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SCHUR CONVEXITY OF BONFERRONI MEANS

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ABSTRACT. Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of the Bonferroni means for n variables are investigated, and some mean value inequalities of n variables are established.
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1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\},$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in R^n_+$ and $p, q \geq 0, p+q \neq 0$. The Bonferroni mean was originally introduced by Bonferroni in [1], which was defined as follows:

$$B^{p,q}(\boldsymbol{x}) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} x_i^p x_j^q \right)^{\frac{1}{p+q}} \tag{1}$$

Obviously, the Bonferroni mean has the following properties:

(i) $B^{p,q}(0, 0, \cdots, 0) = 0$.
(ii) $B^{p,q}(x, x, \cdots, x) = x$, if $x_i = x$, for all i.
(iii) $B^{p,q}(\boldsymbol{x}) \geq B^{p,q}(\boldsymbol{y})$. i.e., the Bonferroni mean is monotonic, if $x_i \geq y_i$, for all i.
(iv) $\min\{x_i\} \leq B^{p,q}(\boldsymbol{x}) \leq \max\{x_i\}$.

Furthermore, if $q = 0$, then by (1), it follows that

$$B^{p,0}(\boldsymbol{x}) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \left(\frac{1}{n-1} \sum_{i,j=1, i\neq j}^{n} x_j^0 \right) \right)^{\frac{1}{p+0}} = \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \tag{2}$$

which is power means of n variables.

if $n = 2$, then by (1), it follows that

$$B^{p,q}(x, y) = \left(\frac{x^p y^q + x^q y^p}{2} \right)^{\frac{1}{p+q}} \tag{3}$$

which is the generalized Muirhead mean of two variables $M(p, q; x, y)$ (See [2]).

In 2010, Yu-ming Chu etal.[3] studied the Schur-convexity, Schur geometric and harmonic convexities of the generalized Muirhead mean $M(p,q;x,y)$, obtained the following results.

Theorem A. [6] *For fixed* $(p,q)\in R^2$,

(*i*) $M(p,q;x,y)$ *is Schur-convex with* $(x,y)\in R_+^2$ *if and only if* $(p,q)\in\{(p,q)\mid (p-q)^2\geq p+q>0$ *and* $pq\leq 0\}$;

(*ii*) $M(p,q;x,y)$ *is Schur-concave with* $(x,y)\in R_+^2$ *if and only if* $(p,q)\in\{(p,q)\mid (p-q)^2\leq p+q,(p,q)\neq(0,0)\}\cup\{(p,q)\mid p+q<0\}$.

Theorem B. [6] *For fixed* $(p,q)\in R^2$,

(*i*) $M(p,q;x,y)$ *is Schur-geometric convex with* $(x,y)\in R_+^2$ *if and only if* $(p,q)\in\{(p,q)\mid p+q>0\}$;

(*ii*) $M(p,q;x,y)$ *is Schur-geometric concave with* $(x,y)\in R_+^2$ *if and only if* $(p,q)\in\{(p,q)\mid p+q<0\}$.

Theorem C. [7] *For fixed* $(p,q)\in R^2$,

(*i*) $M(p,q;x,y)$ *is Schur-harmonic convex with* $(x,y)\in R_+^2$ *if and only if* $(p,q)\in\{(p,q)\mid p+q>0\}\cup\{(p,q)\mid p\leq 0,q\leq 0,(p-q)^2+p+q\leq 0,p^2+q^2\neq 0\}$;

(*ii*) $M(p,q;x,y)$ *is Schur-harmonic concave with* $(x,y)\in R_+^2$ *if and only if* $(p,q)\in\{(p,q)\mid p\geq 0,p+q<0,(p-q)^2+p+q\geq 0\}\cup\{(p,q)\mid q\geq 0,p+q<0,(p-q)^2+p+q\geq 0\}$.

