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System of first order linear (p, q)-difference equations

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Abstract In this paper, we consider a system of first order linear (p, q)-difference equations, using (p, q)-derivative and (p, q)-integral. We find the solutions of a system of linear (p, q)-difference equations which are similar to the general linear difference equations.

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1. Introduction

In the 18th century, first formula of q-calculus is obtained by Euler. Many mathematicians have been studied q-calculus and they found some classical theory and several remarkable results for q-calculus. After Jackson introduced the definition of q-integral in 1910, the subject of q-calculus has been studied by many mathematicians and physicists. The applications of q-calculus has played an important role in the area of approximation theory, number theory and theoretical physics (see [7]-[12]).

Several researchers obtained various other generalizations of operators based on q-calculus (see [3, 6, 13]). Further generalization of

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q-calculus is the post quantum calculus, denoted by (p,q)-calculus. Recently, Mursaleen et al. applied (p,q)-calculus in approximation theory and introduced (p,q)-analogue of Bernstein operators.

In 1991, R. Chakrabarti and R. Jagannathan[3] introduced the two-parameter quantum, that is, (p,q)-number in physics literature. Around the same time, G. Brodimas, et al. and M. Arik, et al. made the (p,q)-number (see [1, 2, 16]), independently and Wachs and White[17] introduced the (p,q)-number in the mathematics by certain combinatorial problems that is irrelevant to the quantum group.

Since (p,q)-number was introduced, many mathematicians have been studied (p,q)-calculus including (p,q)-exponential, integration, series and differentiation from (p,q)-number until the present day. Katriel and Kibler[15] defined the (p,q)-binomial coefficient and derived a (p,q)-binomial theorem. Smirnov and Wehrhahn[5] gave an operator, or noncommutative, version of such a (p,q)-binomial theorem. Burban and Klimyk[6] studied (p,q)-differentiation, (p,q)-integration. In, 2006, R. Jagannathan and K. S. Rao[2] made the (p,q)-extensions of q-identities. P. N. Sadjang[13] represented two appropriate polynomials of the (p,q)-derivative and investigated some properties of these polynomials. In addition, he discovered two (p,q)-Taylor formulas of polynomials and obtained the formula of (p,q)-integration by part.

In [14], we have discussed methods for solving an ordinary differential equation that involves only one dependent variable by using (p,q)-derivative operator. Many applications, however, require the use of two or more dependent variables, each a function of a single independent variable. Such problems lead naturally to a system of simultaneous ordinary differential equations.

In this paper, we consider a system of first order linear (p,q)-difference equations. We introduce some basic notations about (p,q)-calculus which is found in [1, 3, 4, 5, 6, 13, 16, 17].

Definition 1.1. For any $n \in \mathbb{C}, 0 < q < p \leq 1$, the (p,q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{1.1}$$

Note that the (p,q) number is reduced to q-number, that is, $\lim_{p\to 1}[n]_{p,q} = [n]_q$ for $q \neq 1$ and it is clear that (p,q)-number has symmetric property.

The (p,q)-binomial coefficients are defined by $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \dfrac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$, $0 \le k \le n$ where $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}$ for $n = 1, 2, \cdots$, and $[0]_{p,q}! = 1$.

Definition 1.2. Let f be a function on the set of the complex numbers. We define the (p,q)-derivative of the function f as follows

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0, \tag{1.2}$$

and since $D_{p,q}f(0) = f'(0)$, then it is provided that f is differentiable at 0.

Since $D_{p,q}z^n = [n]_{p,q}z^{n-1}$, if $t(x) = \sum_{k=0}^{n} a_k x^k$ then

$$D_{p,q}t(x) = \sum_{k=0}^{n-1} a_{k+1}[k+1]_{p,q}x^k. \tag{1.3}$$

This equation is equivalent to solve the (p,q)-difference equation in q with known f

$$D_{p,q}g(x) = f(x).$$

From the Definition 1.2, we have

$$\frac{1 - T_{p,q}}{\left(1 - \frac{q}{p}\right)x} g(x) = f\left(\frac{1}{p}x\right), \qquad T_{p,q}g(x) = g\left(\frac{q}{p}x\right)$$

