Advanced Studies in Contemporary Mathematics
27 (2017), No. 4, pp. 609 - 624

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2017.27.4.609

DEGENERATE CHANGHEE NUMBERS AND POLYNOMIALS OF THE SECOND KIND

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Abstract. In this paper, we consider the degenerate Changhee numbers and polynomials of the second kind which are different from the previously introduced degenerate Changhee numbers and polynomials by Kwon-Kim-Seo (see [11]). We investigate some interesting identities and properties for these numbers and polynomials. In addition, we give some new relations between the degenerate Changhee polynomials of the second kind and the Carlitz's degenerate Euler polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

$$\begin{aligned} I(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see } [8,19]). \end{aligned} \tag{1.1}$$

From (1.1), we note that

$$I(f_n) + (-1)^{n-1} I(f) = 2\sum_{a=0}^{n-1} (-1)^{n-1-a} f(a), \quad (\text{see } [8,18,19]), \tag{1.2}$$

2010 *Mathematics Subject Classification.* 11B83; 11S80.

Key words and phrases. Changhee polynomials, degenerate Changhee polynomials of the second kind, fermionic p-adic integrals, stirling numbers .

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$. It is well known that the Euler polynomials are defined by the generating function

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [1 - 20])}. \tag{1.3}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

In [2,3], L. Carlitz considered the degenerate Euler polynomials given by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \ (\lambda \in \mathbb{R}). \tag{1.4}$$

When $x = 0$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers. From (1.4), we easily note that

$$\begin{aligned}\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0} \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, we have

$$\lim_{\lambda\to 0} \mathcal{E}_{n,\lambda}(x) = E_n(x), \ (n \geq 0), \quad \text{(see [2])}.$$

As is well known, the Changhee polynomials are defined by the generating function

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \quad \text{(see [7, 9])}. \tag{1.5}$$

When $x = 0$, $Ch_n = Ch_n(0)$, $(n \geq 0)$, are called the Changhee numbers. From (1.2), we note that

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}. \tag{1.6}$$

Thus, by (1.6), we get

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = Ch_n(x), \ (n \geq 0), \quad \text{(see [7])}, \tag{1.7}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

It is not difficult to show that

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.8}$$

By (1.8), we get

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x),\ (n \geq 0), \quad \text{(see [6, 7, 8, 18, 19])}. \tag{1.9}$$

The Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l,\ (n \geq 0), \quad \text{(see [1 - 20])}, \tag{1.10}$$

and those of the second kind are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l,\ (n \geq 0), \quad \text{(see [1, 4, 5, 7, 18])}. \tag{1.11}$$

From (1.6) and (1.8), we note that

$$Ch_n(x) = \sum_{l=0}^{n} E_l(x) S_1(n,l), \tag{1.12}$$

and

$$E_n(x) = \sum_{l=0}^{n} Ch_n(x) S_2(n,l),\ (n \geq 0), \quad \text{(see [7])}.$$

Recently, the degenerate Changhee polynomials are introduced by Kwon-Kim-Seo as

$$\frac{2\lambda}{2\lambda + \log(1+\lambda t)} \left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x = \sum_{n=0}^{\infty} Ch^*_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.13}$$

When $x = 0$, $Ch^*_{n,\lambda} = Ch^*_{n,\lambda}(0)$ are called the degenerate Changhee numbers (see [11]).

Note that $\lim_{\lambda \to 0} Ch^*_{n,\lambda}(x) = Ch_n(x)$, $(n \geq 0)$.

Recently, many researchers have studied Changhee numbers and polynomials (see [1-20]). In this paper, we consider the degenerate Changhee numbers and polynomials of the second kind which are different from the previous introduced degenerate Changhee numbers and polynomials by Kwon-Kim-Seo (see [11]). We give some new and interesting identities and properties for these numbers

and polynomials. In addition, we investigate some relations between the degenerate Changhee polynomials of the second kind and Carlitz's degenerate Euler polynomials.