In recent years, the research on Schur convexity of all kinds of means in n variables is more and more active and fruitful (see [10] - [19]). In this paper, we for the case of $n\geq 3$, discuss the Schur-convexity, Schur geometric and harmonic convexities of the Bonferroni mean $B^{p,q}(\boldsymbol{x})$. Our main results are as follows:

Theorem 1. *For* $n\geq 3$ *and fixed* $(p,q)\in R^2$,

(*i*) *if* $0\leq q\leq p\leq 1$ *and* $p-q\leq\sqrt{p+q},p+q\neq 0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-concave with* $\boldsymbol{x}\in R_+^n$;

(*ii*) *if* $q\leq p\leq 0$ *and* $p+q\neq 0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-concave with* $\boldsymbol{x}\in R_+^n$;

(*iii*) *if* $p\geq 1,q\leq 0$ *and* $p+q>0$, $B^{p,q}(\boldsymbol{x})$ *is Schur-convex with* $\boldsymbol{x}\in R_+^n$;

(*iv*) *if* $p\geq 1,q\leq 0$ *and* $p+q<0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-concave with* $\boldsymbol{x}\in R_+^n$.

Theorem 2. *For* $n\geq 3$ *and fixed* $(p,q)\in R^2$,

(*i*) *if* $p+q>0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-geometric convex with* $\boldsymbol{x}\in R_+^n$;

(*ii*) *if* $p+q<0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-geometric concave with* $\boldsymbol{x}\in R_+^n$.

Theorem 3. *For* $n\geq 3$ *and fixed* $(p,q)\in R^2$,

(*i*) *if* $p\geq q\geq 0$ *and* $p+q\neq 0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-harmonic convex with* $\boldsymbol{x}\in R_+^n$;

(*ii*) *if* $0\geq p\geq q\geq -1$ *and* $p+q\neq 0,(p-q)^2+p+q\leq 0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-harmonic convex with* $\boldsymbol{x}\in R_+^n$;

(*iii*) *if* $p\geq 0,q\leq -1$ *and* $p+q>0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-harmonic convex with* $\boldsymbol{x}\in R_+^n$;

(*iv*) *if* $p\geq 0,q\leq -1$ *and* $p+q<0$, *then* $B^{p,q}(\boldsymbol{x})$ *is Schur-harmonic concave with* $\boldsymbol{x}\in R_+^n$.

2. Definitions and Lemmas

For convenience, we introduce some definitions as follows.

Definition 1. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, the function φ: $\Omega \to \mathbb{R}$ is said to be Schur-convex on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

(*i*) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \leq \alpha \leq 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \cdots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be convex on Ω if

$$\varphi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0, 1]$. The function φ is said to be concave on Ω if and only if $-\varphi$ is convex function on Ω.

Definition 4. (*i*) A set $\Omega \subset \mathbb{R}^n$ is called symmetric, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

Lemma 1. [4, p. 84] *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex$ $(Schur-concave)$ $function$ if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{4}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

The first systematical study of the functions preserving the ordering of majorization was made by Issai Schur in 1923. In Schur's honor, such functions are said to be "Schur-convex". It can be used extensively in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields. See[11].

Definition 5. [6], [7] Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}_+^n$.

(*i*) $\Omega \subset \mathbb{R}_+^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \cdots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$,$\boldsymbol{y} \in \Omega$ and α,$\beta \in [0, 1]$ such that $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}_+^n$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur geometrically convex function on Ω if $(\log x_1, \log x_2, \cdots, \log x_n) \prec (\log y_1, \log y_2, \cdots, \log y_n)$

on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex function.

Lemma 2. [6] *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric and geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{5}$$

holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then φ is a Schur geometrically convex (Schur geometrically concave) function.

The Schur geometric convexity was proposed by Zhang [6] in 2004, we also note that some authors use the term "Schur multiplicative convexity".

In 2009, Chu [8], [9] introduced the notion of Schur harmonically convex function, and some interesting inequalities were obtained.

Definition 6. [8] Let $\Omega \subset \mathbb{R}_+^n$.