and

$$D_{p,q}f(x) = D_{1,\frac{q}{p}}f(px). \tag{1.4}$$

Thus, we can see that

$$\begin{aligned} g(x) &= \left(1 - \frac{q}{p}\right)\sum_{i=0}^{\infty} T_{p,q}^i \left\{xf\left(\frac{1}{p}x\right)\right\} \\ &= \left(1 - \frac{q}{p}\right)x\sum_{i=0}^{\infty}\left(\frac{q}{p}\right)^i f\left(\frac{q^i}{p^{i+1}}x\right). \end{aligned}$$

If the series in the right hand side of above is convergent then we can find the previous calculus is obviously valid. Let f be an arbitrary function. In [13], we note that the definition of (p,q)- integral is

$$\int f(x)d_{p,q}x = (p-q)x\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}x\right). \tag{1.5}$$

The operators of (p,q)-difference equation have the following theorem.

Theorem 1.1. The operators, $D_{p,q}$, has that

(i) Derivative of a product

$$\begin{aligned}D_{p,q}(f(x)g(x)) &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)\\ &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x),\end{aligned}$$

(ii) Derivative of a ratio

$$\begin{aligned}D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}\\ &= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.\end{aligned}$$

A general linear (p,q)-difference equations of first order is represented

$$D_{p,q}y(x) = a(x)y(qx) + b(x), \tag{1.6}$$

a non homogeneous equation while the corresponding homogeneous one has

$$D_{p,q}y(x) = a(x)y(qx). \tag{1.7}$$

We also get the solutions of the homogeneous equations in the exponential functions.

The most important aim of this paper is to find solutions of system of first order (p,q)-differential equations. In Section 2, we investigate solutions about the system of first order linear (p,q)-differential equations in various case. In Section 3, we derive solutions about the autonomous systems of first order nonlinear (p,q)-differential equations in some case and also include each examples.

2. The system of linear (p, q)-difference equations

In this section, we investigate solutions about the system of first order linear (p, q)-differential equations in various cases. Consider a simple case of the system of linear (p, q)-difference equations

$$\begin{pmatrix} D_{p,q}y_1(x) \\ D_{p,q}y_2(x) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1(qx) \\ y_2(qx) \end{pmatrix},$$

then we have the solution as follows

$$\begin{aligned} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} &= \begin{pmatrix} 1 + a_{11}(1-\frac{q}{p})x & a_{12}(1-\frac{q}{p})x \\ a_{21}(1-\frac{q}{p})x & 1 + a_{22}(1-\frac{q}{p})x \end{pmatrix} \begin{pmatrix} y_1(\frac{q}{p}x) \\ y_2(\frac{q}{p}x) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-\frac{q}{p})x \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1(qx) \\ y_2(qx) \end{pmatrix}. \end{aligned}$$

Now, we can write the general case of the system of linear (p, q)-difference equations, that is,

$$\begin{pmatrix} D_{p,q}y_1(x) \\ D_{p,q}y_2(x) \\ \vdots \\ D_{p,q}y_k(x) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} y_1(qx) \\ y_2(qx) \\ \vdots \\ y_k(qx) \end{pmatrix}.$$

From the above equations, we obtain a simple representation of the system of linear (p, q)-difference equations as below

$$D_{p,q}y(x) = A(x)y(qx) + b(x), \tag{2.1}$$

where $y(x) = (y_1(x), \cdots, y_k(x))^t$, $b(x) = (b_1(x), \cdots, b_k(x))^t \in \mathbb{R}^k$, $A(x) = (a_{i,j}(x))_{i,j=1}^k$.

Note that, the system is said to be non homogeneous equation and the corresponding homogeneous equation is

$$D_{p,q}y(x) = A(x)y(qx). \tag{2.2}$$

From equations (2.1) and (2.2), we have following results by definitions of (p, q)-derivative formula, respectively,

$$y(x) = \left[I - (1-\frac{q}{p})xA(\frac{1}{p}x)\right] y(\frac{q}{p}x) + x(1-\frac{q}{p})b(\frac{1}{p}x), \tag{2.3}$$

$$y(x) = \left[I - (1-\frac{q}{p})xA(\frac{1}{p}x)\right] y(\frac{q}{p}x). \tag{2.4}$$

By using the recurrence relation, we get the next theorem from equation (2.4).

