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SEMI-SYMMETRIC SEMI-METRIC CONNECTION IN A LORENTZIAN β-KENMOTSU MANIFOLD

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Abstract. In the present paper, we consider a semi-symmetric semi-metric connection in a Lorentzian β-Kenmotsu manifold. We investigate the curvature tensor and the Ricci tensor of a Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection. Moreover, we consider pseudo projectively flat, ξ-pseudo projectively flat and ϕ-pseudo projectively semisymmetric Lorentzian β-Kenmotsu manifolds with a semi-symmetric semi-metric connection and obtain the scalar curvature r in each case.

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1. Introduction

In 1969, S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [10]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $R \times_f C$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [5] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [5].

In the Gray-Hervella classification of almost Hermitian manifolds [4], there appears a class W_4 of Hermitian manifolds, which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on a manifold $\bar{M}$ is called a trans-Sasakian structure [7], if the product manifold $(\bar{M} \times R, J, G)$ belongs to the class W_4 [4], where J is the almost complex structure on $\bar{M} \times R$ defined by

$$J(X, ad/dt) = (\phi X - a\xi, \eta(X)ad/dt)$$

for all vector fields X on $\bar{M}$ and smooth function a on $\bar{M} \times R$ and G is the product metric on $\bar{M} \times R$. This may be expressed by the condition [1]

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{1.1}$$

for some smooth functions α and β on $\bar{M}$ and we say that the trans-Sasakian structure is of type (α, β).

From the condition (1.1) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{1.2}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y)\xi + \beta g(\phi X, \phi Y). \tag{1.3}$$

In particular from (1.1), one has the notion of a β-Kenmotsu structure which may be defined by

$$(\nabla_X \phi)Y = \beta[g(\phi X, Y)\xi - \eta(Y)\phi X],$$

where β is a non-zero constant. Also, we have

$$\nabla_X \xi = \beta[X - \eta(X)\xi].$$

Thus $\alpha = 0$ and therefore a trans-Sasakian structure of type (α, β) with β a non-zero constant is always β-Kenmotsu manifold. If $\beta = 1$, then β-Kenmotsu manifold is Kenmotsu manifold.

A linear connection $\bar{\nabla}$ in a Riemannian manifold $\bar{M}$ is said to be a semi-symmetric connection [3,9] if its torsion tensor T of the connection $\bar{\nabla}$

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] \tag{1.4}$$

satisfies

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \tag{1.5}$$

where η is a 1-form. If moreover, a semi-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y, Z) = 2\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y) \tag{1.6}$$

for all $X, Y, Z \in \chi(\bar{M})$, where $\chi(\bar{M})$ is the Lie algebra of vector fields of the manifold $\bar{M}$, then $\bar{\nabla}$ is said to be a semi-symmetric semi-metric connection.

2. Lorentzian β-Kenmotsu manifolds

A differentiable manifold $\bar{M}$ of dimension n is called Lorentzian β-Kenmotsu manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \; g(X, \xi) = \eta(X), \; g(\phi X, Y) = -g(X, \phi Y) \tag{2.2}$$

for all $X, Y \in \chi(\bar{M})$. Then such a structure (ϕ, η, ξ, g) is termed as Lorentzian para-contact structure and the manifold $\bar{M}$ with a Lorentzian para-contact structure is called a Lorentzian para-contact manifold [6]. On a Lorentzian para-contact manifold, we also have

$$(\nabla_X \phi)(Y) = \beta[g(\phi X, Y)\xi + \eta(Y)\phi X] \tag{2.3}$$

for any $X, Y \in \chi(\bar{M})$, where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Thus a Lorentzian para-contact manifold satisfying (2.3) is called a Lorentzian β-Kenmotsu manifold [11]. From (2.3), it is easy to obtain that

$$\nabla_X \xi = -\beta\phi^2 X = -\beta[X + \eta(X)\xi], \tag{2.4}$$

$$(\nabla_X \eta)Y = \beta g(\phi X, \phi Y) = \beta[g(X, Y) + \eta(X)\eta(Y)]. \tag{2.5}$$