(i) A set Ω is said to be harmonically convex if $\dfrac{\boldsymbol{xy}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\dfrac{1}{\boldsymbol{x}} = (\dfrac{1}{x_1}, \dfrac{1}{x_2}, \cdots, \dfrac{1}{x_n})$.

(ii) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur harmonically convex on Ω if $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Lemma 3. [8] *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \to \mathbb{R}_+$ be a continuously symmetric function which is differentiable on Ω^0. Then φ is Schur harmonically convex (Schur harmonically concave) on Ω if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2^2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 \quad (\leq 0), \quad \boldsymbol{x} \in \Omega^0. \tag{6}$$

Lemma 4. *For $z \geq 1$, let*

$$f(z) = -qz^{p-q+1} + pz^{p-q} - pz + q \tag{7}$$

(i) *if $p \geq q \geq 0$ and $(p-q)^2 \leq p+q$, then $f(z) \leq 0$;*
(ii) *if $q \leq p \leq 0$, then $f(z) \geq 0$;*
(iii) *if $p \geq 0, q \leq 0$ and $(p-q)^2 \geq p+q$, then $f(z) \geq 0$;*
(iv) *if $p \geq 0, q \leq 0$ and $p+q \leq 0$, then $f(z) \geq 0$.*

Proof.

$$f'(z) = -q(p-q+1)z^{p-q} + p(p-q)z^{p-q-1} - p.$$

$$f'(1) = -q(p-q+1) + p(p-q) - p = (p-q)^2 - (p+q).$$

$$\begin{aligned} f''(z) &= -q(p-q+1)(p-q)z^{p-q-1} + p(p-q)(p-q-1)z^{p-q-2} \\ &= z^{p-q-2}h(z) \end{aligned}$$

where

$$h(z) = -q(p-q+1)(p-q)z + p(p-q)(p-q-1). \tag{8}$$

$$h(1) = -q(p-q+1)(p-q) + p(p-q)(p-q-1) = (p-q)[(p-q)^2 - (p+q)].$$

$$h^{'}(z) = -q(p-q+1)(p-q)$$

- (i) if $p \geq q \geq 0$ and $(p-q)^2 \leq p+q$, then $h^{'}(z) \leq 0$ and $h(1) \leq 0$, therefore $h(z) \leq 0$ for $z \geq 1$, meanwhile $f^{''}(z) \leq 0$, but $f^{'}(1) \leq 0$, so that $f^{'}(z) \leq 0$, from $f(1) = 0$, it follows that $f(z) \leq 0$ for $z \geq 1$.
- (ii) if $q \leq p \leq 0$, then $h^{'}(z) \geq 0$ and $h(1) \geq 0$, therefore $h(z) \geq 0$ for $z \geq 1$, meanwhile $f^{''}(z) \geq 0$, but $f^{'}(1) \geq 0$, so that $f^{'}(z) \geq 0$, from $f(1) = 0$, it follows that $f(z) \geq 0$ for $z \geq 1$.

Proving propositions (iii) and (iv) is similar to proposition (ii), so it is omitted. The proof of lemma 4 is complete. □

Lemma 5. *For $z \geq 1$, let*

$$g(z) = pz^{p-q+1} - qz^{p-q} + qz - p \tag{9}$$

- (i) *if $p \geq q \geq 0$, then $g(z) \geq 0$;*
- (ii) *if $0 \geq p \geq q$ and $(p-q)^2 + p + q \leq 0$, then $g(z) \leq 0$;*
- (iii) *if $p \geq 0 \geq q$ and $(p-q)^2 + p + q \geq 0$, then $g(z) \geq 0$.*

Proof.

