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POSITIVITY OF SUMS FOR n−CONVEX FUNCTIONS VIA TAYLOR'S FORMULA AND GREEN FUNCTION

A. R. KHAN[1,2], J. PEČARIĆ[2,3], M. PRALJAK[4], AND S. VAROŠANEC[5]

Abstract. Conditions under which the inequality $\sum_{i=1}^{m} p_i f(x_i) \geq 0$ holds for every n-convex function f are considered. We are using two approaches: one by the Taylor formula and other using the Green function. Integral analogues and some related results for n-convex functions at a point are also given, as well as bounds for the integral remainders which occur in identities associated with the obtained inequalities.

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1. Introduction

In the forties of the last century, T. Popoviciu studied necessary and sufficient conditions on numbers $x_1, \ldots, x_m, p_1, \ldots, p_m$ under which the inequality $\sum_{i=1}^{m} p_i f(x_i) \geq 0$ holds for any convex function f. Nowdays such results are known as Popoviciu type inequalities and a number of them is given in [7, Chap. 9]. We are interested in such results involving n-convex functions, so let us first recall the definition and some properties of n-convex functions.

Definition 1.1. The n-th order divided difference of a function $f : [a, b] \to \mathbb{R}$ at distinct points $x_i, x_{i+1}, \ldots, x_{i+n} \in [a, b] \subset \mathbb{R}$ for some $i \in \mathbb{N}$ is defined recursively by:

$$\begin{aligned} [x_j; f] &= f(x_j), \quad j \in \{i, \ldots, i+n\} \\ [x_i, \ldots, x_{i+n}; f] &= \frac{[x_{i+1}, \ldots, x_{i+n}; f] - [x_i, \ldots, x_{i+n-1}; f]}{x_{i+n} - x_i}. \end{aligned}$$

We say that f is n-convex or convex of the n-th order if all the n-th order divided differences of the function f are non-negative, i.e. if

$$[x_i, \ldots, x_{i+n}; f] \geq 0$$

for any mutually distinct points $x_i, \ldots, x_{i+n} \in [a, b]$.

It is clear that a 1-convex function is in fact a nondecreasing function, and a 2-convex function is a convex function in the classical sense. So, the concept of n-convexity is a generalization of convexity. It is a known fact that if the n-th order derivative $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geq 0$. For $1 \leq k \leq n-2$, a function f is n-convex if and only if $f^{(k)}$ exists and is $(n-k)$-convex. Furthermore, f is n-convex if and only if $f \in C^{(n-2)}$, $f^{(n-1)}$ exists everywhere except in at most countable many points and $f^{(n-1)}$ is nondecreasing. In the further text we use notation $(x-s)_+^k$, $k \in \mathbb{N}_0$, for the following

$$(x-s)_+^k = \begin{cases} (x-s)^k, & \text{if } x \geq s \\ 0, & \text{if } x < s. \end{cases}$$

The following result for n-convex functions is due to T. Popoviciu (see [7, p.262] and [8]).

Proposition 1.2. *The inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0 \tag{1}$$

holds for all n-convex functions $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, if and only if the $m-$tuples $\mathbf{x} = (x_1, \ldots, x_m) \in [a,b]^m$, $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$ satisfy

$$\sum_{i=1}^{m} p_i x_i^k = 0, \qquad \textit{for all } k \in \{0, 1, \ldots, n-1\} \tag{2}$$

$$\sum_{i=1}^{m} p_i (x_i - s)_+^{n-1} \geq 0, \qquad \textit{for every } s \in [a,b], \tag{3}$$

In fact, Popoviciu proved a result in which $x_1 < x_2 < \ldots < x_m$ and (3) holds for every $s \in [x_{(1)}, x_{(m-n+1)}]$, but, as discussed in [2], it also holds in the form given in the above proposition. The integral analogue is given in the next proposition.

Proposition 1.3. *Let $p : [\alpha, \beta] \to \mathbb{R}$ and $g : [\alpha, \beta] \to [a,b]$ be integrable. Then the inequality*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \geq 0 \tag{4}$$

holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if

$$\begin{aligned} &\int_{\alpha}^{\beta} p(x) g(x)^k\, dx = 0, \qquad \textit{for all } k \in \{0, 1, \ldots, n-1\} \\ &\int_{\alpha}^{\beta} p(x) \left(g(x) - s\right)_+^{n-1}\, dx \geq 0, \qquad \textit{for every } s \in [a,b]. \end{aligned} \tag{5}$$

Remark 1.4. Case $n = 2$ was of particular interest and in [4] (see also [7, p.262]) it is proved that if $n = 2$ conditions (2) and (3) can be replaced with

$$\sum_{i=1}^{m} p_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i |x_i - x_k| \geq 0 \text{ for } k \in \{1, \ldots, m\}. \tag{6}$$

Finally, let us mention the Taylor formula which has a crucial role in our work. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}$, $a, b \in I$, $a < b$. Then for $c, x \in [a,b]$ the following formula holds

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{1}{(n-1)!} \int_c^x f^{(n)}(s)(x-s)^{n-1} ds. \tag{7}$$

The outline of the paper is as follows: in Section 2 we use the Taylor formula (7) to obtain inequalities of type (1) and (4) for n-convex functions. In section 3 we obtain Popoviciu type inequalities using the Green function and the Taylor formula. In Section 4 we give related inequalities for n-convex functions at a point, a generalization of the class of n-convex functions introduced in [6]. In Section 5 we give bounds for the integral reminders which occur in identities obtained in earlier sections by using the pre-Grüss inequality. In the last section we prove certain properties of linear functionals associated with the obtained inequalities which follow from exponentially convexity and log-convexity.

2. Popoviciu type identities and inequalities via Taylor formula

Our first result is an identity which is a basic tool for our subsequent results.

