}$.

Proof. The desired homeomorphism sends $s \in \Gamma^{M \times N,\ c}_{(p_1,p_2),(q_1,q_2)}(t) \subseteq \Gamma^{M \times N,\ c}_{(p_1,p_2),(q_1,q_2)}$ to the pair

$$(s|_{c^{-1}[k]},\ s|_{c^{-1}[l]}) \ \in\ \Gamma^{N,\ c|_{c^{-1}[k]}}_{p_1,\ q_1}(a) \ \times\ \Gamma^{N,\ c|_{c^{-1}[l]}}_{p_2,\ q_2}(t-a), \quad \text{where} \ \ a = \sum_{i \in c^{-1}[k]} s_i.$$

□

Next we consider the moduli spaces of directed paths on quotient manifolds. Let M be a smooth manifold, G a compact Lie group acting freely on M, and assume that the directed manifold $(M, v_1, ..., v_k)$ is invariant under the action of G, i.e. $d_p g(v_i) = v_i(gp)$ for all $p \in M$, $g \in G$. Then M/G is a smooth manifold and it comes with a smooth quotient map $pi : M \longrightarrow M/G$, which induces a surjective map $d\pi : TM \longrightarrow T(M/G)$, and canonical isomorphisms $\overline{d_p\pi} : T_pM/T_p(Gp) \longrightarrow T_{\overline{p}}(M/G)$. Note also that we have isomorphisms $T_{\overline{p}}(M/G) \simeq \left(\bigoplus_{g\in G} T_{gp}M\right)/G$. Thus we obtain the directed manifold $(M/G,\ \overline{v}_1,\ \dots\ ,\overline{v}_k)$ with $\overline{v}_i = d\pi(v_i)$.

Theorem 16. Let $(M, v_1, ..., v_k)$ be a directed manifold, invariant under the action of the compact Lie group G, and let $p, q \in M$. Then $(M/G,\ \overline{v}_1,\ \dots\ ,\overline{v}_k)$ with $\overline{v}_i = d\pi(v_i)$ is a directed manifold, G acts naturally on $\Gamma^M_{G_p,G_q}(t)$, and we have that $\Gamma^{M/G}_{\overline{p},\overline{q}}(t) \simeq \left(\Gamma^M_{G_p,G_q}(t)\right)/G$.

Proof. The result follows from the fact that there are G-equivariant homeomorphisms $\Gamma^M_{p,Gq}(t) \longrightarrow \Gamma^{M/G}_{\overline{p},\overline{q}}(t)$ and $\Gamma^M_{p,Gq}(t) \longrightarrow \left(\Gamma^M_{Gp,Gq}(t)\right)/G$. As the vector fields v_i are G-invariant, the corresponding flows φ_i are also G-invariant: $\varphi_i(gp,t) = g\varphi_i(p,t)$, and therefore $\varphi_{c,s}(gp,t) = g\varphi_{c,s}(p,t)$ for any pattern and time distribution (c,s). This shows that G acts on $\Gamma^M_{Gp,Gq}(t)$, and that $\Gamma^M_{p,q}(t) \simeq \Gamma^M_{gp,gq}(t)$ for $p, q \in M$. A pair (c,s) defines a directed path from $\overline{p}$ to $\overline{q}$ in M/G if and only if $\overline{\varphi}_c(\overline{p}, s) = \overline{q}$. If the latter equation holds we have that $\pi\varphi_c(p,s) = \overline{\varphi}_c(\overline{p},s) = \overline{q}$ and thus $\varphi_c(p,s) \in Gq$. Therefore (c,s) defines a directed path from $\overline{p}$ to $\overline{q}$ if and only if (c,s) defines an indirect influence from p to Gq. So we have shown that the map $\Gamma^M_{p,Gq}(t) \longrightarrow \Gamma^{M/G}_{\overline{p},\overline{q}}(t)$ is a G-equivariant homeomorphism. Similarly, if $a \in Gp$, then $\overline{\varphi}_c(\overline{p},s) = \overline{q}$ if and only if $\varphi_c(a,s) = \varphi_c(gp,s) = g\varphi_c(p,s)$ belongs to Gq. Thus the map $\Gamma^M_{p,Gq}(t) \longrightarrow \left(\Gamma^M_{G_p,G_q}(t)\right)/G$ is a G-equivariant homeomorphism. $\square$

6 Directed Paths for Constant Vector Fields

As a first and pretty workable example, linking the theory of indirect influences on directed manifolds with linear programming techniques, we consider constant vector fields on affine spaces. Fix a directed manifold $(\mathbb{R}^d, v_1, ..., v_k)$ where the vector fields $v_j = \sum_{j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}$, have constant coefficients $a_{ij} \in \mathbb{R}$ for $i \in [d]$, $j \in [k]$.