$$g^{'}(z) = p(p-q+1)z^{p-q} - q(p-q)z^{p-q-1} + q.$$

$$g^{'}(1) = p(p-q+1) - q(p-q) + q = (p-q)^2 + p + q$$

$$\begin{aligned} g^{''}(z) &= p(p-q+1)(p-q)z^{p-q-1} - q(p-q)(p-q-1)z^{p-q-2} \\ &= (p-q)z^{p-q-2}m(z) \end{aligned}$$

where

$$m(z) = p(p-q+1)z - q(p-q-1). \tag{10}$$

$$m(1) = (p-q)^2 + p + q$$

$$m^{'}(z) = p(p-q+1)$$

- (i) if $p \geq q \geq 0$, then it is easy to see that $m^{'}(z) \geq 0$ for $z \geq 1$, but $m(1) \geq 0$, therefore $m(z) \geq 0$ for $z \geq 1$, meanwhile $g^{''}(z) \leq 0$, but $g^{'}(1) > 0$, so that $g^{'}(z) \geq 0$, from $g(1) = 0$, it follows that $g(z) \geq 0$ for $z \geq 1$.
- (ii) if $0 \geq p \geq q$ and $(p-q)^2 + p + q \leq 0$, then $m^{'}(z) \leq 0$ and $m(1) \leq 0$, therefore $m(z) \leq 0$ for $z \geq 1$, meanwhile $g^{''}(z) \leq 0$, but $g^{'}(1) \leq 0$, so that $g^{'}(z) \leq 0$, from $g(1) = 0$, it follows that $g(z) \leq 0$ for $z \geq 1$.

Proving propositions (iii) is similar to proposition (i), so it is omitted. The proof of lemma 5 is complete. □

Lemma 6. [4] *Let $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $A_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i$. Then*

$$\mathbf{u} = \Bigg(\underbrace{A_n(\boldsymbol{x}), A_n(\boldsymbol{x}), \cdots, A_n(\boldsymbol{x})}_{n}\Bigg) \prec (x_1, x_2, \cdots, x_n) = \boldsymbol{x}. \tag{11}$$

Lemma 7. [4, p. 189] *If $x_i > 0, i = 1, 2, \ldots, n$, then for all nonnegative constants c satisfying $0 < c < \frac{1}{n}\sum_{i=1}^{n} x_i$*

$$\left(\frac{x_1}{\sum_{j=1}^{n} x_j}, \ldots, \frac{x_n}{\sum_{j=1}^{n} x_j}\right) \prec \left(\frac{x_1 - c}{\sum_{j=1}^{n}(x_j - c)}, \ldots, \frac{x_n - c}{\sum_{j=1}^{n}(x_j - c)}\right) \tag{12}$$

3. Proof of Theorems

Proof of Theorem 1: Let

$$b(\boldsymbol{x}) = \frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} x_i^p x_j^q. \tag{13}$$

Then

$$\frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_1} = \frac{1}{p+q}\left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} \cdot \left[p x_1^{p-1}\left(x_2^q + x_3^q + \cdots + x_n^q\right) + q x_1^{q-1}\left(x_2^p + x_3^p + \cdots + x_n^p\right)\right]$$

$$\frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_2} = \frac{1}{p+q}\left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} \cdot \left[p x_2^{p-1}\left(x_1^q + x_3^q + \cdots + x_n^q\right) + p x_2^{q-1}\left(x_1^p + x_3^p + \cdots + x_n^p\right)\right]$$

It is easy to see that $B^{p,q}(\boldsymbol{x})$ is symmetric on R_+^n, without loss of generality, we may assume that $x_1 \geq x_2 > 0$. Let $z = \frac{x_1}{x_2}$. Then $z \geq 1$.

$$\begin{aligned}
\Delta_1 :&= (x_1 - x_2)\left(\frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_1} - \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_2}\right) \\
&= (x_1 - x_2)\frac{1}{p+q}\left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)}[p(x_3^q + \cdots + x_n^q)(x_1^{p-1} - x_2^{p-1}) \\
&+ q(x_3^p + \cdots + x_n^p)(x_1^{q-1} - x_2^{q-1}) + x_1^{p-1}x_2^{q-1}(px_2 - qx_1) + x_1^{q-1}x_2^{p-1}(qx_2 - px_1)] \\
&= (x_1 - x_2)\frac{1}{p+q}\left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)}[p(x_3^q + \cdots + x_n^q)(x_1^{p-1} - x_2^{p-1}) \\
&+ q(x_3^p + \cdots + x_n^p)(x_1^{q-1} - x_2^{q-1}) + x_2^{p+q-1}(z^{p-1}(p - qz) + z^{q-1}(q - pz))] \\
&= (x_1 - x_2)\frac{1}{p+q}\left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)}[p(x_3^q + \cdots + x_n^q)(x_1^{p-1} - x_2^{p-1}) \\
&+ q(x_3^p + \cdots + x_n^p)(x_1^{q-1} - x_2^{q-1}) + x_2^{p+q-1}z^{q-1}f(z)].
\end{aligned} \tag{14}$$