Theorem 2.1. *Let $n, m \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}$, $a, b \in I$, $a < b$. Furthermore, let $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$. Then*

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i (x_i - a)^k + \frac{1}{(n-1)!} \int_a^b f^{(n)}(s) \left(\sum_{i=1}^{m} p_i (x_i - s)_+^{n-1} \right) ds \tag{8}$$

and

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \sum_{i=1}^{m} p_i (b - x_i)^k + \frac{(-1)^n}{(n-1)!} \int_a^b f^{(n)}(s) \left(\sum_{i=1}^{m} p_i (s - x_i)_+^{n-1} \right) ds.$$

Proof. Using notation $(x-s)_+$ we get

$$\int_a^x f^{(n)}(s)(x-s)^{n-1} ds = \int_a^b f^{(n)}(s)(x-s)_+^{n-1} ds$$

for $x \in [a,b]$ and applying the Taylor formula (7) for $c = a$ we get

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(n-1)!} \int_a^b f^{(n)}(s)(x-s)_+^{n-1} ds. \tag{9}$$

Putting in (9) $x = x_i$ for $i = 1, 2, \ldots, m$, multiplying each equation with the corresponding p_i, and adding all m equations we get

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) &= \sum_{i=1}^{m} p_i \left[\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x_i - a)^k \right.\\ &\quad \left. + \frac{1}{(n-1)!}\int_a^b f^{(n)}(s)(x_i - s)_+^{n-1} ds\right] \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i (x_i - a)^k \\ &\quad + \frac{1}{(n-1)!}\int_a^b f^{(n)}(s)\left(\sum_{i=1}^{m} p_i (x_i - s)_+^{n-1}\right) ds\end{aligned}$$

which is the desired identity (8). The second identity is proved in a similar manner using the fact that for $x \in [a, b]$

$$\int_b^x f^{(n)}(s)(x-s)^{n-1} ds = (-1)^n \int_a^b f^{(n)}(s)(s-x)_+^{n-1} ds$$

and applying the Taylor formula for $c = b$. □

We may state its integral version as follows.

Theorem 2.2. *Let $g : [\alpha, \beta] \to [a, b]$ and $p : [\alpha, \beta] \to \mathbb{R}$ be integrable functions. Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}$, $a, b \in I$, $a < b$. Then*

$$\begin{aligned}\int_\alpha^\beta p(x) f(g(x))\, dx &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \int_\alpha^\beta p(x)(g(x) - a)^k dx \\ &\quad + \frac{1}{(n-1)!}\int_a^b f^{(n)}(s) \int_\alpha^\beta p(x)\,(g(x) - s)_+^{n-1}\, dx ds,\end{aligned}$$

$$\begin{aligned}\int_\alpha^\beta p(x) f(g(x))\, dx &= \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \int_\alpha^\beta p(x)\,(b - g(x))^k dx \\ &\quad + \frac{(-1)^n}{(n-1)!}\int_a^b f^{(n)}(s) \int_\alpha^\beta p(x)\,(s - g(x))_+^{n-1}\, dx ds.\end{aligned}$$

Proof. Our required identities are obtained by using the Taylor formulae for $c = a$ and $c = b$ in the expression

$$\int_\alpha^\beta p(x) f(g(x))\, dx$$

and then using the Fubini theorem. □

Now we state inequalities derived from the obtained identities. In the rest of the paper we use the following notation:

$$\Omega_1^{[a,b]}(m, \mathbf{x}, \mathbf{p}, s) := \sum_{i=1}^{m} p_i\,(x_i - s)_+^{(n-1)}, \tag{10}$$

$$\Omega_2^{[a,b]}(m,\mathbf{x},\mathbf{p},s) := (-1)^n \sum_{i=1}^{m} p_i \left(s-x_i\right)_+^{(n-1)}, \tag{11}$$

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) := \sum_{i=1}^{m} p_i f(x_i) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i (x_i-a)^k, \tag{12}$$

$$A_2^{[a,b]}(m,\mathbf{x},\mathbf{p},f) := \sum_{i=1}^{m} p_i f\left(x_i\right) - \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}\left(b\right)}{k!} \sum_{i=1}^{m} p_i (b-x_i)^k. \tag{13}$$

Theorem 2.3. *Let $n,m \in \mathbb{N}$, $x_i \in [a,b]$, I is an interval, $[a,b] \subset I$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$.*

(i) If

$$(U_1) \qquad \Omega_1^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0, \quad \text{for all } s \in [a,b],$$

then for every $n-$convex function $f : I \to \mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on I the following inequality holds

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0. \tag{14}$$

If in (U_1) reversed sign of inequality holds, then inequality (14) is also reversed.

(ii) If

$$(U_2) \qquad \Omega_2^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0, \quad \text{for all } s \in [a,b],$$

then for every $n-$convex function $f : I \to \mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on I the following inequality holds

$$A_2^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0. \tag{15}$$

If in (U_2) reversed sign of inequality holds, then inequality (15) is also reversed.

If the condition "f is n-convex" is replaced by "f is n-concave", then under the same assumptions about Ω_1 and Ω_2, inequalities (14) and (15) hold in the reversed direction.

Proof. We prove (i). Let $\Omega_1^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0$ for all $s \in [a,b]$ and let f be n-convex. Then, $f^{(n)} \geq 0$ and

$$\int_a^b f^{(n)}(s) \left(\sum_{i=1}^{m} p_i (x_i - s)_+^{n-1} \right) ds \geq 0.$$

By Theorem 2.1

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) = \frac{1}{(n-1)!} \int_a^b f^{(n)}(s) \left(\sum_{i=1}^{m} p_i (x_i - s)_+^{n-1} \right) ds \geq 0$$

and we get (14). Other cases are proved in a similar manner. □

Now we state an important consequence.

Theorem 2.4. *Let $n \in \mathbb{N}$, $n \geq 2$, $[a,b] \subset I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous. Additionally, let $j \in \mathbb{N}$*

be fixed, $2 \le j \le n$ *and let* $(x_1, \ldots, x_m) \in [a, b]^m$, $(p_1, \ldots, p_m) \in \mathbb{R}^m$ *satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0 \quad \text{for} \quad k = 0, 1, \ldots, j-1, \tag{16}$$

$$\sum_{i=1}^{m} p_i (x_i - s)_+^{j-1} \ge 0, \quad \text{for } s \in [a, b]. \tag{17}$$

If f *is* n*-convex, then*

$$\sum_{i=1}^{m} p_i f(x_i) \ge \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i (x_i - a)^k \tag{18}$$

with agreement that for $j = n$*, we put* $\sum_{k=j}^{n-1} = 0$.