Theorem 17. Consider the directed manifold $(\mathbb{R}^d, v_1, ..., v_k)$. Fix a pattern $c \in D(n,k)$ and points $p, q \in \mathbb{R}^d$. The space of directed paths $\Gamma^c_{p,q}(t)$ is the convex polytope given on the variables $s \in \mathbb{R}^{n+1}_{\geq 0}$ by the system of equations: $a_{ic(0)}s_0 + \cdots + a_{ic(n)}s_n = q_i - p_i$, for $i \in [d]$, and $s_0 + \cdots + s_n = 1$.

Proof. The result follows from the fact that the solutions of the differential equation $\dot{p} = v$, where v is constant are of the form $p(t) = a + vt$, with initial condition $p(0) = a$. $\square$

Theorem 18. Consider the directed manifold $(\mathbb{R}^d, v_1, ..., v_k)$. For $p, q \in \mathbb{R}^d$, the volume of the space of directed paths $\Gamma^c_{p,q}(t)$ is given by $\mathrm{vol}(\Gamma^c_{p,q}(t)) = \mathrm{vol}(\mathrm{Conv}(u_I))$, where $\mathrm{Conv}(u_I)$ is the convex hull of the vectors u_I defined by the following conditions:

- $I \subseteq [d]$ is a subset of cardinality n. The entries of $u_I \in \mathbb{R}^{n+1}_{\geq 0}$ vanish for indexes not in I.
- For a matrix A we let A_I be its restriction to the columns with indexes in I. The set I must be such that $\det \begin{pmatrix} A_c \\ 1 \end{pmatrix}_I \neq 0$.
- u_I is the unique solution of the linear system $\begin{pmatrix} A_c \\ 1 \end{pmatrix}_I u_I \ = \ \begin{pmatrix} q-p \\ t \end{pmatrix}$.

Proof. From Theorem 17 and results from linear programming [16, 22] one have $\Gamma^c_{p,q}(t) = \mathrm{Conv}(u_I)$. □

6.1 Dimension One

Below we use the following well-known identity involving the classical beta B and gamma Γ functions:

$$\int_0^1 s^n(1-s)^m ds \ = \ \mathrm{B}(n+1,m+1) \ = \ \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} \ = \ \frac{n!m!}{(n+m+1)!}.$$

Theorem 19. Consider the directed manifold $(\mathbb{R}, \frac{d}{dx}, -\frac{d}{dx})$. For $x, y \in \mathbb{R}$ we have that $\mathrm{vol}(\Gamma_{0,x}(t)) = 0$ if $|x| > t$, $\mathrm{vol}(\Gamma_{0,x}(t)) = 1$ if $|x| = t$, and otherwise:

$$\mathrm{vol}(\Gamma_{0,x}(t)) \ = \ \sum_{n=0}^{\infty} \Big[2\frac{(t+x)^n(t-x)^n}{n!^2} \ + \ t\frac{(t+x)^n(t-x)^n}{(n+1)!n!} \Big] 2^{-2n}.$$

Furthermore, $\mathrm{vol}(\Gamma_{x,0}(t)) \ = \ \mathrm{vol}(\Gamma_{0,x}(t))$ and $\mathrm{vol}(\Gamma_{x,y}(t)) \ = \ \mathrm{vol}(\Gamma_{0,y-x}(t))$. The wave of influences for $t > 0$ is given by $u(x,t) \ = \ 4(e^t - 1)$.

Proof. Fix $x \in \mathbb{R}$ and a pattern $c \in D(n,k)$. The space of directed paths $\Gamma^c_{0,x}(t)$ is the polytope given by $\sum_{i=0}^n (-1)^{c_i} s_i = x$ and $\sum_{i=0}^n s_i = t$. Since we have just two vector fields, a pattern $(c_0, ..., c_n)$ is determined by its initial value c_0. Figure 3 shows the directed path associated to the tuple $(7,5,3,7) \in \Gamma^{(1,2,1,2)}_{(0,-2)}$.

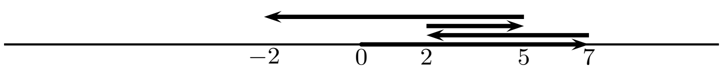

Figure 3: Directed path associated to the tuple $(7,5,3,7) \ \in \ \Gamma^{(1,2,1,2)}_{(0,-2)}$.