Moreover the Riemann curvature tensor R, the Ricci tensor S and the Ricci operator Q on a Lorentzian β-Kenmotsu manifold $\bar{M}$ with respect to the Levi-Civita connection satisfy the following equations [11]:

$$R(X,Y)\xi = \beta^2[\eta(Y)X - \eta(X)Y], \tag{2.6}$$

$$R(\xi,X)Y = \beta^2[g(X,Y)\xi - \eta(Y)X], \tag{2.7}$$

$$R(\xi,X)\xi = \beta^2[\eta(X)\xi + X], \tag{2.8}$$

$$S(X,\xi) = (n-1)\beta^2\eta(X), S(\xi,\xi) = -(n-1)\beta^2, \tag{2.9}$$

$$Q\xi = (n-1)\beta^2\xi, \tag{2.10}$$

$$S(\phi X,\phi Y) = -S(X,Y) - (n-1)\beta^2\eta(X)\eta(Y), \tag{2.11}$$

where $X, Y \in \chi(\bar{M})$ and $S(X,Y) = g(QX,Y)$.

Definition 2.1. [12] *A Lorentzian β-Kenmotsu manifold $\bar{M}$ is said to be an η-Einstein manifold if its Ricci tensor S of type $(0,2)$ satisfies*

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2\eta(X)\eta(Y), \tag{2.12}$$

where λ_1 and λ_2 are smooth functions on $\bar{M}$. In particular, if $\lambda_2 = 0$, then an η-Einstein manifold is an Einstein manifold.

Contracting (2.12), we have

$$r = n\lambda_1 - \lambda_2. \tag{2.13}$$

On the other hand, putting $X = Y = \xi$ and using (2.9) in (2.12), we also have

$$-(n-1)\beta^2 = -\lambda_1 + \lambda_2. \tag{2.14}$$

Hence it follows from (2.13) and (2.14) that

$$\lambda_1 = \frac{r-(n-1)\beta^2}{n-1},\ \lambda_2 = \frac{r-n(n-1)\beta^2}{n-1}.$$

So the Ricci tensor S of a Lorentzian β-Kenmotsu manifold is given by

$$S(X,Y) = \frac{r-(n-1)\beta^2}{n-1}g(X,Y) + \frac{r-n(n-1)\beta^2}{n-1}\eta(X)\eta(Y). \tag{2.15}$$

The relation between the semi-symmetric semi-metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on M is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi. \tag{2.16}$$

3. Curvature tensor of Lorentzian β-Kenmotsu manifolds with a semi-symmetric semi-metric connection

Let $\bar{M}$ be an n-dimensional Lorentzian β-Kenmotsu manifold. The curvature tensor $\bar{R}$ of $\bar{M}$ with respect to a semi-symmetric semi-metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

From (2.2), (2.16) and (3.1), we obtain

$$\begin{aligned}\bar{R}(X,Y)Z &= (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\\ &+(\nabla_Y\eta)(X)Z - (\nabla_X\eta)(Y)Z + \eta(X)g(Y,Z)\xi\end{aligned} \tag{3.2}$$

$$-\eta(Y)g(X,Z)\xi + g(Y,Z)\nabla_X\xi - g(X,Z)\nabla_Y\xi.$$

Using (2.4) and (2.5) in (3.2), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z - (\beta-1)\eta(X)g(Y,Z)\xi \quad (3.3)$$
$$+(\beta-1)\eta(Y)g(X,Z)\xi - \beta g(Y,Z)X + \beta g(X,Z)Y,$$

where $X, Y, Z \in \chi(\bar{M})$ and

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor of the connection ∇.
From (3.3), it follows that $\bar{R}$ satisfies