Theorem 2.1. Consider the homogeneous equations $D_{p,q}y(x) = A(x)y(qx)$, then we obtain

$$y(x) = y(x_0) \prod_{t=(\frac{q}{p})^{-1}x_0}^{x} \left[I + (1-\frac{q}{p})tA(\frac{1}{p}t)\right].$$

Proof. By the definition of (p,q)-derivative, the homogeneous equations, $D_{p,q}y(x) = A(x)y(qx)$, one has

$$y(x) = \left[I - (1-\frac{q}{p})xA(\frac{1}{p}x)\right] y(\frac{q}{p}x).$$

And using the recurrence relation, Theorem 2.1 is proved.

$$\begin{aligned} y(x) &= y((\frac{q}{p})^N x) \prod_{i=0}^{N-1} \left[I + (1-\frac{q}{p})\frac{q^i}{p^i}xA(\frac{q^i}{p^{i+1}}x)\right] \\ &= y(x_0) \prod_{t=(\frac{q}{p})^{-1}x_0}^{x} \left[I + (1-\frac{q}{p})tA(\frac{1}{p}t)\right]. \end{aligned} \tag{2.5}$$

Taking $x_0 = 0$, we have

$$y(x) = y(0)\left[I + (1-\frac{q}{p})(\frac{q}{p})^i xA(\frac{q}{p})^i\frac{1}{p}x\right] \tag{2.6}$$

□

Corollary 2.1. If $N \to \infty$ for $0 < \frac{q}{p} < 1$, then $(\frac{q}{p})^N$ approaches 0.

$$y(x) = y(0)\prod_{i=0}^{\infty}\left[I + (1-\frac{q}{p})(\frac{q}{p})^i xA((\frac{q}{p})^i\frac{1}{p}x\right].$$

From above theorem, we can see that there exists a unique solution of homogeneous equations satisfying $y(x_0) = v_0$, for any vector $v_0 \in \mathbb{R}^k$.

Let $Y(x)$ be the matrix that column are constituted by the vectors $y_1(x), \cdots, y_k(x)$ which is a system of k vectors in $\mathbb{R}^k$. There exists a unique matrix solution of homogeneous equations satisfying $T(x_0) = V_0$, for any $k \cdot k$ matrix V_0.

Therefore, we can write

$$Y(x) = V_0 \prod_{t=(\frac{q}{p})^{-1}x_0}^{x}\left[I + (1-\frac{q}{p})tA(\frac{1}{p}t)\right],$$

and for $x_0 = 0$, we have

$$Y(x) = V_0\prod_{i=0}^{\infty}\left[I + (1-\frac{q}{p})(\frac{q}{p})^i xA((\frac{q}{p})^i\frac{1}{p}x\right]$$

Definition 2.1. Let $\{y_1(x), \cdots, y_k(x)\}$ be a set of k-linear independent solutions. It is said to be a fundamental system of solutions and the corresponding non singular matrix $Y(x)$ is said to be a fundamental matrix of the system.

Theorem 2.2. Let Y and Z be such that

$$D_{p,q}Y(x) = A(x)Y(px)$$

$$D_{p,q}Z(x) = -Z(qx)A(x)$$

$$Y(x_0)Z(x_0) = I$$

then $Y(x)Z(x) = I$ where I is the unit matrix.

Proof.

$$\begin{aligned} D_{p,q}Y(x) &= A(x)Y(px) \\ &= Y(px)D_{p,q}Z(x) + Z(qx)D_{p,q}Y(x) \\ &= -Y(px)Z(qx)A(x) + Z(qx)A(x)Y(px) \\ &= 0 \end{aligned}$$

Therefore, $Z(x)Y(x)$ is a constant and $Y(x)Z(x) = I$.

□

In Similar method, we easily have the following theorem.