Furthermore, if $n - j$ *is even, then*

$$\sum_{i=1}^{m} p_i f(x_i) \ge \sum_{k=j}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \sum_{i=1}^{m} p_i (b - x_i)^k \tag{19}$$

while if $n - j$ *is odd, then the reversed inequality in (19) holds.*

Proof. Let $s \in [a, b]$ be fixed. Notice that for $j = n$ we just get Proposition 1.2. For $j \le n - 2$ we get

$$\frac{d^j}{dx^j}(x-s)_+^{n-1} = \begin{cases} (n-1)(n-2)\cdots(n-j)(x-s)^{n-j-1}, & s \le x \le b, \\ 0, & a \le x < s, \end{cases}$$

and

$$(-1)^j \frac{d^j}{dx^j}(s-x)_+^{n-1} = \begin{cases} (n-1)(n-2)\cdots(n-j)(s-x)^{n-j-1}, & a \le x \le s, \\ 0, & s < x \le b, \end{cases}$$

The functions $x \mapsto \frac{d^j}{dx^j}(x-s)_+^{n-1}$ and $x \mapsto (-1)^j \frac{d^j}{dx^j}(s-x)_+^{n-1}$ are non-negative. Hence the functions $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^j (s-x)_+^{n-1}$ are j-convex.

If $j = n - 1$, then we consider the functions $x \mapsto \frac{d^{n-3}}{dx^{n-3}}(x-s)_+^{n-1}$ and $x \mapsto (-1)^{n-1} \frac{d^{n-3}}{dx^{n-3}}(s-x)_+^{n-1}$. They are 2-convex, so $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^{n-1}(s-x)_+^{n-1}$ are $(n-1)$-convex. Hence if $2 \le j \le n-1$, the functions $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^j (s-x)_+^{n-1}$ are j-convex.

Using Proposition 1.2 for the j-convex functions $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^j (s-x)_+^{n-1}$, we get that

$$\sum_{i=1}^{m} p_i (x_i - s)_+^{(n-1)} \ge 0 \tag{20}$$

and

$$(-1)^j \sum_{i=1}^{m} p_i (s - x_i)_+^{(n-1)} \ge 0.$$

Multiplying the last inequality with $(-1)^{n-j}$ (it is positive for even $n - j$) we get

$$(-1)^n \sum_{i=1}^{m} p_i (s - x_i)_+^{(n-1)} \ge 0. \tag{21}$$

Inequalities (20) and (21) mean that assumptions of Theorem 2.3 (i) and (ii) are satisfied, hence inequalities (14) and (15) hold respectively. Moreover, due to assumption (16), $\sum_{i=1}^m p_i P(x_i) = 0$ for every polynomial P of degree $\leq j-1$, so the first j terms in the inner sum in (12) and (13) vanish, i.e. we get inequalities (18) and (19). □

Theorem 2.5. *Let $n \in \mathbb{N}, n \geq 3$. Let $j \in \{2, 3, \ldots, n-1\}$ be fixed number and let $m-$tuples $\mathbf{x} = (x_1, \ldots, x_m) \in [a, b]^m$, $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^m p_i x_i^k = 0, \qquad \text{for all } k \in \{0, 1, \ldots, j-1\} \tag{22}$$

$$\sum_{i=1}^m p_i (x_i - s)_+^{j-1} \geq 0, \qquad \text{for every } s \in [a, b]. \tag{23}$$

If $[a, b] \subset I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ is n-convex such that $f^{(n-1)}$ is absolutely continuous with at least one of the following two properties

(i) $\displaystyle\sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{(k-j)!}(x-a)^{k-j} \geq 0$ *for all $x \in [a, b]$*

(ii) $\displaystyle\sum_{k=j}^{n-1} (-1)^{k-j} \frac{f^{(k)}(b)}{(k-j)!}(b-x)^{k-j} \geq 0$ *for all $x \in [a, b]$ with even $n-j$,*

then the inequality

$$\sum_{i=1}^m p_i f(x_i) \geq 0 \tag{24}$$

holds.

Proof. Let us suppose that f satisfies property (i). Define H by

$$H(x) = \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Then

$$H^{(j)}(x) = \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{(k-j)!}(x-a)^{k-j}$$

and $H^{(j)}(x) \geq 0$, $x \in [a, b]$. Hence H is j-convex. Using Proposition 1.2 for the j-convex function H we obtain

$$\sum_{i=1}^m p_i H(x_i) \geq 0.$$

That conclusion and the previous theorem give

$$\sum_{i=1}^m p_i f(x_i) \geq \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^m p_i (x_i - a)^k = \sum_{i=1}^m p_i H(x_i) \geq 0$$

which is desired inequality (24). If f satisfies property (ii), then we consider the function $H(x) = \displaystyle\sum_{k=j}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!}(b-x)^k$ and proceed in the similar manner. □

Remark 2.6. Let us consider the case: $j = n-1$. Then for an n-convex f under the assumptions $f^{(n-1)}(a) \geq 0$ and (22), (23) we get $\sum_{i=1}^{m} p_i f(x_i) \geq 0$. In comparison with Proposition 1.2, we see that one condition is added and (2), (3) are valid not for n, but for $n-1$. So, this result is an improvement of one direction given in Proposition 1.2.

In the rest of the section we state integral versions of the previous results, the proofs of which are analogous to the discrete case.

Theorem 2.7. *Let $g : [\alpha, \beta] \to [a, b]$ and $p : [\alpha, \beta] \to \mathbb{R}$ be integrable functions and let $f : I \to \mathbb{R}$, $[a, b] \subset I$, be such that $f^{(n-1)}$ is absolutely continuous.*

If

$$(U_3) \qquad \Omega_3^{[a,b]}([\alpha,\beta], g, p, s) := \int_\alpha^\beta p(x)\,(g(x)-s)_+^{(n-1)}\,dx \geq 0,$$

for all $s \in [a, b]$, then for every $n-$convex function f the following inequality holds

$$A_3^{[a,b]}([\alpha,\beta], g, p, f) := \int_\alpha^\beta p\,(x)\,f(g(x))\,dx$$
$$-\sum_{k=0}^{n-1} \frac{f^{(k)}\,(a)}{k!} \int_\alpha^\beta p(x)(g(x)-a)^k dx \geq 0, \tag{25}$$

If in (U_3) reversed sign of inequality holds, then inequality (25) is also reversed.