2. Degenerate Changhee numbers and polynomials of the second kind

From (1.2), we note that

$$\int_{\mathbb{Z}_p} \big(1+\lambda\log(1+t)\big)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) = \frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}, \tag{2.1}$$

where $\lambda \in \mathbb{C}_p$ with $|\lambda|_p \leq 1$. Now, we define the degenerate Changhee polynomials of the second kind by

$$\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.2}$$

Thus, by (2.1), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} \big(1+\lambda\log(1+t)\big)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) &= \sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l} d\mu_{-1}(y)\lambda^l\big(\log(1+t)\big)^l \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l} l!d\mu_{-1}(y)\lambda^l\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Comparing the coefficients on both sides of (2.2) and (2.3), we have

$$\sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l} l!d\mu_{-1}(y)\lambda^l = Ch_{n,\lambda}(x),\ (n\geq 0). \tag{2.4}$$

The λ-analogue of falling factorial sequence is given by

$$(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda),\ (n\geq 1),\ (x)_{0,\lambda}=1. \tag{2.5}$$

Thus, by (2.4) and (2.5), we get

$$\sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}(x+y)_{l,\lambda} d\mu_{-1}(y) = Ch_{n,\lambda}(x),\ (n\geq 0). \tag{2.6}$$

Theorem 2.1. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} \int_{\mathbb{Z}_p} (x+y)_{l,\lambda} d\mu_{-1}(y) S_1(n,l) = Ch_{n,\lambda}(x).$$

It is not difficult to show that

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

From (2.7), we note that

$$\int_{\mathbb{Z}_p} (x+y)_{n,\lambda} d\mu_{-1}(y) = \mathcal{E}_{n,\lambda}(x), \ (n \geq 0). \tag{2.8}$$

Thus, from Theorem 2.1 and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} S_1(n,l) \mathcal{E}_{l,\lambda}(x) = Ch_{n,\lambda}(x).$$

By replacing t by $e^t - 1$ in (2.2), we get

$$\begin{aligned} \sum_{m=0}^{\infty} Ch_{m,\lambda}(x) \frac{1}{m!}(e^t-1)^m &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.9}$$

On the other hand,

$$\begin{aligned} \sum_{m=0}^{\infty} Ch_{m,\lambda}(x) \frac{1}{m!}(e^t-1)^m &= \sum_{m=0}^{\infty} Ch_{m,\lambda}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Ch_{m,\lambda}(x) S_2(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}(x) = \sum_{m=0}^{n} Ch_{m,\lambda}(x) S_2(n,m).$$

When $x = 0$, $Ch_{n,\lambda}(x) = Ch_{n,\lambda}(0)$, $(n \geq 0)$, are called the degenerate Changhee numbers of the second kind.

From (2.2), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}} \\
&= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}\lambda^m\big(\log(1+t)\big)^m\right) \\
&= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!}\right) \\
&= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(x)_{m,k}S_1(k,m)\right)\frac{t^k}{k!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_1(k,m)Ch_{n-k,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.11}$$

By comparing the coefficients on both sides of (2.11), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$Ch_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_1(k,m)Ch_{n-k,\lambda}.$$

By (1.2) , we easily get

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0). \tag{2.12}$$

Thus, by (2.12), we get

$$\int_{\mathbb{Z}_p}\big(1+\lambda\log(1+t)\big)^{\frac{x+1}{\lambda}}d\mu_{-1}(x) + \int_{\mathbb{Z}_p}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}d\mu_{-1}(x) = 2. \tag{2.13}$$

From (2.2) and (2.13), we have

$$\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}} + \frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}} = 2. \tag{2.14}$$

From (2.2) and (2.14), we have

$$\sum_{n=0}^{\infty}\big(Ch_{n,\lambda}(1)+Ch_{n,\lambda}\big)\frac{t^n}{n!}=2. \tag{2.15}$$

Comparing the coefficients on both sides of (2.15), we obtain the following theorem.