We distinguish four cases taking into account the initial value c_0 and the parity of n. Consider the pattern $(1,2,...,1,2)$ of length $2n$, for $n \geq 1$. Then $\Gamma^c_{0,x}(t)$ is the polytope given by $\sum_{i=0}^{2n-1} (-1)^i s_i \ = \ x$ and $\sum_{i=0}^{2n-1} s_i \ = \ t$. Setting $\sum_{i=0}^{n-1} s_{2i} \ = \ a$ and $\sum_{i=0}^{n-1} s_{2i+1} \ = \ b$, the previous equations become $a-b=x$ and $a+b=t$, with solutions $a=\frac{t+x}{2}$ and $b=\frac{t-x}{2}$. By definition $a,b\geq 0$, thus we must have $|x|<t$ in order that $\Gamma^c_{0,x}(t)\neq\emptyset$. For $|x|<t$, we have that $\Gamma^c_{0,x}(t)=\Delta_{n-1}(\frac{t+x}{2})\times\Delta_{n-1}(\frac{t-x}{2})$, and therefore $\mathrm{vol}(\Gamma^c_{0,x}(t)) = \frac{(t+x)^{n-1}(t-x)^{n-1}}{2^{2n-2}(n-1)!^2}$. For the pattern $(1,2,...,1,2,1)$ of length $2n+1$, with $n\geq 1$, we get

$$\mathrm{vol}(\Gamma^c_{0,x}(t)) \;=\; \mathrm{vol}\Big[\Delta_n(\frac{t+x}{2}) \;\times\; \Delta_{n-1}(\frac{t-x}{2})\Big] \;=\; \frac{(t+x)^n(t-x)^{n-1}}{2^{2n-1}n!(n-1)!}.$$

The pattern $c=(2,1,\cdots,2,1)$ of length $2n$, with $n\geq 1$, leads to

$$\mathrm{vol}(\Gamma^c_{0,x}(t)) \;=\; \mathrm{vol}\Big[\Delta_{n-1}(\frac{t-x}{2}) \;\times\; \Delta_{n-1}(\frac{t+x}{2})\Big] \;=\; \frac{(t+x)^{n-1}(t-x)^{n-1}}{2^{2n-2}(n-1)!^2}.$$

For the pattern $c=(2,1,\cdots,2,1,2)$ of length $2n+1$, with $n\geq 1$, we get that

$$\mathrm{vol}(\Gamma^c_{0,x}(t)) \;=\; \mathrm{vol}\Big[\Delta_n(\frac{t-x}{2}) \;\times\; \Delta_{n-1}(\frac{t+x}{2})\Big] \;=\; \frac{(t+x)^{n-1}(t-x)^n}{2^{2n-1}(n-1)!n!}.$$

Therefore $\mathrm{vol}(\Gamma_{0,x}(t))$ is for $|x|<t$ given by:

$$\sum_{n=1}^{\infty}\Big[4\frac{(t+x)^{n-1}(t-x)^{n-1}}{(n-1)!^2} \;+\; \frac{(t+x)^n(t-x)^{n-1}}{n!(n-1)!} \;+\; \frac{(t+x)^{n-1}(t-x)^n}{n!(n-1)!}\Big]2^{1-2n}$$

yielding the desired result.

Applying Theorem 11 we obtain that $\mathrm{vol}(\Gamma_{x,y}(t))=\mathrm{vol}(\Gamma_{x-x,y-x}(t))=\mathrm{vol}(\Gamma_{0,y-x}(t))$. In particular we get that $\mathrm{vol}(\Gamma_{x,0}(t))=\mathrm{vol}(\Gamma_{0,-x}(t))$. The explicit formula for $\mathrm{vol}(\Gamma_{x,y}(t))$ given above yields $\mathrm{vol}(\Gamma_{0,-x}(t))=\mathrm{vol}(\Gamma_{0,x}(t))$. Making the change of variables $y-x\to y$ we get:

$$u(x,t)=\int_{x-t}^{x+t}\mathrm{vol}(\Gamma_{0,x-y}(t))dy=\int_{-t}^{t}\mathrm{vol}(\Gamma_{0,-y}(t))dy=\int_{-t}^{t}\mathrm{vol}(\Gamma_{0,y}(t))dy=u(0,t).$$

To compute $u(0,t)$ we make the change of variable $y=t(2s-1)$ in the integral

$$\int_{-t}^{t}\sum_{n=0}^{\infty}\Big[2\frac{(t+y)^n(t-y)^n}{n!^2} \;+\; t\frac{(t+y)^n(t-y)^n}{(n+1)!n!}\Big]2^{-2n}dy \;=$$

$$4\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!} \;+\; 4\sum_{n=0}^{\infty}\frac{t^{2n+2}}{(2n+2)!} \;=\; 16\sinh(t) \;+\; 16(\cosh(t)-1) \;=\; 4(e^t-1).$$

$\square$

6.2 Dimension Two

Consider the directed manifold $(\mathbb{R}^2, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, and let $\Gamma(x,y) = \Gamma_{(0,0),(x,y)}$ be the moduli space of directed paths from $(0,0)$ to (x,y). Note that such influences can only happen at time $t = x+y$, and thus there is no need to include the time variable in the notation. Figure 4 shows the directed path associated to the tuple $(1,3,2,1) \in \Gamma^{(2,1,2,1)}(4,3)$.