If

$$(U_4) \qquad \Omega_4^{[a,b]}([\alpha,\beta], g, p, s) := (-1)^n \int_\alpha^\beta p(x)\,(s-g(x))_+^{(n-1)}\,dx \geq 0,$$

for all $s \in [a, b]$, then for every $n-$convex function f the following inequality holds

$$A_4^{[a,b]}([\alpha,\beta], g, p, f) := \int_\alpha^\beta p\,(x)\,f(g(x))\,dx$$
$$-\sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}\,(b)}{k!} \int_\alpha^\beta p(x)(b-g(x))^k dx \geq 0. \tag{26}$$

If in (U_4) reversed sign of inequality holds, then inequality (26) is also reversed.

If the condition "f is n-convex" is replaced by "f is n-concave", then under the same assumptions about Ω_3 and Ω_4, inequalities (25) and (26) hold in the reversed direction.

Theorem 2.8. *Suppose all the assumptions from Theorem 2.2 hold. Additionally, let $j \in \mathbb{N}$, $2 \leq j \leq n$ and let $p : [\alpha, \beta] \to \mathbb{R}$ and $g : [\alpha, \beta] \to [a, b]$ satisfy*

$$\int_\alpha^\beta p(x) g(x)^k\,dx = 0, \qquad \text{for all } k \in \{0, 1, \ldots, j-1\}$$

$$\int_\alpha^\beta p(x)\,(g(x)-s)_+^{j-1}\,dx \geq 0, \qquad \text{for every } s \in [a, b].$$

If f is n-convex, then

$$\int_{\alpha}^{\beta} p(x) f(g(x)) dx \geq \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{k!} \int_{\alpha}^{\beta} p(x)(g(x)-a)_{+}^{k} dx.$$

If, in addition $n-j$ is even, then

$$\int_{\alpha}^{\beta} p(x) f(g(x)) dx \geq \sum_{k=j}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \int_{\alpha}^{\beta} p(x)(b-g(x))_{+}^{k} dx \tag{27}$$

while if $n-j$ is odd, then the reversed sign of inequality holds in (27).

3. Popoviciu type inequalities via Green function

In this section we obtain another identity and the corresponding linear inequality using the Green function and applying again the Taylor formula. The Green function is a function $G:[a,b]\times[a,b]\to\mathbb{R}$ defined by

$$G(s,t)=\begin{cases} \frac{(s-b)(t-a)}{b-a} & \text{for } a\leq t\leq s,\\ \frac{(t-b)(s-a)}{b-a} & \text{for } s\leq t\leq b.\end{cases}$$

The function G is convex with respect to each variable. The next theorem contains two identities in which the sum $\sum_{i=1}^{m} p_i f(x_i)$ is expressed with the n-th derivative of the function f and the values of the first $n-3$ derivatives of f in the points a and b.

Theorem 3.1. *Let $n\in\mathbb{N}$, $n\geq 3$, and $f: I\to\mathbb{R}$, $[a,b]\subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous. Furthermore, let $m\in\mathbb{N}$, $x_i\in[a,b]$ and $p_i\in\mathbb{R}$ for $i\in\{1,2,\ldots,m\}$ be such that*

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i x_i = 0.$$

Then

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) &= \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \sum_{i=1}^{m} p_i G(x_i,t)(t-a)^k dt\\ &+ \frac{1}{(n-3)!}\int_a^b f^{(n)}(s)\left(\int_s^b \sum_{i=1}^{m} p_i G(x_i,t)(t-s)^{n-3} dt\right) ds \end{aligned} \tag{28}$$

and

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) &= \sum_{k=0}^{n-3} (-1)^k \frac{f^{(k+2)}(b)}{k!} \int_a^b \sum_{i=1}^{m} p_i G(x_i,t)(b-t)^k dt\\ &- \frac{1}{(n-3)!}\int_a^b f^{(n)}(s)\left(\int_a^s \sum_{i=1}^{m} p_i G(x_i,t)(t-s)^{n-3} dt\right) ds. \end{aligned} \tag{29}$$

Proof. Using integration by parts the following is valid

$$f(x)=\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)+\int_a^b G(x,t)f''(t)dt.$$

Putting in the above equality $x = x_i$, multiplying with p_i, adding all the equalities for $i = 1, \ldots, m$ and using the conditions that $\sum_{i=1}^m p_i = 0$, $\sum_{i=1}^m p_i x_i = 0$ we get

$$\sum_{i=1}^m p_i f(x_i) = \int_a^b \left(\sum_{i=1}^m p_i G(x_i, t) \right) f''(t) dt.$$

Differentiating (7) twice we get

$$f''(x) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(c)}{k!} (x-c)^k + \frac{1}{(n-3)!} \int_c^x f^{(n)}(s)(x-s)^{n-3} ds. \tag{30}$$

Putting in (30) $c = a$ and $c = b$ respectively we get

$$\sum_{i=1}^m p_i f(x_i) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \left(\sum_{i=1}^m p_i G(x_i, t) \right) (t-a)^k dt$$
$$+ \frac{1}{(n-3)!} \int_a^b \int_a^t f^{(n)}(s)(t-s)^{n-3} \left(\sum_{i=1}^m p_i G(x_i, t) \right) ds dt$$

and

$$\sum_{i=1}^m p_i f(x_i) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(b)}{k!} \int_a^b \left(\sum_{i=1}^m p_i G(x_i, t) \right) (t-b)^k dt$$
$$+ \frac{1}{(n-3)!} \int_a^b \int_b^t f^{(n)}(s)(t-s)^{n-3} \left(\sum_{i=1}^m p_i G(x_i, t) \right) ds dt.$$