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0 \quad (3.4)$$

and

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z \quad (3.5)$$

which implies that $\bar{R}$ satisfies the first Bianchi identity and skew-symmetric property with respect to the first two variables with respect to a semi-symmetric semi-metric connection.
On contracting X in (3.3), we get

$$\bar{S}(Y,Z) = S(Y,Z) + (2\beta - n\beta - 1)g(Y,Z) + (\beta-1)\eta(Y)\eta(Z), \quad (3.6)$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on $\bar{M}$. This gives

$$\bar{Q}Y = QY + (2\beta - n\beta + 1)Y + (\beta-1)\eta(Y)\xi, \quad (3.7)$$

where $\bar{Q}$ and Q are the Ricci operators of the connections $\bar{\nabla}$ and ∇, respectively on $\bar{M}$.
Contracting again Y and Z in (3.6), it follows that

$$\bar{r} = r - (n-1)[(n-1)\beta + 1], \quad (3.8)$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on $\bar{M}$.
From (3.6), it follows that

$$\bar{S}(Y,Z) = \bar{S}(Z,Y). \quad (3.9)$$

Thus the Ricci tensor $\bar{S}$ in a Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection is symmetric.

Lemma 3.1. *Let $\bar{M}$ be an n-deminsional Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection $\bar{\nabla}$, then*

$$\bar{R}(X,Y)\xi = \beta(\beta-1)[\eta(Y)X - \eta(X)Y], \quad (3.10)$$

$$\bar{R}(\xi,X)Y = (\beta^2-1)g(X,Y)\xi - \beta(\beta-1)\eta(X)Y + (\beta-1)\eta(X)\eta(Y)\xi, \quad (3.11)$$

$$\bar{R}(\xi,X)\xi = \beta(\beta-1)[\eta(X)\xi + X], \quad (3.12)$$

$$\bar{S}(Y,\xi) = [(n-1)\beta^2 - (n+1)\beta + 2n]\eta(Y), \quad (3.13)$$

$$\bar{Q}\xi = [\beta(\beta-1)(n-1) + 2]\xi, \quad (3.14)$$

where $X, Y \in \chi(\bar{M})$.

Proof. From equations (2.1), (2.2), (2.6)-(2.10), (3.3), (3.6) and (3.7), we find equations (3.10)-(3.14) easily. □

Lemma 3.2. *Let $\bar{M}$ be an n-deminsional Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection $\bar{\nabla}$, then*

$$(\bar{\nabla}_X\phi)Y = (\beta-1)g(\phi X, Y)\xi + \beta\eta(Y)\phi X, \tag{3.15}$$

$$\bar{\nabla}_X\xi = -\beta[X+\eta(X)\xi], \tag{3.16}$$

$$\begin{aligned}\bar{S}(\phi X, \phi Y) = &-S(X,Y) - (2\beta - n\beta - 1)g(X,Y)\\ &-[(n-1)\beta^2 - n\beta + 2n - 1]\eta(X)\eta(Y),\end{aligned} \tag{3.17}$$

where $X, Y \in \chi(\bar{M})$.

Proof. By the covariant differentiation of ϕY with respect to X, we have

$$\bar{\nabla}_X\phi Y = (\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_X Y)$$

which by using (2.1) and (2.16) takes the form

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y + g(X, \phi Y)\xi.$$

In view of (2.3), the last equation gives

$$(\bar{\nabla}_X\phi)(Y) = (\beta-1)g(\phi X, Y)\xi + \beta\eta(Y)\xi.$$

To prove (3.16), we replace $Y = \xi$ in (2.16) and we have

$$\bar{\nabla}_X\xi = \nabla_X\xi - \eta(X)\xi + g(X,\xi)\xi.$$

Using (2.2) and (2.4), it follows that

$$\bar{\nabla}_X\xi = -\beta(X+\eta(X)\xi).$$

In order to prove (3.17), we replace $X = \phi X$ and $Y = \phi Y$ in $\bar{S}(X,Y) = g(\bar{Q}X, Y)$ and we have

$$\bar{S}(\phi X, \phi Y) = g(\bar{Q}\phi X, \phi Y).$$

Using properties $g(X, \phi Y) = -g(\phi X, Y)$, $\phi Q = Q\phi$ and (3.13) in the last equation, we get

$$\bar{S}(\phi X, \phi Y) = -\bar{S}(X,Y) - [(n-1)\beta^2 - (n+1)\beta + 2n]\eta(X)\eta(Y)$$

