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Note on a new proof of Bolzano-Weierstrass Theorem

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Abstract. In the paper, a new proof of the well-known Bolzano-Weierstrass Theorem is proposed.
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Bolzano-Weierstrass Theorem asserts that every bounded infinite sequence of real numbers has at least one limit point [1, p.53].

Below, we give a new proof of this Theorem.

Proof. Let $\{a_n^{(1)}\}_{n=1}^{\infty}$ be an infinite sequence of real numbers, that is bounded. Since $\mathcal{R}$ satisfies the Completeness Axiom (see [2, p.16]): *Any non-empty set which is bound above has a least upper bound* (called *supremum*), then $\{a_n^{(1)}\}_{n=1}^{\infty}$ has

$$\sup\{a_n^{(1)}\}_{n=1}^{\infty} = l_1.$$

Let us assume that $\{a_n^{(1)}\}_{n=1}^{\infty}$ does not have a limit point. Then $l_1 \in \{a_n^{(1)}\}_{n=1}^{\infty}$ and moreover, according to our assumption, the set K of all different k, such that

$$a_k^{(1)} = l_1$$

is finite. We delete all these members $a_k^{(1)}$ from the sequence $\{a_n^{(1)}\}_{n=1}^{\infty}$ and denote the rest infinite sequence by $\{a_n^{(2)}\}_{n=1}^{\infty}$. It is clear that $\{a_n^{(2)}\}_{n=1}^{\infty}$ is a subsequence of $\{a_n^{(1)}\}_{n=1}^{\infty}$. Then $\{a_n^{(2)}\}_{n=1}^{\infty}$ is bounded above and we denote

$$l_2 = \sup\{a_n^{(2)}\}_{n=1}^{\infty}.$$

It is clear that $\{a_n^{(2)}\}_{n=1}^{\infty}$ does not have a limit point, because of our assumption for $\{a_n^{(1)}\}_{n=1}^{\infty}$.

Therefore, $l_2 \in \{a_n^{(2)}\}_{n=1}^{\infty}$ and obviously,

$$l_2 < l_1.$$

Repeating all considerations, that we made for $\{a_n^{(1)}\}_{n=1}^{\infty}$, to $\{a_n^{(2)}\}_{n=1}^{\infty}$, we obtain the infinite sequence $\{a_n^{(3)}\}_{n=1}^{\infty}$ (which is a subsequence of $\{a_n^{(1)}\}_{n=1}^{\infty}$) and setting

$$l_3 = \sup\{a_n^{(3)}\}_{n=1}^{\infty}$$

we have

$$l_3 < l_2 < l_1.$$

Making this process infinite, we obtain the sequence $\{a_n^{(k)}\}_{n=1}^{\infty}$, $k = 1, 2, 3, \dots$, such that for every fixed k, $\{a_n^{(k)}\}_{n=1}^{\infty}$ is a subsequence of $\{a_n^{(1)}\}_{n=1}^{\infty}$,

$$l_k = \sup\{a_n^{(k)}\}_{n=1}^{\infty},$$

$l_k \in \{a_n^{(k)}\}_{n=1}^{\infty}$, and $\{a_n^{(k)}\}_{n=1}^{\infty}$ does not have a limit point.

Thus, we constructed the infinite sequence $\{l_n\}_{n=1}^{\infty}$ that is strictly monotonic decreasing. But all terms of this sequence are terms of $\{a_n^{(1)}\}_{n=1}^{\infty}$, too, i.e., $\{l_n\}_{n=1}^{\infty}$ is a subsequence of $\{a_n^{(1)}\}_{n=1}^{\infty}$. Therefore, according to our assumption, $\{l_n\}_{n=1}^{\infty}$ does not have a limit point.

On the other hand, $\{l_n\}_{n=1}^{\infty}$ is bounded from below, since $\{a_n^{(1)}\}_{n=1}^{\infty}$ is bounded from below. Therefore, $\{l_n\}_{n=1}^{\infty}$ converges to a finite real number

$$l = \inf\{l_n\}_{n=1}^{\infty}.$$

The proof of this fact is obvious and does not depend of Bolzano-Weierstrass Theorem. But evidently, l is a limit point of the infinite sequence $\{l_n\}_{n=1}^{\infty}$ and therefore, it is a limit point of the sequence $\{a_n^{(1)}\}_{n=1}^{\infty}$, too. The last contradicts to our assumption that the infinite sequence $\{a_n^{(1)}\}_{n=1}^{\infty}$ does not have a limit point.

Thus, Bolzano-Weierstrass Theorem is completely proved.

References

[1] McShane, E. J. and T. A. Botts. Real analysis. Van Nostrand, New York, 1959.

[2] Aliprantis, C. D. and O. Burkinshaw. Principles of real analysis. Academic Press, Toronto.