Using the Fubini theorem we obtain identities (28) and (29). □

Theorem 3.2. *Let $n, m \in \mathbb{N}$, $n \geq 3$, $\mathbf{x} = (x_1, \ldots, x_m) \in [a,b]^m$ and $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$ be such that*

$$\sum_{i=1}^m p_i = 0, \quad \sum_{i=1}^m p_i x_i = 0. \tag{31}$$

(i) If

$$(U_5) \qquad \Omega_5^{[a,b]}(m, \mathbf{x}, \mathbf{p}, s) := \int_s^b \sum_{i=1}^m p_i G(x_i, t)(t-s)^{n-3} dt \geq 0 \text{ for all}$$

$s \in [a,b]$, then for every $n-$convex function $f : I \to \mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on $I \subseteq [a,b]$ the following inequality holds

$$A_5^{[a,b]}(m, \mathbf{x}, \mathbf{p}, f) :=$$
$$\sum_{i=1}^m p_i f(x_i) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \sum_{i=1}^m p_i G(x_i, t)(t-a)^k dt \geq 0. \tag{32}$$

If in (U_5) reversed sign of inequality holds, then inequality (32) is also reversed.

(ii) If

$$(U_6) \qquad \Omega_6^{[a,b]}(m, \mathbf{x}, \mathbf{p}, s) := \int_a^s \sum_{i=1}^m p_i G(x_i, t)(t-s)^{n-3} dt \leq 0 \text{ for all}$$

$s \in [a,b]$, then for every $n-$convex function $f : I \to \mathbb{R}$ such that $f^{(n-1)}$ is

absolutely continuous on $I \subseteq [a,b]$ the following inequality holds

$$A_6^{[a,b]}(m,\mathbf{x},\mathbf{p},f) :=$$

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{k=0}^{n-3}(-1)^k \frac{f^{(k+2)}(b)}{k!}\int_a^b \sum_{i=1}^{m} p_i G(x_i,t)(b-t)^k dt \geq 0. \tag{33}$$

If in (U_6) reversed sign of inequality holds, then inequality (33) is also reversed.

(iii) If the condition "f is n-convex" is replaced by "f is n-concave", then under the same assumptions about Ω_5 and Ω_6, inequalities (32) and (33) hold in the reversed direction.

Proof. If f is n-convex, then $f^{(n)} \geq 0$. Using this fact and the identities from Theorem 3.1 we get the required results. □

If we add an additional condition on $\mathbf{x}$, then in the previous statements we can remove assumptions about Ω_5 and Ω_6. More precisely, we have the following result.

Theorem 3.3. *Let $n \in \mathbb{N}$, $n \geq 3$, and $f : I \to \mathbb{R}$, $[a,b] \subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous. Furthermore, let $m \in \mathbb{N}$, $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$ such that*

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i |x_i - x_k| \geq 0 \text{ for } k = 1,2,\ldots,m.$$

If f is n-convex, then (32) holds. If n is even, then (33) is valid, while if n is odd, then a reversed sign in inequality (33) holds.

If f is n-concave, then reversed (32) holds. If n is even, then reversed (33) holds, while if n is odd, then inequality (33) holds.

Proof. By Remark 1.4 m-tuples $\mathbf{x}$, $\mathbf{p}$ satisfy the assumptions of Proposition 1.2. Since G is convex with respect to the first variable, using Proposition 1.2 we conclude that

$$\sum_{i=1}^{m} p_i G(x_i,t) \geq 0 \quad \text{for } t \in [a,b].$$

Note that $(t-s)^{n-3} \geq 0$ for $t \in [s,b]$ so we get $\Omega_5^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0$. By Theorem 3.2 (i), we have that $A_5^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0$. Other parts are proved in the similar manner. □

The integral versions of the previous three theorems may also be stated. Since the proofs of these results are similar, we omit the details.

Theorem 3.4. *Let $g : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be integrable functions such that*

$$\int_\alpha^\beta p(x)dx = 0, \quad \int_\alpha^\beta p(x)g(x)dx = 0. \tag{34}$$

Let $n \geq 3$ and $f : I \to \mathbb{R}$, $[a,b] \subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous. Then we get the following identities

$$\int_\alpha^\beta p(x) f(g(x))\, dx = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (t-a)^k \, dt + \frac{1}{(n-3)!} \int_a^b f^{(n)}(s) \left(\int_s^b \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (t-s)^{n-3} \, dt \right) ds,$$

$$\int_\alpha^\beta p(x) f(g(x))\, dx = \sum_{k=0}^{n-3} (-1)^k \frac{f^{(k+2)}(b)}{k!} \int_a^b \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (b-t)^k dt - \frac{1}{(n-3)!} \int_a^b f^{(n)}(s) \left(\int_a^s \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (t-s)^{n-3} dt \right) ds.$$

Theorem 3.5. *Let g, p, n satisfy assumptions of Theorem 3.4 hold.*

(i) If

$$(U_7) \quad \Omega_7^{[a,b]}([\alpha,\beta], g, p, s) := \int_s^b \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (t-s)^{n-3} dt \geq 0$$

for all $s \in [a,b]$, then for every $n-$convex function $f : I \to \mathbb{R}$, $[a,b] \subset I$, such that $f^{(n-1)}$ is absolutely continuous, the following inequality holds

$$A_7^{[a,b]}([\alpha,\beta], g, p, f) := \int_\alpha^\beta p(x) f(g(x))\, dx - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (t-a)^k \, dt \geq 0. \tag{35}$$

If in (U_7) reversed sign of inequality holds, then inequality (35) is also reversed.

(ii) If

$$(U_8) \quad \Omega_8^{[a,b]}([\alpha,\beta], g, p, s) := \int_a^s \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (t-s)^{n-3} dt \leq 0$$

for all $s \in [a,b]$, then for every $n-$convex function $f : I \to \mathbb{R}$, $[a,b] \subset I$, such that $f^{(n-1)}$ is absolutely continuous, the following inequality holds

$$A_8^{[a,b]}([\alpha,\beta], g, p, f)) := \int_\alpha^\beta p(x) f(g(x))\, dx - \sum_{k=0}^{n-3} (-1)^k \frac{f^{(k+2)}(b)}{k!} \int_a^b \left(\int_\alpha^\beta p(x) G(g(x),t) dx \right) (b-t)^k dt \geq 0. \tag{36}$$

If in (U_8) reversed sign of inequality holds, then inequality (36) is also reversed.