For $n \geq 3$ and fixed $(p, q) \in R^2$,

(i) if $0 \leq q \leq p \leq 1$ and $p - q \leq \sqrt{p+q}, p + q \neq 0$, then by ($i$) in Lemma 4, it follows $f(z) \leq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $x_1^{p-1} - x_2^{p-1} \leq 0$ and $x_1^{q-1} - x_2^{q-1} \leq 0$. Hence from (12), we conclude that $\Delta_1 \leq 0$, by Lemma 1, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-concave with $\boldsymbol{x} \in R_+^n$.

(ii) if $q \leq p \leq 0$ and $p + q \neq 0$, then by (ii) in Lemma 4, it follows $f(z) \geq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $p(x_1^{p-1} - x_2^{p-1}) \geq 0$ and $q(x_1^{q-1} - x_2^{q-1}) \geq 0$. Notice that $\frac{1}{p+q} < 0$, from (14), we conclude that $\Delta_1 \leq 0$, by Lemma 1, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-concave with $\boldsymbol{x} \in R_+^n$. then $B^{p,q}(\boldsymbol{x})$ is Schur-concave with $\boldsymbol{x} \in R_+^n$

(iii) if $p \geq 1, q \leq 0$ and $p+q>0$, then $(p-q)^2 \geq p-q \geq p \geq p+q$, and then by (iii) in Lemma 4, it follows $f(z) \geq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $p(x_1^{p-1} - x_2^{p-1}) \geq 0$ and $q(x_1^{q-1} - x_2^{q-1}) \geq 0$. Notice that $\frac{1}{p+q} > 0$, from (14), we conclude that $\Delta_1 \geq 0$, by Lemma 1, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-convex with $\boldsymbol{x} \in R_+^n$.

(iv) if $p \geq 1, q \leq 0$ and $p+q<0$, then by (iv) in Lemma 4, it follows $f(z) \geq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $p(x_1^{p-1} - x_2^{p-1}) \geq 0$ and $q(x_1^{q-1} - x_2^{q-1}) \geq 0$. Notice that $\frac{1}{p+q} < 0$, from (14), we conclude that $\Delta_1 \leq 0$, by Lemma 1, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-concave with $\boldsymbol{x} \in R_+^n$.

The proof of Theorem 1 is completed.

Proof of Theorem 2:

$$\begin{aligned} x_1 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_1} &= \frac{1}{p+q} \left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} \\ &\quad \cdot \left[p x_1^p \left(x_2^q + x_3^q + \cdots + x_n^q\right) + q x_1^p \left(x_2^p + x_3^p + \cdots + x_n^p\right)\right] \end{aligned}$$

$$\begin{aligned} x_2 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_2} &= \frac{1}{p+q} \left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} \\ &\quad \cdot \left[p x_2^p \left(x_1^q + x_3^q + \cdots + x_n^q\right) + q x_2^p \left(x_1^p + x_3^p + \cdots + x_n^p\right)\right] \end{aligned}$$

Without loss of generality, we may assume that $x_1 \geq x_2 > 0$. Let $z = \frac{x_1}{x_2}$. Then $z \geq 1$.