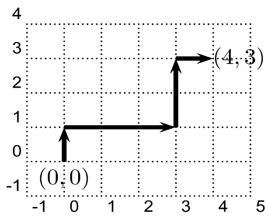

Figure 4. Directed path associated to $(1,3,2,1) \in \Gamma^{(2,1,2,1)}(4,3)$.

In our next results we use the following notation. For $k \in \mathbb{N}$ we set

$$i_k(x,y) = \sum_{n=0}^{\infty} \frac{x^n y^{n+k}}{n!(n+k)!} \quad \text{and} \quad i_{-k}(x,y) = i_k(y,x) = \sum_{n=0}^{\infty} \frac{x^{n+k} y^n}{(n+k)!n!}.$$

Lemma 20. For $l, m \in \mathbb{N}$ and $k \in \mathbb{Z}$ we have that $\frac{\partial^l}{\partial x^l}\frac{\partial^m}{\partial y^m} i_k(x,y) = i_{l-m+k}(x,y)$. For $k \in \mathbb{N}$, the function $i_k(x,y)$ is given in terms of the modified Bessel function $I_k(z)$ by $i_k(x,y) = x^{-\frac{k}{2}} y^{\frac{k}{2}} I_k(2\sqrt{xy})$, where we recall that $I_v(z) = (\frac{z}{2})^v \sum_{n=0}^{\infty} \frac{(z^2/4)^n}{n!\Gamma(v+n+1)}$. For $k \in \mathbb{N}$, we have that $I_k(z) = i_k(\frac{z}{2}, \frac{z}{2})$.

Theorem 21. Consider the directed manifold $(\mathbb{R}^2, \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y})$.

1. There are no directed paths from $(0,0)$ to a point $(x,y) \notin \mathbb{R}^2_{\geq 0}$.
2. $\mathrm{vol}(\Gamma(x,0)) = \mathrm{vol}(\Gamma(0,x)) = 1$, for $x \in \mathbb{R}_{>0}$.
3. For $(x,y) \in \mathbb{R}^2_{>0}$, the moduli space $\Gamma(x,y)$ of directed paths from $(0,0)$ to (x,y) has volume
$$\mathrm{vol}(\Gamma(x,y)) = i_{-1}(x,y) + 2i_0(x,y) + i_1(x,y) = \sum_{n=0}^{\infty} \Big(\frac{x^{n+1}y^n}{(n+1)!n!} + 2\frac{x^n y^n}{n!^2} + \frac{x^n y^{n+1}}{n!(n+1)!} \Big).$$
4. The derivatives of the function $\mathrm{vol}(\Gamma) = \mathrm{vol}(\Gamma(x,y))$ are given by:
$$\frac{\partial^l}{\partial x^l}\frac{\partial^m}{\partial y^m}\mathrm{vol}(\Gamma) = i_{l-m-1}(x,y) + 2i_{l-m}(x,y) + i_{l-m+1}(x,y).$$

5. We have that $\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathrm{vol}(\Gamma) = \mathrm{vol}(\Gamma)$.
6. Only points $(x, y) \in \mathbb{R}^2_{\geq 0}$ on the segment $x + y = t$ receive an influence from $(0, 0)$ at time $t \geq 0$. Among the points on this segment, the highest influence from $(0, 0)$ is exerted on the point $(\frac{t}{2}, \frac{t}{2})$; the volume of the moduli space of directed paths from $(0, 0)$ along the line of maximal influences is given by

$$\mathrm{vol}(\Gamma(t, t)) \ = \ 2\sum_{n=0}^{\infty} \binom{n}{\lfloor n/2 \rfloor} \frac{t^n}{n!}.$$

7. The wave of influences $u(x, y, t)$ is given for $t > 0$ by $u(x, y, t) = 2(e^t - 1)$.

Proof. Item 1 is clear, and item 2 simply counts the influences that arise, respectively, from the patterns (1) and (2). Let us show 3. Since $k = 2$, a pattern $(c_0, ..., c_n)$ is determined by its initial value c_0. For $(x, y) \in \mathbb{R}^2_{>0}$ we distinguish four cases taking into account the initial value c_0 and the parity of n.