(iii) If the condition "f is n-convex" is replaced by "f is n-concave", then under the same assumptions about Ω_7 and Ω_8, inequalities (35) and (36) hold in the reversed direction.

Theorem 3.6. *Let all the assumptions of Theorem 3.4 hold. Additionally, let*

$$\int_\alpha^\beta p(x)(g(x)-t)_+^{n-1} dx \geq 0 \quad \textit{for all } t \in [a,b].$$

If f is n-convex, then (35) holds. If n is even, then (36) holds, while if n is odd, then a reversed sign in inequality (36) holds.

If f is n-concave, then reversed (35) holds. If n is even, then reversed (36) holds, while if n is odd, then inequality (36) holds.

4. Related inequalities for n-convex functions at a point

In this section we give related results for the class of n-convex functions at a point which is introduced in [6].

Definition 4.1. Let I be an interval in $\mathbb{R}$, c a point in the interior of I and $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is said to be n-convex at point c if there exists a constant K such that the function

$$F(x) = f(x) - \frac{K}{(n-1)!}x^{n-1}$$

is $(n-1)$-concave on $I \cap (-\infty, c]$ and $(n-1)$-convex on $I \cap [c, \infty)$. A function f is said to be n-concave at point c if the function $-f$ is n-convex at point c.

It is known that a function is n-convex on an interval if and only if it is n-convex at every point of the interval. Necessary and sufficient conditions on two linear functionals $A : C[a, c] \to \mathbb{R}$ and $B : C[c, b] \to \mathbb{R}$ so that the inequality $A(f) \le B(f)$ holds for every function f that is n-convex at c are studied in [6]. In this section we give inequalities of this type for particular linear functionals related to inequalities obtained in the previous section.

Let e_i denote the monomials $e_i(x) = x^i$, $i \in \mathbb{N}_0$. First we state our main theorem of this section for the discrete case.

Theorem 4.2. *Let $c \in (a, b)$, $\mathbf{x} \in [a, c]^m$, $\mathbf{y} \in [c, b]^l$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{q} \in \mathbb{R}^l$ and $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous.*

(i) If $k = 1, 2$ let $A_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, s)$ be defined as in (10)–(13) and satisfy the following conditions:

$$\Omega_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, s) \ge 0, \quad \text{for every } s \in [a, c], \tag{37}$$

$$\Omega_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, s) \ge 0, \quad \text{for every } s \in [c, b], \tag{38}$$

and

$$A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, e_n) = A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, e_n). \tag{39}$$

If f is $(n+1)$-convex at point c, then

$$A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, f) \le A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, f). \tag{40}$$

If inequalities in (37) *and* (38) *are reversed, then* (40) *holds with the reversed sign of inequality.*

(ii) If $k = 5, 6$ let $A_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, s)$ be defined as in Theorem 3.2 and let assumption (31) holds. For $k = 5$, if (37), (38) and (39) are valid, then for an $(n+1)$-convex function f at point c, $(n \ge 3)$, inequality (40) holds. For $k = 6$, if (39) holds and reversed (37), (38) are valid, then inequality (40) holds.

Proof. (i) Let $k \in \{1, 2\}$ and (37), (38), (39) hold. Since f is $(n+1)$-convex at point c there exists a constant K such that the function $F = f - \frac{K}{n!}e_n$ is n-concave on $[a, c]$ and n-convex on $[c, b]$.

Applying Theorem 2.3 to F on the interval $[a, c]$ and on the interval $[c, b]$ we have

$$A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, F) \le 0 \le A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, F).$$

Using definition of F we obtain that

$$A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, f) - \frac{K}{n!}A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, e_n) \le A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, f) - \frac{K}{n!}A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, e_n)$$

$$A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, f) \le A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, f) - \frac{K}{n!}\Big[A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, e_n) - A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, e_n)\Big].$$

Since equality (39) is valid we get

$$A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, f) \le A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, f).$$

□

Remark 4.3. A closer look at the proof of Theorem 4.2 gives us that similar result holds if instead of equality (39) we consider the condition

$$K\left(A_k^{[c,b]}(l, \mathbf{y}, \mathbf{q}, e_n) - A_k^{[a,c]}(m, \mathbf{x}, \mathbf{p}, e_n)\right) \ge 0.$$

Corollary 4.4. *Let $j_1, j_2, n \in \mathbb{N}$, $2 \le j_1, j_2 \le n$ and let $f : [a, b] \to \mathbb{R}$ be $(n+1)$-convex at point c. Let m-tuples $\mathbf{x} \in [a, c]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{l} p_i x_i^k = 0, \qquad \text{for all } k \in \{0, 1, \ldots, j_1 - 1\}$$

$$\sum_{i=1}^{l} p_i (x_i - s)_+^{j_1 - 1} \ge 0, \qquad \text{for every } s \in [a, c]$$

and let l-tuples $\mathbf{y} \in [c, b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ satisfy

$$\sum_{i=1}^{l} q_i y_i^k = 0, \qquad \text{for all } k \in \{0, 1, \ldots, j_2 - 1\}$$

$$\sum_{i=1}^{l} q_i (y_i - s)_+^{j_2 - 1} \ge 0, \qquad \text{for every } s \in [c, b]$$

and let identity (39) *holds.*

Then

$$A_1^{[a,c]}(m, \mathbf{x}, \mathbf{p}, f) \le A_1^{[c,b]}(l, \mathbf{y}, \mathbf{q}, f)$$

and if $n - j_1, n - j_2$ are even, then

$$A_2^{[a,c]}(m, \mathbf{x}, \mathbf{p}, f) \le A_2^{[c,b]}(l, \mathbf{y}, \mathbf{q}, f).$$