Theorem 2.3. Let Y and Z be such that

$$D_{p,q}Y(x) = A(x)Y(qx)$$

$$D_{p,q}Z(x) = -Z(px)A(x)$$

$$Y(x_0)Z(x_0) = I$$

then $Y(x)Z(x) = I$.

Theorem 2.4. The general solution of the non homogeneous (p,q)-difference equations (2.1) is

$$\begin{aligned} y(x) &= Y(x)C(x) \\ &= Y(x)C + \int_{x_0}^{x} Y(x)Y^{-1}(t)b(t)d_{p,q}t \end{aligned} \tag{2.7}$$

where $C = Y^{-1}(x_0)y(x_0)$.

Proof. Consider the non homogeneous equation (2.1), $D_{p,q}y(x) = A(x)y(qx) + b(x)$.

From the method of variations of constants, we have the general solution under the form

$$y(x) = Y(x)C(x), \tag{2.8}$$

where $Y(x)$ is a fundamental matrix for the corresponding homogeneous system and $C(x)$ is an unknown k-dimensional vector.

Equation (2.1) gives the system

$$Y(px)D_{p,q}C(x) = b(x). \tag{2.9}$$

The result reads

$$C(x) = \int_{x_0}^{x} Y^{-1}(pt)b(t)d_{p,q}t + C. \tag{2.10}$$

Hence, from (2.8) and (2.10), we have the above result.

□

Also, as equivalent result, we get next theorem.

Theorem 2.5. Non homogeneous (p, q)-difference equations have the solution,

$$y(x) = \phi(x, x_0)y(x_0) + \int_{x_0}^{x} \phi(x, pt)b(t)d_{p,q}t \tag{2.11}$$

where $\phi(x, y) = Y(x)Y^{-1}(y)$ is the (p, q)-state transition matrix.

And we can write convenient form in the controllability theory, as below,

$$y(x) = \phi(x, x_0)\left[y(x_0) + \int_{x_0}^{x} \phi(x_0, pt)b(t)d_{p,q}t\right]. \tag{2.12}$$

Corollary 2.2. When $x_0 = 0$, the equations (2.7), (2.11), (2.12) are expressed next forms

$$y(x) = Y(x)C + (p-q)x\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}Y(x)Y^{-1}(\frac{q^i}{p^i}x)b(\frac{q^i}{p^{i+1}}x),$$

$$y(x) = \phi(x,0)y(0) + (p-q)x\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}\phi(x,\frac{q^i}{p^i}x)b(\frac{q^i}{p^{i+1}}x),$$

and

$$y(x) = \phi(x,0)\left[(y(0) + (p-q)x\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}\phi(0,\frac{q^i}{p^i}x)b(\frac{q^i}{p^{i+1}}x)\right].$$

In the equation (2.11), the function

$$\int_{x_0}^{x}\phi(x,t)b(t)d_{p,q}t$$

is a particular solution of the equation(2.1). Therefore, the solution of the non homogeneous (p,q)-difference equations is represented by a sum of its particular and the general solution that derives from homogeneous equations.

3. Autonomous systems of (p,q)-difference equations

In this section, we derive solutions about the autonomous systems of first order non linear (p,q)-difference equations in some case and also include each example.

Theorem 3.1. Let A is a constant matrix. The equations

$$D_{p,q}y(x) = Ay(px)$$

have its solution in series form, $y(x) = \sum_{n=0}^{\infty} C_n x^n$, where C_n is a k-dimensional vector. And one gets

$$y_{p,q}(x) = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}A^nC_0}{[n]_{p,q}!}x^n = C_0e_{p,q}(Ax).$$

Proof. Let $D_{p,q}y(x) = Ay(px)$ with A is a constant matrix. From the definition of (p,q)- difference equation, we obtain

$$y(qx) = (I - (p-q)xA)\, y(px). \tag{3.1}$$

And by the equation(3.1) and $y(x) = \sum_{n=0}^{\infty} C_n x^n$, we get

$$\sum_{n=0}^{\infty} C_n (qx)^n = (I - (p-q)xA) \sum_{n=0}^{\infty} C_n (px)^n. \tag{3.2}$$