which by using (3.6) gives (3.17). □

4. Pseudo projectively flat Lorentzian β-Kenmotsu manifolds with a semi-symmetric semi-metric connection

Definition 4.1. *The Pseudo projective curvature tensor $\bar{P}$ of an n-dimensional Lorentzian β-Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection $\bar{\nabla}$ is given by [8]*

$$\begin{aligned}\bar{P}(X,Y)Z = &a\bar{R}(X,Y)Z + b[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]\\ &-\frac{\bar{r}}{n}(\frac{a}{n-1}+b)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{4.1}$$

where a and b are constants such that $a, b \neq 0$ and $\bar{R}$, $\bar{S}$ and $\bar{r}$ are the curvature tensor, the Ricci tensor and the scalar curvature with a semi-symmetric semi-metric connection on $\bar{M}$.

Let us assume that the manifold $\bar{M}$ with a semi-symmetric semi-metric connection is pseudo projectively flat, then

$$g(\bar{P}(X,Y)Z,\phi W)=0 \tag{4.2}$$

and hence

$$ag[\bar{R}(X,Y)Z,\phi W]+b[\bar{S}(Y,Z)g(X,\phi W)-\bar{S}(X,Z)g(Y,\phi W)] \\ -\frac{\bar{r}}{n}(\frac{a}{n-1}+b)[g(Y,Z)g(X,\phi W)-g(X,Z)g(Y,\phi W)]=0. \tag{4.3}$$

Putting $Y=Z=\xi$ and using (2.1), (2.2), (3.12) and (3.13) in (4.3), we have

$$-a\beta(\beta-1)g(X,\phi W)-((n-1)\beta^2-(n+1)\beta+2n)bg(X,\phi W) \\ +\frac{\bar{r}}{n}(\frac{a}{n-1}+b)g(X,\phi W)=0 \tag{4.4}$$

which on using (3.8) gives

$$r=(n-1)[(n-1)\beta+1]+\frac{n(n-1)[a\beta(\beta-1)+((n-1)\beta^2-(n+1)\beta+2n)b]}{a+b(n-1)}, \tag{4.5}$$

as $g(X,\phi W)\neq 0$.

Thus we can state the following theorem:

Theorem 4.2. *If a Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection is pseudo projectively flat, then the scalar curvature r is given by (4.5).*

5. ξ-pseudo projectively flat Lorentzian β-Kenmotsu manifolds with a semi-symmetric semi-metric connection

If the Lorentzian β-Kenmotsu manifolds with a semi-symmetric semi-metric connection is ξ-pseudo projectively flat, then

$$\bar{P}(X,Z)\xi=0 \tag{5.1}$$

and hence

$$a\bar{R}(X,Y)\xi+b[\bar{S}(Y,\xi)X-\bar{S}(X,\xi)Y]-\frac{\bar{r}}{n}(\frac{a}{n-1}+b)[g(Y,\xi)X-g(X,\xi)Y]. \tag{5.2}$$

Taking inner product of (5.2) with U, we have

$$ag(\bar{R}(X,Y)\xi,U)+b[\bar{S}(Y,\xi)g(X,U)-\bar{S}(X,\xi)g(Y,U)] \\ -\frac{\bar{r}}{n}(\frac{a}{n-1}+b)[\eta(Y)g(X,U)-\eta(X)g(Y,U)]. \tag{5.3}$$

Taking $Y=\xi$ and using (2.1), (2.2), (3.12) and (3.13) in (5.3), we get

$$ag(\bar{R}(X,Y)\xi,U)=0\Rightarrow \bar{R}(X,Y)\xi=0. \tag{5.4}$$

Using (2.2), (3.13) and (5.4) in (5.2), it follows that

$$[((n-1)\beta^2-(n+1)\beta+2n)b-\frac{\bar{r}}{n}(\frac{a}{n-1}+b)](\eta(Y)X-\eta(X)Y)=0 \tag{5.5}$$

which on using (3.8) gives

$$r=(n-1)[(n-1)\beta+1]+\frac{n(n-1)[(n-1)\beta^2-(n+1)\beta+2n]b}{a+b(n-1)}. \tag{5.6}$$

Thus we can state the following theorem:

Theorem 5.1. *If a Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection is ξ-pseudo projectively flat, then $\bar{R}(X,Y)\xi = 0$ and the scalar curvature r is given by* (5.6).