- Patterns $(1, 2, ..., 1, 2)$ and $(2, 1, ..., 2, 1)$ of length $2n$, for $n \geq 1$, have a contribution of $\mathrm{vol}(\Delta^x_{n-1})\ \mathrm{vol}(\Delta^y_{n-1}) = \frac{x^{n-1}y^{n-1}}{(n-1)!^2}$ to the volume of the moduli space of directed paths.
- The pattern $(1, 2, ..., 1, 2, 1)$ of length $2n + 1$, for $n \geq 1$, have a contribution of $\mathrm{vol}(\Delta^x_n)\ \mathrm{vol}(\Delta^y_{n-1}) = \frac{x^n y^{n-1}}{n!(n-1)!}$ to the volume of the moduli space of directed paths.
- The pattern $(2, 1, ..., 2, 1, 2)$ of length $2n + 1$, for $n \geq 1$, have a contribution of $\mathrm{vol}(\Delta^x_{n-1})\ \mathrm{vol}(\Delta^y_n) = \frac{x^{n-1}y^n}{(n-1)!n!}$ to the volume of the moduli space of directed paths.

Putting together the three summands we obtain that

$$\mathrm{vol}(\Gamma(x, y)) \ = \ \sum_{n=1}^{\infty}\Big(2\frac{x^{n-1}y^{n-1}}{(n-1)!^2} \ + \ \frac{x^n y^{n-1}}{n!(n-1)!} \ + \ \frac{x^{n-1}y^n}{(n-1)!n!}\Big),$$

an expression equivalent to our desired result after a change of variables. Clearly, $\mathrm{vol}(\Gamma(x, y))$ is symmetric in x and y, thus item 4 follows.

Item 5 follows from item 3 and Lemma 20. Item 6 is a particular case of item 5. Let us show item 7. Let $\mathrm{vol}_n(\Gamma(x, y))$ be the n-th coefficient in the series expansion of $\mathrm{vol}(\Gamma(x, y))$ from item 3. The points influenced by $(0, 0)$ at time t are of the form $(s, t - s)$ with $0 < s < t$. Thus:

$$\mathrm{vol}_n(\Gamma(s, t - s)) \ = \ (st - s^2)^{n-1}\Big(\frac{2}{(n-1)!^2} \ + \ \frac{t}{(n-1)!n!}\Big).$$

Therefore $\frac{\partial}{\partial s}\mathrm{vol}_n(\Gamma(s, t-s)) = (n-1)(st-s^2)^{n-2}(t-2s)\Big(\frac{2}{(n-1)!^2} + \frac{t}{(n-1)!n!}\Big)$. The sign of this expression is determined by the sign of $(t - 2s)$, as the other factors are positive. Thus the volume of the moduli space of directed paths from $(0,0)$ exerted on time t achieves a global maximum at the point $(\frac{t}{2}, \frac{t}{2})$, and we have that $\mathrm{vol}(\Gamma(t,t)) = 2\sum_{n=0}^{\infty}\Big(\frac{t^{2n}}{n!^2} + \frac{t^{2n+1}}{(n+1)!n!}\Big) =$

$$2\sum_{n=0}^{\infty}\Big(\binom{2n}{n}\frac{t^{2n}}{(2n)!} + \binom{2n+1}{n}\frac{t^{2n+1}}{(2n+1)!}\Big) = 2\sum_{n=0}^{\infty}\binom{n}{\lfloor n/2\rfloor}\frac{t^n}{n!}.$$

Item 8. By translation invariance the wave of influence is independent of x, y. Thus we have that

$$u(x,y,t) = u(0,0,t) = \int_0^t \mathrm{vol}(\Gamma_{(-s,s-t),(0,0)}(t))ds = \int_0^t \mathrm{vol}(\Gamma_{(0,0),(s,t-s)}(t))ds =$$

$$\int_0^t \Gamma(s,t-s)ds = \sum_{n=0}^{\infty}\int_0^t \Big(2\frac{s^n(t-s)^n}{n!^2} + \frac{s^{n+1}(t-s)^n}{(n+1)!n!} + \frac{s^n(t-s)^{n+1}}{n!(n+1)!}\Big)ds =$$

$$2\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!} + 2\sum_{n=0}^{\infty}\frac{t^{2n+2}}{(2n+2)!} = 2(\sinh(t)+\cosh(t)-1) = 2(e^t-1).$$

□

Next we consider the moduli spaces of directed paths on the torus $T^2 = S^1 \times S^1$. We use coordinates $(x,y) \in \mathbb{R}^2$ representing the point $(e^{2\pi ix}, e^{2\pi iy}) \in T^2$. Consider the vector fields on T^2 given in local coordinates by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. The moduli space of directed paths on the torus T^2 from $(1,1)$ to $(e^{2\pi ix}, e^{2\pi iy})$ exerted in time $t > 0$ is denoted by $\Gamma(e^{2\pi ix}, e^{2\pi iy}, t)$. Recall that $D(e^{2\pi ix}, e^{2\pi iy}, t)$ is the set of one-direction paths.

