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ON THE TENSOR PRODUCT OF UNBOUNDED QUASI-REPRESENTATIONS OF GROUPS

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Abstract. We prove that, in contrast to the case of the tensor product of two bounded quasi-representations, the tensor product of every quasi-representation T (of a group) admitting an invertible defect operator with an unbounded quasi-representation is not a quasi-representation. In particular, the tensor product of two quasi-representations of a group need not be a quasi-representation if one of the quasi-representations is unbounded.

§ 1. Introduction

Recall that any mapping T of a given group G into the group of invertible operators on some Banach space E such that $T(e_G) = 1_E$ (where $E_G = e$ stands for the identity element of G) and the norm $\|T(gh) - T(g)T(h)\|$, $g, h \in G$, is uniformly small on G, which means that

$$\|T(gh) - T(g)T(h)\| \le \delta \text{ for any } g, h \in G \text{ and for some small } \delta > 0,$$

is referred to as a *quasi-representation* (more exactly, as a δ-*quasi-representation*); see [1–2]. The operators of the form $T(gh) - T(g)T(h)$ are called *defect operators.*

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Definition 1. Let T and S be two quasi-representations of a group G in Banach spaces E and F, respectively, and let $E \otimes F$ be some Banach tensor product of the Banach spaces with respect to some tensor norm [3]. The mapping $T \otimes S\colon G \to \mathcal{L}(E \otimes F)$ (where $\mathcal{L}(E)$ means the Banach algebra of bounded linear operators on E) defined by the rule

$$(T \otimes S)(g) = T(g) \otimes S(g), \qquad g \in G,$$

is called the *tensor product* of the quasi-representations T and S.

As was claimed (without proof) already in [1], the following assertion holds.

Theorem 1. *Let T and S be ε- and δ-representations, respectively, and let T and S be bounded, i.e.,*

$$\|T(g)\|_{\mathcal{L}(E)} \le C_T \quad \textit{for any} \quad g \in G, \qquad \|S(g)\|_{\mathcal{L}(E)} \le C_S \quad \textit{for any} \quad g \in G.$$

Then $T \otimes S$ is a bounded mapping,

$$\|T \otimes S(g)\|_{\mathcal{L}(E\otimes F)} \le C_T C_S \quad \textit{for any} \quad g \in G, \tag{1}$$

and $T \otimes S$ is a $C_S\varepsilon + C_T\delta$-quasi-representation of G in $E \otimes F$.

Proof. Inequality (1) is obvious. Further,

$$\begin{aligned}
&\|(T \otimes S)(gh) - (T \otimes S)(g)(T \otimes S)(h)\|_{\mathcal{L}(E\otimes F)} \\
&\quad = \|T(gh) \otimes S(gh) - T(g)T(h) \otimes S(g)S(h)\|_{\mathcal{L}(E\otimes F)} \\
&\quad = \|(T(gh) - T(g)T(h)) \otimes S(gh) \\
&\qquad + T(g)T(h) \otimes (S(gh) - S(g)S(h))\|_{\mathcal{L}(E\otimes F)} \\
&\quad \le \|(T(gh) - T(g)T(h)) \otimes S(gh)\|_{\mathcal{L}(E\otimes F)} \\
&\qquad + \|T(g)T(h) \otimes (S(gh) - S(g)S(h))\|_{\mathcal{L}(E\otimes F)} \\
&\quad \le C_S\varepsilon + C_T\delta,
\end{aligned}$$

as was to be proved.

A natural problem arises: whether or not a tensor product of two (not necessarily bounded) quasi-representations of a group is a quasi-representation again. In the present note, we prove that the tensor product of every quasi-representation T admitting an invertible defect operator with an unbounded quasi-representation of this group is not a quasi-representation of the group. In particular, the tensor product of two not necessarily bounded quasi-representations need not be a quasi-representation.