6. ϕ-pseudo projectively semisymmetric Lorentzian β-Kenmotsu manifolds with a semi-symmetric semi-metric connection

Definition 6.1. *A Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection $(\bar{M}^n, g)$, $n > 1$, is said to be ϕ-pseudo projectively semisymmetric if $\bar{P}(X,Y)\cdot\phi = 0$ on $\bar{M}$ for all X, $Y \in \chi(\bar{M})$.*

Let $\bar{M}$ be an n-dimensional $(n > 1)$ ϕ-pseudo projectively semisymmetric Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection. Therefore $\bar{P}(X,Y)\cdot\phi = 0$ turns into

$$(\bar{P}(X,Y)\cdot\phi)Z = \bar{P}(X,Y)\phi Z - \phi\bar{P}(X,Y)Z = 0 \tag{6.1}$$

for any vector fields X, Y and $Z \in \chi(\bar{M})$. Now from (4.1), we have

$$\begin{aligned}\bar{P}(X,Y)\phi Z &= a\bar{R}(X,Y)\phi Z + b[\bar{S}(Y,\phi Z)X - \bar{S}(X,\phi Z)Y] \\ &\quad -\frac{\bar{r}}{n}(\frac{a}{n-1} + b)[g(Y,\phi Z)X - g(X,\phi Z)Y].\end{aligned} \tag{6.2}$$

In view of (2.1), (3.3), (3.6) and (3.8), (6.2) yields

$$\begin{aligned}\bar{P}(X,Y)\phi Z &= a[\beta^2(g(Y,\phi Z)X - g(X,\phi Z)Y) - (\beta-1)\eta(X)g(Y,\phi Z)\xi \\ &\quad +(\beta-1)\eta(Y)g(X,\phi Z)\xi - \beta g(Y,\phi Z)X + \beta g(X,\phi Z)Y] \\ &\quad +b[S(Y,\phi Z)X - S(X,\phi Z)Y + (2\beta - n\beta - 1)g(Y,\phi Z)X - (2\beta - n\beta - 1)g(X,\phi Z)Y] \\ &\quad -\frac{r-(n-1)((n-1)\beta+1)}{n}(\frac{a}{n-1} + b)[g(Y,\phi Z)X - g(X,\phi Z)Y].\end{aligned} \tag{6.3}$$

Similarly, we have

$$\begin{aligned}\phi\bar{P}(X,Y)Z &= a[\beta^2(g(Y,Z)\phi X - g(X,Z)\phi Y) - \beta g(Y,Z)\phi X + \beta g(X,Z)\phi Y] \\ &\quad +b[S(Y,Z)\phi X - S(X,Z)\phi Y + (2\beta - n\beta - 1)g(Y,Z)\phi X \\ &\quad -(2\beta - n\beta - 1)g(X,Z)\phi Y + (\beta-1)\eta(Y)\eta(Z)\phi X - (\beta-1)\eta(X)\eta(Z)\phi Y] \\ &\quad -\frac{r-(n-1)((n-1)\beta+1)}{n}(\frac{a}{n-1} + b)[g(Y,Z)\phi X - g(X,Z)\phi Y].\end{aligned} \tag{6.4}$$

Putting (6.3) and (6.4) and then taking $Y = \xi$ in (6.1) , we obtain

$$\begin{aligned}&[a(1-\beta^2) - b(\frac{r-(n-1)\beta^2}{n-1} + (2\beta - n\beta - 1)) + \frac{r-(n-1)((n-1)\beta+1)}{n}\cdot \\ &(\frac{a}{n-1} + b)]g(X,\phi Z)\xi + [-a\beta(\beta-1) + b((n-1)\beta^2 + 2\beta - n\beta - 1) - \beta + 1) \\ &-\frac{r-(n-1)((n-1)\beta+1)}{n}(\frac{a}{n-1} + b)]\eta(Z)\phi X = 0.\end{aligned} \tag{6.5}$$