Theorem 2.5. *For $n\geq 0$, we have*

$$Ch_{n,\lambda}(1)+Ch_{n,\lambda}=\begin{cases}2, & \text{if } n=0\\ 0, & \text{if } n\geq 1.\end{cases}$$

By Theorem 2.5, we easily get

$$Ch_{0,\lambda}=1,\ Ch_{1,\lambda}=-\frac{1}{2},\ Ch_{2,\lambda}=\frac{1}{2}(1+\lambda),\cdots.$$

For $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, by (1.2) we have

$$\int_{\mathbb{Z}_p} f(x+d)d\mu_{-1}(x)+\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x)=2\sum_{a=0}^{d-1}(-1)^a f(a). \tag{2.16}$$

Let us take $f(x)=\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}$. Then by (2.16), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}d\mu_{-1}(x)\\ &=\frac{2}{\big(1+\lambda\log(1+t)\big)^{\frac{d}{\lambda}}+1}\sum_{a=0}^{d-1}(-1)^a\big(1+\lambda\log(1+t)\big)^{\frac{a}{\lambda}}\\ &=\sum_{a=0}^{d-1}(-1)^a\frac{2}{1+\big(1+\frac{\lambda}{d}d\log(1+t)\big)^{\frac{d}{\lambda}}}\big(1+\tfrac{\lambda}{d}d\log(1+t)\big)^{\frac{d}{\lambda}\frac{a}{d}}.\end{aligned} \tag{2.17}$$

By (2.7), we easily get

$$\begin{aligned}\frac{2}{1+\big(1+\frac{\lambda}{d}d\log(1+t)\big)^{\frac{d}{\lambda}}}\big(1+\tfrac{\lambda}{d}d\log(1+t)\big)^{\frac{d}{\lambda}\frac{a}{d}}&=\sum_{m=0}^{\infty}\mathcal{E}_{m,\frac{\lambda}{d}}\left(\frac{a}{d}\right)\frac{d^m}{m!}\big(\log(1+t)\big)^m\\ &=\sum_{m=0}^{\infty}\mathcal{E}_{m,\frac{\lambda}{d}}d^m\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}d^m\mathcal{E}_{m,\frac{\lambda}{d}}S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.18}$$

From (2.17) and (2.18), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_{n,\lambda}\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}} d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} d^m \sum_{a=0}^{d-1}(-1)^a \mathcal{E}_{m,\frac{\lambda}{d}} S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2), we have*

$$Ch_{n,\lambda} = \sum_{m=0}^{n} d^m \sum_{a=0}^{d-1}(-1)^a \mathcal{E}_{m,\frac{\lambda}{d}} S_1(n,m).$$

Now, we observe that

$$\begin{aligned}&\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{x+1}{\lambda}} + \frac{2\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}} \\ &= 2\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}.\end{aligned} \tag{2.20}$$

Thus, by (2.2) and (2.20), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(Ch_{n,\lambda}(x+1)+Ch_{n,\lambda}(x)\right)\frac{t^n}{n!} &= 2\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{1}{m!}\big(\log(1+t)\big)^m \\ &= \sum_{n=0}^{\infty}\left(2\sum_{m=0}^{n}(x)_{m,\lambda}S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.21}$$

Thus, by (2.21), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$Ch_{n,\lambda}(x+1)+Ch_{n,\lambda}(x) = 2\sum_{m=0}^{n}(x)_{m,\lambda}S_1(n,m).$$

From (2.16), we have

$$\begin{aligned}&\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{d}{\lambda}} + \frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}} \\ &= 2\sum_{a=0}^{d-1}(-1)^a\big(1+\lambda\log(1+t)\big)^{\frac{a}{\lambda}},\end{aligned} \tag{2.22}$$

where $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2).

By (2.2) and (2.22), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(Ch_{n,\lambda}(d)+Ch_{n,\lambda}\right)\frac{t^n}{n!} &= 2\sum_{a=0}^{d-1}(-1)^a\sum_{m=0}^{\infty}(a)_{m,\lambda}\frac{1}{m!}\big(\log(1+t)\big)^m \\ &= 2\sum_{a=0}^{d-1}(-1)^a\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(a)_{m,\lambda}S_1(n,m)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(2\sum_{a=0}^{d-1}\sum_{m=0}^{n}(a)_{m,\lambda}S_1(n,m)(-1)^a\right)\frac{t^n}{n!}.\end{aligned} \tag{2.23}$$

Therefore, by (2.23), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2), we have*

$$Ch_{n,\lambda}(d)+Ch_{n,\lambda} = 2\sum_{a=0}^{d-1}\sum_{m=0}^{n}(a)_{m,\lambda}S_1(n,m)(-1)^a.$$

Now, we consider the higher-order degenerate Changhee polynomials of the second kind which are derived from the multivariate fermionic p-adic integral on $\mathbb{Z}_p$.