Proof. Since f is $(n+1)$-convex at point c there exists a constant K such that function $F = f - \frac{K}{n!}e_n$ is n-concave on $[a, c]$ and n-convex on $[c, b]$. The number j_1 and m-tuples $\mathbf{x}, \mathbf{p}$ satisfy the assumptions of Theorem 2.4 and for concave F on $[a, c]$ we get

$$A_1^{[a,c]}(m, \mathbf{x}, \mathbf{p}, F) \leq 0.$$

Also, the number j_2 and l-tuples $\mathbf{y}, \mathbf{q}$ satisfy the assumptions of Theorem 2.4 and for convex F on $[c, b]$ we get

$$A_1^{[c,b]}(l, \mathbf{y}, \mathbf{q}, F) \geq 0.$$

Hence

$$A_1^{[a,c]}(m, \mathbf{x}, \mathbf{p}, F) \leq A_1^{[c,b]}(l, \mathbf{y}, \mathbf{q}, F)$$

which is equivalent to

$$A_1^{[a,c]}(m, \mathbf{x}, \mathbf{p}, f) - \frac{K}{n!} A_1^{[a,c]}(m, \mathbf{x}, \mathbf{p}, e_n) \leq A_1^{[c,b]}(l, \mathbf{y}, \mathbf{q}, f) - \frac{K}{n!} A_1^{[c,b]}(l, \mathbf{y}, \mathbf{q}, e_n)$$

and using condition (39) we get the desired inequality. The second statement is proved in the similar manner. □

Integral analogues of the previous theorem may be stated as:

Theorem 4.5. *Let $\alpha \leq \beta, \gamma \leq \delta$, $a < c < b$, $g : [\alpha, \beta] \to [a, c]$, $p : [\alpha, \beta] \to \mathbb{R}$, $h : [\gamma, \delta] \to [c, b]$, $q : [\gamma, \delta] \to \mathbb{R}$ be integrable. Let $f : I \to \mathbb{R}$, $[a, b] \subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous.*

(i) If $k = 3, 4$ let $A_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, s)$ be defined as in Theorem 2.7 *and satisfy the following conditions:*

$$\Omega_k^{[a,c]}([\alpha, \beta], g, p, s) \geq 0, \quad \textit{for every } s \in [a, c], \tag{41}$$

$$\Omega_k^{[c,b]}([\gamma, \delta], h, q, s) \geq 0, \quad \textit{for every } s \in [c, b], \tag{42}$$

$$A_k^{[a,c]}([\alpha, \beta], g, p, e_n) = A_k^{[c,b]}([\gamma, \delta], h, q, e_n). \tag{43}$$

If f is $(n + 1)$-convex at point c, then

$$A_k^{[a,c]}([\alpha, \beta], g, p, f) \leq A_k^{[c,b]}([\gamma, \delta], h, q, f). \tag{44}$$

If the inequalities in (41) *and* 42 *are reversed, then the reversed sign in* (44) *holds.*

(ii) If $k = 7, 8$ let $A_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot, \cdot, \cdot, s)$ be defined as in Theorem 3.4 and let assumption (34) holds. For $k = 7$, if (41), (42) and (43) are valid, then for an $(n + 1)$-convex function f at point c, $(n \geq 3)$, inequality (44) holds. For $k = 8$, if (43) holds and reversed (41), (42) are valid, then inequality (44) holds.

Corollary 4.6. *Let $j_1, j_2, n \in \mathbb{N}$, $2 \leq j_1, j_2 \leq n$, let $f : I \to \mathbb{R}$, $[a, b] \subset I$, be $(n + 1)$-convex at point c, let integrable $p : [\alpha, \beta] \to \mathbb{R}$ and $g : [\alpha, \beta] \to [a, c]$ satisfy* (5) *with n replaced by j_1, let $q : [\gamma, \delta] \to \mathbb{R}$ and $h : [\gamma, \delta] \to [c, b]$ satisfy*

$$\int_\gamma^\delta q(x) h^k(x) = 0, \quad \textit{for all } k \in \{0, 1, \ldots, j_2 - 1\}$$

$$\int_\gamma^\delta q(x)(h(x) - s)_+^{j_2 - 1} dx \geq 0, \quad \textit{for every } s \in [c, b]$$

and let (43) *holds. Then*

$$A_3^{[a,c]}([\alpha,\beta],g,p,f) \le A_3^{[c,b]}([\gamma,\delta],h,q,f).$$

If $n-j_1$ *and* $n-j_2$ *are even, then*

$$A_4^{[a,c]}([\alpha,\beta],g,p,f) \le A_4^{[c,b]}([\gamma,\delta],h,q,f).$$

5. Bounds for $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and R_n^k

In this section we give several estimations connected with the functionals $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$, $k \in \{1,\ldots,8\}$. We use the well-known Hölder inequality and bound for the Čebyšev functional $T(f,h)$ which is defined as:

$$T(f,h) = \frac{1}{b-a}\int_a^b f(x)h(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b h(x)dx.$$

This bound is given in the following proposition in which the pre-Grüss inequality is given.

Proposition 5.1. *([3]) Let* $f,h:[a,b]\to\mathbb{R}$ *be integrable such that* $fh \in L(a,b)$*. If*

$$\gamma \le h(x) \le \Gamma \quad \text{for} \quad x\in[a,b],$$

then

$$|T(f,h)| \le \frac{1}{2}(\Gamma-\gamma)\sqrt{T(f,f)}.$$

Now by using the aforementioned result, we are going to obtain formula for $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and estimations of remainders which occur in this formula. For the sake of brevity, in the present and next two sections we use the notations $A_k(f) = A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\Omega_k(t) = \Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,t)$ for $k\in\{1,2,\ldots,8\}$. Now, we are ready to state main results of this section.