$$\begin{aligned} \Delta_2 :&= (x_1 - x_2)\left(x_1 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_1} - x_2 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_2}\right) \\ &= (x_1 - x_2) \frac{1}{p+q} \left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} [p(x_3^q + \cdots + x_n^q)(x_1^p - x_2^p) \\ &+ q(x_3^p + \cdots + x_n^p)(x_1^q - x_2^q) + p x_1^p x_2^q - p x_1^q x_2^p + q x_1^q x_2^p - q x_1^p x_2^q)] \\ &= (x_1 - x_2) \frac{1}{p+q} \left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} [p(x_3^q + \cdots + x_n^q)(x_1^p - x_2^p) \\ &+ q(x_3^p + \cdots + x_n^p)(x_1^q - x_2^q) + (p-q) x_2^q x_2^p (z^p - z^q)]. \end{aligned} \tag{15}$$

Note that there are always $p(x_1^p - x_2^p) \geq 0$ and $q(x_1^q - x_2^q) \geq 0$. For $z > 1$, the function z^t is increasing with t, so $(p-q)(z^p - z^q) \geq 0$. Thus if $p+q>0$, then from (15), we conclude that $\Delta_2 \geq 0$, by Lemma 2, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-geometric convex with $\boldsymbol{x} \in R_+^n$, if $p+q<0$, then from (15), we conclude that $\Delta_2 \leq 0$, by Lemma 2, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-geometric concave with $\boldsymbol{x} \in R_+^n$.

The proof of Theorem 2 is completed.

Proof of Theorem 3:

$$\begin{aligned} x_1^2 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_1} &= \frac{1}{p+q} \left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} \\ &\quad \cdot \left[p x_1^{p+1} \left(x_2^q + x_3^q + \cdots + x_n^q\right) + q x_1^{p+1} \left(x_2^p + x_3^p + \cdots + x_n^p\right)\right] \end{aligned}$$

$$\begin{aligned} x_2^2 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_2} &= \frac{1}{p+q} \left(b(\boldsymbol{x})\right)^{\frac{1}{p+q}-1} \frac{1}{n(n-1)} \\ &\quad \cdot \left[p x_2^{p+1} \left(x_1^q + x_3^q + \cdots + x_n^q\right) + q x_2^{p+1} \left(x_1^p + x_3^p + \cdots + x_n^p\right)\right] \end{aligned}$$

Without loss of generality, we may assume that $x_1 \geq x_2 > 0$. Let $z = \frac{x_1}{x_2}$. Then $z \geq 1$.

$$\begin{aligned}
\Delta_3 :&= (x_1 - x_2)\left(x_1^2 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial B^{p,q}(\boldsymbol{x})}{\partial x_2}\right) \\
&= (x_1 - x_2)\frac{1}{p+q}(b(\boldsymbol{x}))^{\frac{1}{p+q}-1}\frac{1}{n(n-1)}[p(x_3^q + \cdots + x_n^q)(x_1^{p+1} - x_2^{p+1}) \\
&+ q(x_3^p + \cdots + x_n^p)(x_1^{q+1} - x_2^{q+1}) + px_1^{p+1}x_2^q - px_1^q x_2^{p+1} + qx_1^{q+1}x_2^p - qx_1^p x_2^{q+1})] \\
&= (x_1 - x_2)\frac{1}{p+q}(b(\boldsymbol{x}))^{\frac{1}{p+q}-1}\frac{1}{n(n-1)}[p(x_3^q + \cdots + x_n^q)(x_1^{p+1} - x_2^{p+1}) \\
&+ q(x_3^p + \cdots + x_n^p)(x_1^{q+1} - x_2^{q+1}) + x_2^{p+1}x_2^q z^q(z^{p-q}(pz - q) - (p - qz))] \\
&= (x_1 - x_2)\frac{1}{p+q}(b(\boldsymbol{x}))^{\frac{1}{p+q}-1}\frac{1}{n(n-1)}[p(x_3^q + \cdots + x_n^q)(x_1^{p+1} - x_2^{p+1}) \\
&+ q(x_3^p + \cdots + x_n^p)(x_1^{q+1} - x_2^{q+1}) + x_2^{p+1}x_2^q z^q g(z)]
\end{aligned} \tag{16}$$