Using the method of coefficient comparison in the equation (3.2), we have

$$C_k = \frac{p-q}{p^k - q^k} p^{k-1} AC_{n-1}. \tag{3.3}$$

And from the equation (3.3), we get next result by using recursive calculation.

$$\begin{aligned} C_n &= \prod_{k=1}^{n} \frac{p-q}{p^k - q^k} p^{\binom{n}{2}} A^n C_0 \\ &= \frac{p^{\binom{n}{2}} A^n C_0}{[n]_{p,q}!}. \end{aligned}$$

Therefore, we have the solution that is (p,q)-exponential function,

$$y_{p,q}(x) = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} A^n C_0}{[n]_{p,q}!} x^n = C_0 e_{p,q}(Ax).$$

□

Example 3.1. Consider the (p,q)-difference equation in simple case,

$$\begin{pmatrix} D_{p,q}y_1(x) \\ D_{p,q}y_2(x) \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} y_1(px) \\ y_2(px) \end{pmatrix}.$$

Then we can get the solution as follows

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e_{p,q}(-2px) = \begin{pmatrix} e_{p,q}(-2px) \\ -e_{p,q}(-2px) \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e_{p,q}(6px) = \begin{pmatrix} 3e_{p,q}(6px) \\ 5e_{p,q}(6px) \end{pmatrix}.$$

Theorem 3.2. Let A is a constant matrix. The equations,

$$D_{p,q}y(x) = Ay(qx),$$

have its solution that is represented with exponential function,

$$y_{p^{-1},q^{-1}}(x) = C_0 e_{p^{-1},q^{-1}}(Ax).$$

Proof. By the definition of (p,q)-difference equation, we have next result.

$$y(x) = [I - (p-q)xA]\, y(\frac{q}{p}x),$$

$$\sum_{n=0}^{\infty} C_n(x)^n = [I - (p-q)xA] \sum_{n=0}^{\infty} C_n(\frac{q}{p}x)^n$$

And using recursive calculation, C_n is represented as follows

$$C_n = \prod_{k=1}^{n} \frac{1-\frac{q}{p}}{1-(\frac{q}{p})^k} q^{\binom{n}{2}} A^n C_0 = C_0 \frac{q^{\binom{n}{2}} A^n}{[n]_{1,\frac{q}{p}}!} = C_0 e_{p^{-1},q^{-1}}(Ax).$$

□

Example 3.2. Consider the simple case of Theorem 3.2,

$$\begin{pmatrix} D_{p,q}y_1(x) \\ D_{p,q}y_2(x) \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} y_1(qx) \\ y_2(qx) \end{pmatrix},$$

then we have the solutions with the exponential function,

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e_{p,q}(-2qx) = \begin{pmatrix} e_{p^{-1},q^{-1}}(-2qx) \\ -e_{p^{-1},q^{-1}}(-2qx) \end{pmatrix},$$

and

$$X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e_{p,q}(6qx) = \begin{pmatrix} 3e_{p^{-1},q^{-1}}(6qx) \\ 5e_{p^{-1},q^{-1}}(6qx) \end{pmatrix}.$$

The functions $e_{p,q}(Ax), e_{p^{-1},q^{-1}}(Ax)$ are (p,q)-versions and its inverse of the usual exponential function $e(Ax)$. From Theorem 2.2, we get the following result easily.

Theorem 3.3. Let A is a constant matrix. Then we have

$$e_{p,q}(Ax)e_{p^{-1},q^{-1}}(Ax) = I.$$

Proof. By definition of (p,q)-derivative, we can see that

$$D_{p,q}e_{p,q}(Ax) = Ae_{p,q}(Apx),$$
$$D_{p,q}e_{p^{-1},q^{-1}}(-Ax) = -Ae_{p^{-1},q^{-1}}(Aqx)$$

and $e_{p,q}(Ax_0)e_{p^{-1},q^{-1}}(-Ax_0) = I$ for $x_0 = 0$.

Hence, we can obtain the following result from Theorem 2.2.

$$e_{p,q}(Ax)e_{p^{-1},q^{-1}}(Ax) = I.$$

□

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