For $r \in \mathbb{N}$, we define the higher-order degenerate Changhee polynomials of the second kind which are given by the multivariate fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\big(1+\lambda\log(1+t)\big)^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \left(\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\right)^r\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}Ch_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.24}$$

When $x = 0$, $Ch_{n,\lambda}^{(r)} = Ch_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Changhee numbers of the second kind.

From (2.24), we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\big(1+\lambda\log(1+t)\big)^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\frac{x_1+\cdots+x_r+x}{\lambda}}{m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\lambda^m\big(\log(1+t)\big)^m\\
&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{m,\lambda}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{1}{m!}\big(\log(1+t)\big)^m\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{m,\lambda}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)S_1(n,m)\right)\frac{t^n}{n!}
\end{aligned}\tag{2.25}$$

It is easy to show that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},
\end{aligned}\tag{2.26}$$

where $\mathcal{E}_{n,\lambda}^{(r)}(x)$ are the Carlitz's degenerate Euler polynomials of order r.

Thus, by (2.26), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{m,\lambda}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=\mathcal{E}_{m,\lambda}^{(r)}(x),\ (m\geq 0).\tag{2.27}$$

Therefore, by (2.24), (2.25) and (2.27), we obtain the following theorem.

Theorem 2.9. *For $n\geq 0$, we have*

$$Ch_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}\mathcal{E}_{m,\lambda}^{(r)}(x)S_1(n,m),\ (r\in\mathbb{N}).$$

By replacing t by e^t-1 in (2.24), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=\sum_{m=0}^{\infty}Ch_{m,\lambda}^{(r)}(x)\frac{(e^t-1)^m}{m!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}Ch_{m,\lambda}^{(r)}(x)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.28}$$

Thus, by comparing the coefficients on both sides of (2.28) and (2.26), we obtain the following theorem.

Theorem 2.10. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n} Ch_{m,\lambda}^{(r)}(x) S_2(n,m).$$

The degenerate Stirling numbers of the second kind are defined by Kim as

$$\frac{1}{n!}\big((1+\lambda t)^{\frac{1}{\lambda}} - 1\big)^n = \sum_{m=n}^{\infty} S_{2,\lambda}(m,n)\frac{t^m}{m!}, \quad \text{(see [10])}. \tag{2.29}$$

Here the left hand side of (2.29) is given by

$$\begin{aligned}
&\frac{1}{n!}\big((1+\lambda t)^{\frac{1}{\lambda}} - 1\big)^n = \frac{1}{n!}\left(e^{\frac{1}{\lambda}\log(1+\lambda t)} - 1\right)^n \\
&= \sum_{l=n}^{\infty} S_2(l,n)\lambda^{-l}\frac{1}{l!}\big(\log(1+\lambda t)\big)^l \\
&= \sum_{l=n}^{\infty} S_2(l,n)\lambda^{-l}\sum_{m=l}^{\infty} S_1(m,l)\frac{\lambda^m t^m}{m!} \\
&= \sum_{m=n}^{\infty}\left(\sum_{l=n}^{m} S_2(l,n)\lambda^{m-l}S_1(m,l)\right)\frac{t^m}{m!}.
\end{aligned} \tag{2.30}$$

Comparing (2.29) and (2.30), we have

$$S_{2,\lambda}(m,n) = \sum_{l=n}^{m} S_2(l,n)\lambda^{m-l}S_1(m,l), \tag{2.31}$$

where $m, n \geq 0$ with $m \geq n$.