§ 2. Main Theorem

Theorem. *The tensor product of every quasi-representation T (of a group) admitting an invertible defect operator with any unbounded quasi-representation S of this group is not a quasi-representation of the group.*

Proof. Let T and S be quasi-representations of a group G in Banach spaces E and F, respectively. Note that

$$\begin{aligned}(T\otimes S)(gh) &- (T\otimes S)(g)(T\otimes S)(h)\\ &= (T(gh)-T(g)T(h))\otimes S(gh)\\ &\quad + T(g)T(h)\otimes (S(gh)-S(g)S(h)).\end{aligned}$$

It follows from our assumptions that the norm of the second summand is bounded by $C_T^2\delta$. Therefore, the difference in question has uniformly bounded norm on $G\times G$ if and only if the first summand is uniformly bounded on $G\times G$. However, if the defect operator $T(gh)-T(g)T(h)$ is invertible, then we can consider the operator

$$\begin{aligned}&(T(ghk)-T(gh)T(k))\otimes S(ghk)\\ &\quad = (T(ghk)-T(g)T(hk))\otimes S(ghk)\\ &\qquad + (T(g)T(hk)-T(g)T(h)T(k))\otimes S(ghk)\\ &\qquad + (T(g)T(h)T(k)-T(gh)T(k))\otimes S(ghk).\end{aligned} \tag{2}$$

If the first summand on the right-hand side of (2) is unbounded, then $T\otimes S$ is not a quasi-representation, and the theorem is proved, and therefore we may assume that this summand is bounded. Let us consider the second summand. Let us use the assumption of the theorem and choose some g and h in G such that the defect operator $T(gh)-T(g)T(h)$ is invertible and consider the element $(T(g)T(hk)-T(g)T(h)T(k))\otimes S(ghk)$ as a function of $k\in G$; write

$$\begin{aligned}&(T(g)T(hk)-T(g)T(h)T(k))\otimes S(ghk)\\ &\quad = (T(g)T(hk)-T(g)T(h)T(k))\otimes (S(ghk)-S(gh)S(k))\\ &\qquad + (T(g)T(hk)-T(g)T(h)T(k))\otimes S(gh)S(k)\\ &\quad = T(g)(T(hk)-T(h)T(k))\otimes (S(ghk)-S(gh)S(k))\\ &\qquad + (T(g)\otimes S(gh))((T(hk)-T(h)T(k))\otimes S(k)).\end{aligned}$$

Since g is chosen, it follows that the first summand on the right-hand side is bounded with respect to k by some constant; the first factor in the other summand is fixed and invertible, while the other factor is unbounded together with $S(k)$. If the function $k \mapsto (T(hk) - T(h)T(k)) \otimes S(k)$, $k \in G$, is unbounded, then $T \otimes S$ is not a quasi-representation, and the theorem is proved, and therefore we may assume that this function is bounded. Then the second summand of (2) is bounded, together with the first summand, while the third summand, $(T(g)T(h)T(k) - T(gh)T(k)) \otimes S(ghk) = ((T(g)T(h) - T(gh)) \otimes 1_F)(T(k) \otimes S(ghk))$ has a fixed bounded invertible first factor and an unbounded other factor by the assumption of the theorem, and thus is obviously unbounded. This completes the proof of the theorem.

§ 4. Discussion

Let G be the countable product, over $\mathbb{N}$, of counterparts of the symmetric group S_3 defined for all elements of $\mathbb{N}$. One can readily construct a small perturbation π of the canonical three-dimensional representation of G which is a quasi-representation with invertible defect operator. Let χ be a free ultrafilter on $\mathbb{N}$. Then the formula

$$T(s) = \chi(\pi(s)), \qquad s \in G,$$

where χ is applied entrywise, defines a quasi-representation of G satisfying the conditions of Theorem 2, because the group G obviously admits unbounded quasi-representations.

It is of interest to find a simpler example of a group with a quasi-representation admitting an invertible defect operator.

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