For $n \geq 3$ and fixed $(p, q) \in R^2$,

(i) if $p \geq q \geq 0$ and $p + q \neq 0$, then by (i) in Lemma 5, it follows $g(z) \geq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $p(x_1^{p+1} - x_2^{p+1}) \geq 0$ and $q(x_1^{q+1} - x_2^{q+1}) \geq 0$. Hence from (16), we conclude that $\Delta_3 \geq 0$, by Lemma 3, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-harmonic convex with $\boldsymbol{x} \in R_+^n$;

(ii) if $0 \geq p \geq q \geq -1$ and $p + q \neq 0, (p - q)^2 + p + q \leq 0$ then by (ii) in Lemma 5, it follows $g(z) \leq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $p(x_1^{p+1} - x_2^{p+1}) \leq 0$ and $q(x_1^{q+1} - x_2^{q+1}) \leq 0$. Notice that $\frac{1}{p+q} < 0$, from (16), we conclude that $\Delta_3 \geq 0$, by Lemma 3, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-harmonic convex with $\boldsymbol{x} \in R_+^n$;

(iii) if $p \geq 0, q \leq -1$ and $p + q > 0$, then $(p - q)^2 + p + q \geq 0$, and then by (iii) in Lemma 5, it follows $g(z) \geq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $p(x_1^{p+1} - x_2^{p+1}) \geq 0$ and $q(x_1^{q+1} - x_2^{q+1}) \geq 0$. Notice that $\frac{1}{p+q} > 0$, from (16), we conclude that $\Delta_3 \geq 0$, by Lemma 3, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-harmonic convex with $\boldsymbol{x} \in R_+^n$.

(iv) if $p \geq 0, q \leq -1$ and $p+q < 0$, then $(p-q)^2+p+q = p^2+q(q-p+1)+p(1-q) \geq 0$, by (iv) in Lemma 5, it follows $g(z) \geq 0$. Furthermore, from $x_1 \geq x_2 > 0$, we have $p(x_1^{p+1} - x_2^{p+1}) \geq 0$ and $q(x_1^{q+1} - x_2^{q+1}) \geq 0$. Notice that $\frac{1}{p+q} < 0$, from (16), we conclude that $\Delta_3 \leq 0$, by Lemma 3, it follows that $B^{p,q}(\boldsymbol{x})$ is Schur-harmonic concave with $\boldsymbol{x} \in R_+^n$.

The proof of Theorem 3 is completed.

4. Applications

Theorem 4. *Let $n \geq 3$ and $(p, q) \in R^2$. If $(p, q) \in \{(p, q)|0 \leq q \leq p \leq 1$ and $p + q \neq 0$, $p - q \leq \sqrt{p+q}\ \} \cup \{(p, q)|p + q < 0, p \geq 1, q \leq 0\ \}$, then for $\boldsymbol{x} \in R_+^n$, we have*

$$B^{p,q}(\boldsymbol{x}) \leq A_n(\boldsymbol{x}) \tag{17}$$

if $(p, q) \in \{(p, q)|p \geq 1, q \leq 0$ and $p + q > 0\}$, then the inequality (17) is reversed.

Proof. if $(p, q) \in \{(p, q)|0 \leq q \leq p \leq 1$ and $p + q \neq 0$, $p - q \leq \sqrt{p+q}\ \} \cup \{(p, q)|p + q < 0, p \geq 1, q \leq 0\ \}$, then by Theorem 1, from Lemma 6, we have

$$B^{p,q}(\boldsymbol{u}) \geq B^{p,q}(\boldsymbol{x}),$$

rearranging gives (17), if $(p,q) \in \{(p,q)|p \geq 1, q \leq 0 \text{ and } p+q > 0\}$, then the inequality (17) is reversed.