Theorem 22. Consider the directed manifold $(T^2, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

1. For $x, y \in (0,1)$ we have that $\mathrm{vol}(D(e^{2\pi ix}, e^{2\pi iy}, t)) = 0$.
2. For $x \in (0,1]$ we have that $\mathrm{vol}(D(e^{2\pi ix}, 1, t)) = \mathrm{vol}(D(1, e^{2\pi ix}, t)) = \sum_{m=0}^{\infty}\delta(t, x+m)$.
3. For $(x,y) \in (0,1)^2$, the moduli space $\Gamma(e^{2\pi ix}, e^{2\pi iy}, t)$ of directed paths from $(1,1)$ to $(e^{2\pi ix}, e^{2\pi iy})$ is empty unless $t = x+y+m$ for some $m \geq 0$, and in the latter case we have that: $\mathrm{vol}(\Gamma(e^{2\pi ix}, e^{2\pi iy}, x+y+m))$ is given by
$$\sum_{k+l=m}\sum_{n=0}^{\infty}\Big(2\frac{(x+k)^n(y+l)^n}{n!^2} + (x+y+k+l)\frac{(x+k)^n(y+l)^n}{(n+1)!n!}\Big).$$
4. $\mathrm{vol}(\Gamma(e^{2\pi ix}, e^{2\pi iy}, x+y+m))$ is a symmetric function in x and y.

Proof. We can compute indirect influences on the torus as sums of indirect influences on the plane, indeed we have that $\mathrm{vol}(\Gamma(e^{2\pi ix}, e^{2\pi iy}, x+y+m)) = \sum_{k+l=m}\mathrm{vol}(\Gamma(x+k, y+l, x+y+m)) =$

$$\sum_{k+l=m}\sum_{n=0}^{\infty}\Big(2\frac{(x+k)^n(y+l)^n}{n!^2} + \frac{(x+k)^{n+1}(y+l)^n}{(n+1)!n!} + \frac{(x+k)^n(y+l)^{n+1}}{n!(n+1)!}\Big).$$

□

6.3 Higher Dimensions

Let us first introduce a few combinatorial notions. Given integers $n_1, \ldots, n_k \in \mathbb{N}_{>0}$ we let $\mathrm{Sh}_k(n_1, \ldots, n_k)$ be the set of shuffles of $n_1 + \cdots + n_k$ cards divided into k blocks of cardinalities $n_1, \ldots, n_k$. Recall that a shuffle is a bijection α from the set $[1, n_1+\cdots+n_k] \simeq [1, n_1] \sqcup \cdots \sqcup [1, n_k]$ to itself such that if $i < j \in [1, n_s]$, then $\alpha(i) < \alpha(j) \in [1, n_1+\cdots+n_k]$. When we shuffle a deck of cards split in several blocks, we intertwine the cards in the various blocks, without distorting the order on each block. We say that a shuffle is perfect if no contiguous cards within a block remain contiguous after shuffling, i.e. a shuffle α is called perfect if for $i,\ i+1 \in [1, n_s]$ we have that $\alpha(i) + 1 < \alpha(i+1) \in [1, n_1 + \cdots + n_k]$. Let $\mathrm{PSh}_k(n_1, \ldots, n_k) \subseteq \mathrm{Sh}_k(n_1, \ldots, n_k)$ be the set of perfect shuffles, and psh_k be the corresponding exponential generating series given by

$$\mathrm{psh}_k(x_1, \ldots, x_k) = \sum_{n_1, \ldots, n_k \in \mathbb{N}_{>0}} |\mathrm{PSh}_k(n_1, \ldots, n_k)| \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!}.$$

A subset $A \subseteq [m]$ is called sparse if it does not contain consecutive elements. Let $S_k[m]$ be the set of all sparse subsets of $[m]$ of cardinality k. Let $p(m, k)$ count the numerical partitions of m in k positive summands.

Lemma 23. For $1 \leq k < m \in \mathbb{N}$, we have that $|S_k[m]| = p(m-k, k-1) + 2p(m-k, k) + p(m-k, k+1)$. For $n_1, \ldots, n_k \in \mathbb{N}_{>0}$, then $|\mathrm{PSh}_k(n_1, \ldots, n_k)|$ counts ordered partitions of $n_1 + \ldots + n_k$ with sparse blocks of cardinalities $n_1, \ldots, n_k$.

Consider the map $|\ | : C(n, k) \longrightarrow \mathbb{N}^k$, sending a pattern $c \in C(n, k)$ to its content multi-set given by the sequence $|c| \in \mathbb{N}^k$ such that $|c|_i = |c^{-1}(i)|$. The support of a pattern c is the set $s(c) \subseteq [k]$ with $i \in s(c)$ if and only if $|c|_i \neq 0$.

Lemma 24. Fix $(n_1, \ldots, n_k) \in \mathbb{N}^k_{>0}$. Then $\big|\{c \in C(n, k) \mid |c| = (n_1, ..., n_k)\}\big| = \big|\mathrm{PSh}_k(n_1, \ldots, n_k)\big|$.