Now considering Z to be orthogonal to ξ, then $\eta(Z) = 0$ and $g(X,\phi Z) \neq 0$, which implies that

(6.6)

$$[(-a\beta^2 + a) - b(\frac{r-(n-1)\beta^2}{n-1} + (2\beta - n\beta - 1)) + \frac{r-(n-1)((n-1)\beta+1)}{n}(\frac{a}{n-1} + b)] = 0$$

which on simplifying gives

(6.7)
$$r = \frac{n(n-1)[(2-n)b\beta - (a+b) + (a-b)\beta^2] + (n-1)((n-1)\beta^2 + 1)(a + b(n-1))}{a-b}.$$

Thus we can state the following theorem:

Theorem 6.2. *For an n-dimensional ϕ-pseudo projectively semisymmetric $(n > 1)$ Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection, the curvature tensor r is given by* (6.7)*.*

7. ϕ-Ricci symmetric Lorentzian β-Kenmotsu manifolds with a semi-symmetric semi-metric connection

Definition 7.1. [2] *A Lorentzian β-Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection $\bar{\nabla}$ is said to be ϕ-Ricci symmetric if the Ricci operator $\bar{Q}$ satisfies*
$$\phi^2(\bar{\nabla}_X \bar{Q})(Y) = 0$$
for any vector fields $X, Y \in \chi(\bar{M})$ and $\bar{S}(X,Y) = g(\bar{Q}X, Y)$.

Theorem 7.2. *An n-dimensional ϕ-Ricci symmetric Lorentzian β-Kenmotsu manifold with a semi-symmetric semi-metric connection is an η-Einstein manifold.*

Proof. Let us assume that the manifold is ϕ-Ricci symmetric with a semi-symmetric semi-metric connection. Then we have

$$\phi^2(\bar{\nabla}_X \bar{Q})(Y) = 0 \tag{7.1}$$

which in view of (2.1) becomes

$$(\bar{\nabla}_X \bar{Q})(Y) + \eta((\bar{\nabla}_X \bar{Q})(Y))\xi = 0. \tag{7.2}$$

Taking inner product of (7.2) with Z, we have

$$g[(\bar{\nabla}_X \bar{Q})(Y), Z] + \eta((\bar{\nabla}_X \bar{Q})(Y))\eta(Z) = 0$$

which on simplifying takes the form

$$g[(\bar{\nabla}_X \bar{Q}Y), Z] - \bar{S}[(\bar{\nabla}_X Y), Z] + \eta((\bar{\nabla}_X \bar{Q})(Y))\eta(Z) = 0. \tag{7.3}$$

Replacing $Y = \xi$ and using (3.13), (3.14) and (3.16) in (7.3), we find

$$\begin{aligned} \bar{S}(X,Z) &= [\beta(\beta-1)(n-1)+2]g(X,Z) \\ &+[\beta(\beta-1)(n-1) + 2 - (n-1)\beta^2 + (n+1)\beta - 2n]\eta(X)\eta(Z). \end{aligned} \tag{7.4}$$

By replacing $X = \phi X$ and $Z = \phi Z$ and using (2.2), (7.4) reduces to

$$\bar{S}(\phi X, \phi Z) = [\beta(\beta-1)(n-1)+2]g(\phi X, \phi Z). \tag{7.5}$$

which in view of (2.2) and (3.17) gives

$$\begin{aligned} S(X,Z) &= -[(n-1)\beta^2 - (2n-3)\beta + 1]g(X,Z) \\ &-[2(n-1)\beta^2 - (2n-1)\beta + 2n + 1]\eta(X)\eta(Z). \end{aligned} \tag{7.6}$$

Equation (7.6) is of the form $S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y)$,
where $\lambda_1 = -[(n-1)\beta^2 - (2n-3)\beta + 1]$ and $\lambda_2 = -[2(n-1)\beta^2 - (2n-1)\beta + 2n + 1]$. This result shows that the manifold under the consideration is an η-Einstein manifold. □

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