The proof is complete. $\square$

Theorem 5. *Let $n \geq 3$ and $(p,q) \in R^2$, $\boldsymbol{x} \in R_+^n$, and the constant c satisfying $0 < c < \min\{x_i\}, i = 1,2,\ldots,n$. If $(p,q) \in \{(p,q)|0 \leq q \leq p \leq 1$ and $p+q \neq 0$, $p-q \leq \sqrt{p+q}\ \} \cup \{(p,q)|p+q < 0, p \geq 1, q \leq 0\ \}$, then we have*

$$B^{p,q}(x_1 - c, x_2 - c, \ldots, x_n - c) \leq \left(1 - \frac{c}{A_n(\boldsymbol{x})}\right) B^{p,q}(x_1, x_2, \ldots, x_n) \tag{18}$$

if $(p,q) \in \{(p,q)|p \geq 1, q \leq 0$ and $p+q > 0\}$, then the inequality (18) is reversed.

Proof. Note that $0 < c < \min\{x_i\} < \frac{1}{n}\sum_{i=1}^n x_i$, if $(p,q) \in \{(p,q)|0 \leq q \leq p \leq 1$ and $p+q \neq 0$, $p-q \leq \sqrt{p+q}\ \} \cup \{(p,q)|p+q < 0, p \geq 1, q \leq 0\ \}$, then by Theorem 1, from Lemma 7, we have

$$B^{p,q}\left(\frac{x_1}{\sum_{j=1}^n x_j}, \ldots, \frac{x_n}{\sum_{j=1}^n x_j}\right) \geq B^{p,q}\left(\frac{x_1 - c}{\sum_{j=1}^n (x_j - c)}, \ldots, \frac{x_n - c}{\sum_{j=1}^n (x_j - c)}\right)$$

rearranging gives (18), if $(p,q) \in \{(p,q)|p \geq 1, q \leq 0 \text{ and } p+q > 0\}$, then the inequality (18) is reversed.

The proof is complete. $\square$

Theorem 6. *Let $n \geq 3$ and $(p,q) \in R^2$. If $p+q > 0$, then for $\boldsymbol{x} \in R_+^n$, we have*

$$B^{p,q}(\boldsymbol{x}) \geq G_n(\boldsymbol{x}) \tag{19}$$

where $G_n(\boldsymbol{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$. If $p+q < 0$, then the inequality (19) is reversed.

Proof. From Lemma 6, we have

$$(\log G_n(\boldsymbol{x}), \log G_n(\boldsymbol{x}), \ldots, \log G_n(\boldsymbol{x})) \prec (\log x_1, \log x_2, \ldots, \log x_n).$$

If $p+q < 0$, then by Theorem 2, we have

$$G_n(\boldsymbol{x}) = B^{p,q}(G_n(\boldsymbol{x}), G_n(\boldsymbol{x}), \ldots, G_n(\boldsymbol{x})) \leq B^{p,q}(x_1, x_2, \ldots, x_n) = B^{p,q}(\boldsymbol{x}).$$

If $p+q < 0$, then the inequality (19) is reversed.

The proof is complete. $\square$

Theorem 7. *Let $n \geq 3$ and $(p,q) \in R^2$. If $p \geq q \geq 0$ and $p+q \neq 0$, then for $\boldsymbol{x} \in R_+^n$, we have*

$$B^{p,q}(\boldsymbol{x}) \geq H_n(\boldsymbol{x}) \tag{20}$$

where $H_n(\boldsymbol{x}) = \frac{n}{\sum_{i=1}^n x_i^{-1}}$.

Proof. From Lemma 6, we have

$$\left(\frac{1}{H_n(\boldsymbol{x})}, \frac{1}{H_n(\boldsymbol{x})}, \ldots, \frac{1}{H_n(\boldsymbol{x})}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right).$$

If $p \geq q \geq 0$ and $p+q \neq 0$, then by Theorem 3, we have

$$H_n(\boldsymbol{x}) = B^{p,q}(H_n(\boldsymbol{x}), H_n(\boldsymbol{x}), \ldots, H_n(\boldsymbol{x})) \leq B^{p,q}(x_1, x_2, \ldots, x_n) = B^{p,q}(\boldsymbol{x}).$$

The proof is complete. $\square$

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