Consider the directed manifold $(\mathbb{R}^k, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})$. The moduli space of directed paths from $(0, \ldots, 0)$ to $(x_1, \ldots, x_k)$ is denoted by $\Gamma(x_1, \ldots, x_k)$. Such paths can only happen at time $t = x_1 + \cdots + x_k$.

Theorem 25. Consider the directed manifold $(\mathbb{R}^k, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})$.

1. There are no directed paths from $(0, \ldots, 0)$ to any point $(x_1, \ldots, x_k) \notin \mathbb{R}^k_{\geq 0}$.
2. $\mathrm{vol}(D(0, \ldots, 0, \underset{i\uparrow}{x}, 0, \ldots, 0) = 1$, for $x \in \mathbb{R}_{\geq 0}$ and $i \in [k]$.
3. For $(x_1, \ldots, x_k) \in \mathbb{R}^k_{\geq 0}$, with at least two positive entries, the moduli space $\Gamma(x_1, \ldots, x_k)$ of directed paths from $(0, \ldots, 0)$ to $(x_1, \ldots, x_k)$ has volume

$$\mathrm{vol}(\Gamma(x_1, \ldots, x_k)) = \sum_{\substack{A \subseteq [k] \\ |A| \geq 2}} \frac{\partial^{|A|}}{\partial x_A} \mathrm{psh}_{|A|}(x_A).$$

4. $\mathrm{vol}(\Gamma(x_1,\ldots,x_k))$ is a symmetric function in the variables $x_1,\ldots,x_k$.

Proof. Properties 1 and 2 are clear, let us prove 3. Recall that

$$\mathrm{vol}(\Gamma(x_1,\ldots,x_k)) \;=\; \sum_{n=1}^{\infty}\sum_{c\in C(n,k)}\mathrm{vol}(\Gamma^c(x_1,\ldots,x_k)),$$

where the volume of the moduli space of directed paths with a fix pattern $c\in C(n,k)$ is given by

$$\mathrm{vol}(\Gamma^c(x_1,\ldots,x_k)) \;=\; \prod_{j\in s(c)}\frac{x_j^{|c|_j-1}}{(|c|_j-1)!}.$$

Thus a pattern $c\in C(n,k)$ with support $s(c) = A \subseteq [k]$, with $|A|\geq 2$, contributes to the monomial $\frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1!\cdots n_k!}$, if and only if $|c|_i = n_i+1$ for $i\in A$, and $n_i=0$ for $i\notin A$. Therefore the total contribution of the patterns with support A to this monomial is given by $|\mathrm{PSh}_{|A|}(n_A+1)|\prod_{j\in A}\frac{x_j^{n_j}}{n_j!}$, where n_A is the vector obtained from the tuple $(n_1,...,n_k)$ by erasing the zero entries, and n_A+1 is the vector obtain from n_A by adding 1 to each entry.

Summing over the n_j, and setting $x_A=(x_j)_{j\in A}$, we obtain that the total contribution of the patterns with support A to the volume of the moduli space of direct ed paths is given by

$$\sum_{n_j\in\mathbb{N};\ j\in A}|\mathrm{PSh}_{|A|}(n_A+1)|\prod_{j\in A}\frac{x_j^{n_j}}{n_j!} \;=\; \frac{\partial^{|A|}}{\partial x_A}\mathrm{psh}_{|A|}(x_A).$$

Adding over all possible supports $A\subseteq[k]$, with $|A|\geq 2$, we obtain the desired result.

4. For a permutation $\sigma\in S_n$ we have that $\mathrm{vol}(\Gamma(x_{\sigma(1)},\ldots,x_{\sigma(k)})$ is given by

$$\sum_{\substack{A\subseteq[k]\\|A|\geq 2}}\frac{\partial^{|\sigma A|}}{\partial x_{\sigma A}}\mathrm{psh}_{|\sigma A|}(x_{\sigma A}) \;=\; \sum_{\substack{A\subseteq[k]\\|A|\geq 2}}\frac{\partial^{|A|}}{\partial x_A}\mathrm{psh}_{|A|}(x_A) \;=\; \mathrm{vol}(\Gamma(x_1,\ldots,x_k)).$$

□

Next we consider directed paths on the k-dimensional torus $T^k = S^1\times\cdots\times S^1$. We use coordinates $(x_1,\ldots,x_k)\in\mathbb{R}^k$ representing the point $(e^{2\pi i x_1},\ldots,e^{2\pi i x_k})\in T^k$. Consider the constant vector fields on T^k given in local coordinates by $\frac{\partial}{\partial x_1},\ \cdots\ ,\frac{\partial}{\partial x_k}$. The moduli space of directed paths on T^k from $(1,...,1)$ to $(e^{2\pi i x_1},\ldots,e^{2\pi i x_k})$ exerted in time $t>0$ is denoted by $\Gamma(e^{2\pi i x_1},\ldots,e^{2\pi i x_k},t)$. Recall that the set of one-direction paths is denoted by $D(e^{2\pi i x_1},\ldots,e^{2\pi i x_k},t)$.

