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THE BROCARD-RAMANUJAN DIOPHANTINE EQUATION

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Abstract. Using integer structure analysis in the modular ring Z_4 it is shown that the only integer solutions of the Brocard-Ramanujan equation, $n!+1 = m^2$, occur for $(n, m) \in \{(4, 5), (5, 11), (7, 71)\}$. When $3 \mid m$ the rows in the ring can never be equal, but when $3 \nmid m$ the rows of $n!$ always satisfy an exponential function while the rows of m^2 satisfy two quadratic forms. The intersections of the exponential with the quadratic occur only for (n, m) above.

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1. Introduction

In 1876 and 1885 Henri Brocard [2, 3] asked how many solutions are there for the equation

$$n! + 1 = m^2. \tag{1}$$

This was independently posed by Srinivasa Ramanujan [7] in 1913, but to date there are only three solutions: 'Brown pairs' $(n, m) \in \{(4, 5), (5, 11), (7, 71)\}$. Computational studies have shown that there are no further solutions up to $n \leq 10^9$ [1, 6, 14].

A mixture of theoretical [12] and experimental approaches have included that of Helou and Haddad [8] who proved the existence of many infinite families such that the left hand side of (1) is not a perfect square and gave criteria for it to be non-square. Dabrowski and Ulas [5] have also studied a type of generalization of the left hand side of (1). In this paper we use Integer Structure Analysis (ISA) [9] via the Modular Ring Z_4 (Table 1) to show that (1) has only the three solutions already found.

Table 1. Classes and rows for Z_4

Row $\mathbf{r_i} \downarrow$ / **Class** $\mathbf{i} \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	**Comments**
0	0	1	2	3	$N = 4r_i + i$
1	4	5	6	7	even $\bar{0}_4$, $\bar{2}_4$
2	8	9	10	11	$(N^n, N^{2n}) \in \bar{0}_4$
3	12	13	14	15	odd $\bar{1}_4$, $\bar{3}_4$; $N^{2n} \in \bar{1}_4$

2. Integer Structure Analysis: $3 \nmid m$

The row structure may be found in [10, 11] where it is proved that all odd squares belong to Class $\bar{1}_4$, that is,

$$m^2 = 4R_1 + 1 \tag{2}$$

where, for m with no factor 3,

$$R_1 = 3t(3t \pm 1). \tag{3}$$

Furthermore, all values of $n!$ are even and all belong to class $\bar{0}_4$ when $n \geq 4$; that is,

$$n! = 4r_0. \tag{4}$$

That is,

$$\begin{aligned} n! + 1 &= 4r_0 + 1 \\ &= 4R_1 + 1 \end{aligned} \tag{5}$$

so that if (5) is true, then

$$\begin{aligned} r_0 &= R_1 \\ &= 3t(3t \pm 1). \end{aligned} \tag{6}$$

But r_0 is exponential in effect (Figure 1) so that the only solutions of (1) occur when the exponential function crosses either of the two quadratics at an integer point, so that there can be no further integer solutions other than $(4, 5)$, $(5, 11)$, and $(7, 71)$. The case for $3|m$ is presented in the next section.

Some representative values of t and R_1 are displayed in Table 2. Corresponding values of n and r_0 are set out in Table 3.

Table 2. t vs R_1

$\mathbf{t}$	$\mathbf{R_1' = 3t(3t+1)}$	$\mathbf{m'}$	$\mathbf{R_1 = 3t(3t-1)}$	$\mathbf{m}$	$\mathbf{n}$
1	12‡	7	6*	5	4
2	42‡	13	30*	11	5
3	90	19	72‡	17	
4	156‡	25	132‡	23	
5	240	31	210	29	
6	342‡	37	306‡	35	
7	462‡	43	342	41	
8	600	49	552‡	47	
9	750	55	702‡	53	
10	930	61	870	59	
11	1122‡	67	1056‡	65	
12	1332‡	73	1260*	71	7
13	1560	79	1482‡	77	
14	1806‡	85	1722‡	83	
15	2070	91	1980	89	

‡: incompatible with (1) since right-end-digit $r_0^* = 0$, $n > 4$

*: equal to r_0

The original problem is referenced at Sloane (A050216), with $\{m'\}$ and $\{m\}$ as A016921 ($m'_n = 6n+1$), and A016969 ($m_n = 6n-1$), respectively.

TABLE 3. *Values of n which satisfy (1)

n	**n!**	**Class**	**r**
1	1	$\bar{1}_4$	$r_1 = 0$
2	2	$\bar{2}_4$	$r_2 = 0$
3	6	$\bar{2}_4$	$r_2 = 0$
4*	24	$\bar{0}_4$	$r_0 = 6$
5*	120	$\bar{0}_4$	$r_0 = 30$
6	720	$\bar{0}_4$	$r_0 = 180$
7*	5040	$\bar{0}_4$	$r_0 = 1260$
8	40320	$\bar{0}_4$	$r_0 = 10080$
9	362880	$\bar{0}_4$	$r_0 = 90720$

In the range where $r_0 = R_1$

$$t = \frac{71}{3} - 11n + \frac{4}{3}n^2 \tag{7}$$

which applies only if $r_0 = R_1$ (Table 4 and Figure 2).

TABLE 4. r_0 vs R_1

n	**t** eq'n (7)	$\mathbf{r_0}$	$\mathbf{R_1}$ $\mathbf{3t(3t-1)}$
4	1	6	6
5	2	30	30
6	$5\frac{2}{3}$	180	272
7	12	1260	1260
8	21	10080	3906

3. Integer Structure Analysis: $3 \mid m$

In an earlier generalization Dabrowski [4] showed that

$$n! + A = m^2 \tag{8}$$

has only finitely many solutions for general A. This replacement of 1 from the Brocard-Ramanujan equation merely shifts the r_0 curve up in Figures 1 and 2. These indicate that while the intersection points increase in number they are still severely restricted.

When $3|m$ [10]

$$R_1 = 2 + 18\left(\tfrac{1}{2}j(j+1)\right), \quad j = 0, 1, 2, \ldots \tag{9}$$

The row of $n!$ always has a factor 3 but equation (9) does not, so that the rows can never be equal.

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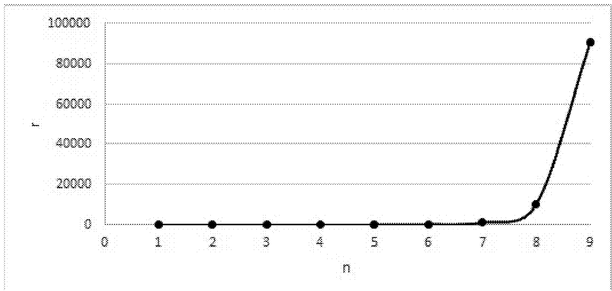

FIGURE 1. n versus r

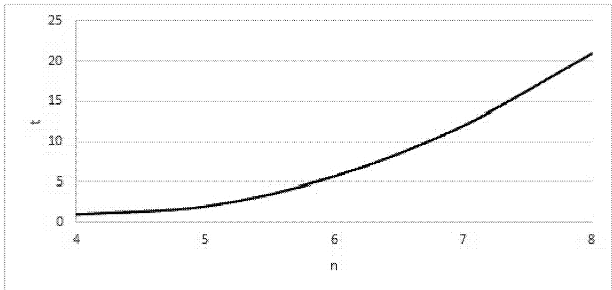

FIGURE 2. Curvilinear trend for n versus t

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