Now, we observe that

$$\begin{aligned}(1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}} &= \left((1+\lambda t)^{\frac{1}{\lambda}}-1+1\right)^{x_1+\cdots+x_r+x} \\ &= \sum_{m=0}^{\infty}\binom{x_1+\cdots+x_r+x}{m}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^m \\ &= \sum_{m=0}^{\infty}(x_1+\cdots+x_r+x)_m\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)\,(x_1+\cdots+x_r+x)_m\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)(x_1+\cdots+x_r+x)_m\right)\frac{t^n}{n!}\end{aligned} \tag{2.32}$$

Thus, by (2.28) and (2.32), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_m d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)Ch_m^{(r)}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.33}$$

Therefore, by (2.28) and (2.33), we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}Ch_m^{(r)}(x)S_{2,\lambda}(n,m) = \sum_{m=0}^{n}Ch_{m,\lambda}^{(r)}(x)S_2(n,m).$$

From the generating function of the higher-order degenerate Changhee numbers of the second kind, we note that

$$\begin{aligned}
&\left(\frac{2}{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}+1}\right)^{r}=\left(\frac{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}-1}{2}+1\right)^{-r}\\
&=\sum_{m=0}^{\infty}\binom{-r}{m}2^{-m}\left(\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}-1\right)^{m}\\
&=\sum_{m=0}^{\infty}(-1)^{m}2^{-m}\binom{r+m-1}{m}m!\sum_{k=m}^{\infty}S_{2,\lambda}(k,m)\frac{1}{k!}\left(\log(1+t)\right)^{k} \qquad (2.34)\\
&=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(-1)^{m}m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(k,m)\right)\sum_{n=k}^{\infty}S_{1}(n,k)\frac{t^{n}}{n!}\\
&=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^{m}m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(k,m)S_{1}(n,k)\right\}\frac{t^{n}}{n!}.
\end{aligned}$$

and

$$\left(\frac{2}{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}+1}\right)^{r}=\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(r)}\frac{t^{n}}{n!}. \qquad (2.35)$$

Therefore, by (2.34) and (2.35), we obtain the following theorem.

Theorem 2.12. *For $n\geq 0$, $r\in\mathbb{N}$, we have*

$$Ch_{n,\lambda}^{(r)}=\sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^{m}m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(k,m)S_{1}(n,k).$$

Let $\mathcal{E}_{m,\lambda}^{(r)}$ be the higher-order degenerate Euler numbers defined by $\mathcal{E}_{n,\lambda}^{(r)}=\mathcal{E}_{n,\lambda}^{(r)}(0)$, $(n\geq 0)$.

Then, by (2.26), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}\frac{t^n}{n!} &= \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r = \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{2}+1\right)^{-r} \\
&= \sum_{m=0}^{\infty}\binom{-r}{m}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^m 2^{-m} \\
&= \sum_{m=0}^{\infty}(-1)^m\binom{r+m-1}{m}2^{-m}m!\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(-1)^m m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.36}$$

Thus, by (2.36), we get

$$\mathcal{E}_{n,\lambda}^{(r)} = \sum_{m=0}^{n}(-1)^m m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(n,m). \tag{2.37}$$

From (2.24), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{2}{1+\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}}\right)^r\left(1+\lambda\log(1+t)\right)^{\frac{x}{\lambda}} \\
&= \left(\frac{2}{1+\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}}\right)^k\left(\frac{2}{1+\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}}\right)^{r-k}\left(1+\lambda\log(1+t)\right)^{\frac{x}{\lambda}} \\
&= \left(\sum_{l=0}^{\infty}Ch_{l,\lambda}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}Ch_{m,\lambda}^{(r-k)}(x)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}Ch_{l,\lambda}^{(k)}Ch_{n-l,\lambda}^{(r-k)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.38}$$

Therefore, by (2.38), we obtain the following convolution result.

Theorem 2.13. *For $n \geq 0$, $r \in \mathbb{N}$, we have*

$$Ch_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}Ch_{l,\lambda}^{(k)}Ch_{n-l,\lambda}^{(r-k)}(x).$$

Remark. By (2.24), we easily get

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{2}{1+\big(1+\lambda\log(1+t)\big)^{\frac{1}{\lambda}}}\right)^r \big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}} \\
&= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda}^{(r)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{1}{m!}\big(\log(1+t)\big)^m\right) \\
&= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda}^{(r)}\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(x)_{m,k\lambda}S_1(k,m)\right)\frac{t^k}{k!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_1(k,m)Ch_{n-k,\lambda}^{(r)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.39}$$

Comparing the coefficients on both sides of (2.39), we have

$$Ch_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{m,\lambda}S_1(k,m)Ch_{n-k,\lambda}^{(r)}.$$

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