Theorem 26. Consider the directed manifold $(T^k, \frac{\partial}{\partial x_1}, \ \cdots \ , \frac{\partial}{\partial x_k})$.

1. For $x_1, \ldots, x_k \in (0,1]$, with at least two entries in $(0,1)$, we have $\mathrm{vol}(D(e^{2\pi i x_1}, \ldots, e^{2\pi i x_k}, t)) = 0$.
2. For $x \in (0,1]$ we have that:

$$\mathrm{vol}(D(1, \ldots, \underset{i\uparrow}{e^{2\pi i x}}, \ldots, 1, t)) \ = \ \sum_{m=0}^{\infty} \delta(t, x_i + m).$$

3. For $x_1, \ldots, x_k \in (0,1]$, with at least two entries in $(0,1)$, the moduli space $\Gamma(e^{2\pi i x_1}, \ldots, e^{2\pi i x_k}, t)$ of directed paths from $(1, \ldots, 1)$ to $(e^{2\pi i x_1}, \ldots, e^{2\pi i x_k})$ is empty unless $t = x_1 + \cdots + x_k + m$ for some $m \geq 0$, and in the latter case we have that:

$$\mathrm{vol}(\Gamma(e^{2\pi i x_1}, \ldots, e^{2\pi i x_k}, x_1 + \cdots + x_k + m)) \ = \ \sum_{m_1 + \ldots + m_k = m} \ \sum_{\substack{A \subseteq [d] \\ |A| \geq 2}} \frac{\partial^{|A|}}{\partial x_A} \mathrm{psh}_{|A|}(x_A \ + \ m_A).$$

4. $\mathrm{vol}(\Gamma(e^{2\pi i x_1}, \ldots, e^{2\pi i x_k}, x_1 + \cdots + x_k + m))$ is a symmetric function on $x_1, \ldots, x_k$.

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References

[1] A. Agrachev, Y. Sachkov, Control theory from the geometric viewpoint, Springer-Verlag, Berlin 2004.

[2] V. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York 1989.

[3] S. Brin, L. Page, R. Motwani, T. Winograd, The Anatomy of a Large-Scale Hypertextual Web Search Engine, Comp. Netw. ISDN Sys. 30 (1998) 107-117.

[4] L. Cano, R. Díaz, Continuous Analogues for the Binomial Coefficients and the Catalan Numbers, preprint, arXiv:1602.09132.

[5] M. Chas, D. Sullivan, String Topology, preprint, arXiv:math/9911159.

[6] F. Chung, The heat kernel as the pagerank of a graph, Proc. Natl. Acad. Sci. U.S.A. 104 (2007) 19735-19740.

[7] R. Díaz, Indirect Influences, Adv. Stud. Contemp. Math. 23 (2013) 29-41.

[8] R. Díaz, L. Gómez, Indirect Influences in International Trade, Netw. Heterog. Media 10 (2015) 149-165.

[9] R. Díaz, A. Vargas, On the Stability of the PWP method, preprint, arXiv:1504.03033.

[10] E. Estrada, N. Hatano, Communicability in complex networks, Phys. Rev. E 77 (2008) 036111.

[11] M. Godet, De l'Anticipation à l'Action, Dunod, París 1992.

[12] M. Grandis, Directed Algebraic Topology, Models of non-reversible worlds, Camb. Univ. Press, Cambridge 2009.

[13] V. Guillemin, A. Pollack, Differential topology, Springer-Verlag, New York 1983.

[14] L. Katz, A new status index derived from sociometric analysis, Psychmetrika 18 (1953) 39-43.

[15] A. Langville, C. Meyer, Deeper Inside PageRank, Internet Math. 1 (2004) 335-400.

[16] J. Matoušek, B. Gaertner, Understanding and Using Linear Programming, Springer-Verlag, Berlin 2007.

[17] R. Penrose, Techniques of Differential Topology in Relativity, Soc. Industrial Appl. Math., Bristol 2014.

[18] B. Simon, Functional integration and quantum physics, Amer. Math. Soc, Providence 2005.

[19] E. Sontag, Mathematical control theory, Springer-Verlag, New York 1998.

[20] C. Vignat, T. Wakhare, On the continuous binomial coefficients by L. Cano and R. Diaz, preprint, arXiv:1606.01986.

[21] F. Warner, Foundations of differentiable manifolds and Lie groups, Springer-Verlag, New York 1983.

[22] X. Yang, Introduction to mathematical optimization, Cambridge Int. Science Publishing, Cambridge 2008.

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