Theorem 5.2. *(i) Let* $k\in\{1,2,3,4\}$*. Let* $f: I\to\mathbb{R}$*,* $[a,b]\subset I$*, be such that* $f^{(n-1)}$ *is an absolutely continuous function and*

$$\gamma \le f^{(n)}(x) \le \Gamma \quad \text{for } x\in[a,b].$$

Then

$$A_k(f) = \frac{\left[f^{n-1}(b)-f^{n-1}(a)\right]}{(n-1)!(b-a)}\int_a^b \Omega_k(s)ds + R_n^k(f;a,b), \tag{45}$$

where the remainder $R_n^k(f;a,b)$ *satisfies the estimation*

$$|R_n^k(f;a,b)| \le \frac{b-a}{2(n-1)!}(\Gamma-\gamma)\sqrt{T(\Omega_k,\Omega_k)}. \tag{46}$$

(ii) Let $k\in\{5,6,7,8\}$*. Let assume that condition (31) holds if* $k=5,6$*, or condition (34) holds if* $k=7,8$*.*

If assumptions of (i) hold with $n\ge 3$*, then (45) and (46) hold with* $(n-3)!$ *instead of* $(n-1)!$ *in the denominator of* $A_k(f)$ *and in the bound for* R_n^k*.*

Proof. Fix $k \in \{1, 2, 3, 4\}$. Using the definition of A_k and results from the second section we have

$$\begin{aligned} A_k(f) &= \frac{1}{(n-1)!}\int_a^b f^{(n)}(s)\Omega_k(s)ds \\ &= \frac{1}{(n-1)!(b-a)}\int_a^b f^{(n)}(s)ds\int_a^b \Omega_k(s)ds + R_n^k(f;a,b) \\ &= \frac{\left[f^{n-1}(b)-f^{n-1}(a)\right]}{(n-1)!(b-a)}\int_a^b \Omega_k(s)ds + R_n^k(f;a,b), \end{aligned}$$

where

$$R_n^k(f;a,b) = \frac{1}{(n-1)!}\left(\int_a^b f^{(n)}(s)\Omega_k(s)ds - \frac{1}{b-a}\int_a^b f^{(n)}(s)ds\int_a^b \Omega_k(s)ds\right).$$

Applying Proposition 5.1 for $f \to \Omega_k$ and $h \to f^{(n)}$, we obtain

$$|R_n^k(f;a,b)| = |T(\Omega_k, f^{(n)}| \leq \frac{b-a}{2(n-1)!}(\Gamma-\gamma)\sqrt{T(\Omega_k,\Omega_k)}.$$

The proof for $k \in \{5, 6, 7, 8\}$ is done in a similar manner. □

Using the same method as in the previous theorem and other type of bounds for the Čebyšev functional we are able to give another estimation for a remainder. Now we state some Ostrowski-type inequalities related to the generalized linear inequalities. As usual, by symbol $L_p[a,b]$, $(1 \leq p < \infty)$, we denote the space of functions f on $[a,b]$ with the property

$$\|f\|_{p,} = \left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}} < \infty.$$

Theorem 5.3. *(i) Let $k \in \{1, 2, 3, 4\}$. Let (q, r) be a pair of conjugate exponents, i.e., $1 \leq q, r \leq \infty$, $\frac{1}{q}+\frac{1}{r}=1$. Let $f^{(n)} \in L_q[a,b]$ for some $n \geq 2$. Then*

$$|A_k(f)| \leq \frac{1}{(n-1)!}\|f^{(n)}\|_q \|\Omega_k\|_r . \tag{47}$$

The constant on the right hand side of (47) *is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.*

(ii) Let $k \in \{5, 6, 7, 8\}$. Let assume that condition (31) holds if $k = 5, 6$, or condition (34) holds if $k = 7, 8$.

If assumptions of (i) hold with $n \geq 3$, then the statement holds with $(n-3)!$ instead of $(n-1)!$ in the denominator of the bound for A_k.

Proof. Fix $k \in \{1, 2, 3, 4\}$. From the definition of A_k and results from the second section, applying the Hölder inequality we get

$$|A_k(f)| = \left|\frac{1}{(n-1)!}\int_a^b f^{(n)}(s)\Omega_k(s)ds\right| \leq \|f^{(n)}\|_q \|\lambda_k\|_r .$$

Let us denote the quotient $\frac{1}{(n-1)!}\Omega_k$ by λ_k. For the proof of the sharpness of $\left(\int_a^b |\lambda_k(t)|^r\, dt\right)^{1/r}$, let us find a function f for which the equality in (47) is obtained.

For $1 < q < \infty$ take f to be such that

$$f^{(n)}(t) = \text{sgn } \lambda_k(t) \cdot |\lambda_k(t)|^{1/(q-1)}.$$

For $q = \infty$, take f such that

$$f^{(n)}(t) = \text{sgn } \lambda_k(t).$$

The fact that (47) is the best possible for $q = 1$ can be proved as in [2, Thm 12]. Proof for $k \in \{5, 6, 7, 8\}$ is done in the similar manner. □

6. Mean Value Theorems and Exponential Convexity

In this section we give several mean-value theorems and apply a general method for obtaining new exponentially convex functions related to the functionals A_k defined in previous sections. As we said in the previous section we use notation $A_k(f) := A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$, $k \in \{1, \ldots, 8\}$ where $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ is defined in the second and third sections. Since theorems in this section contain results for $k = 1, \ldots, 8$, we use this agreement throughout this section: if $k \in \{1, 2, 3, 4\}$, then $n \in \mathbb{N}$; if $k \in \{5, 6\}$, then $n \geq 3$ and (31) holds; if $k \in \{7, 8\}$, then $n \geq 3$ and (34) holds.

Theorem 6.1. *Let $k \in \{1, \ldots, 8\}$ and let us consider A_k as a functional on $C^n[a, b]$. If (U_k) holds, then there exists $\xi_k \in [a, b]$ such that*

$$A_k(f) = f^{(n)}(\xi_k)A_k(f_0), \tag{48}$$

where $f_0(x) = \frac{x^n}{n!}$.

Proof. Since $f^{(n)}$ is continuous on $[a, b]$, so $f^{(n)}([a, b]) = [L, M]$, where $L = \min_{x\in[a,b]} f^{(n)}(x)$ and $M = \max_{x\in[a,b]} f^{(n)}(x)$.

Therefore the function

$$F(x) = M\frac{x^n}{n!} - f(x) = Mf_0(x) - f(x)$$

satisfies $F^{(n)}(x) = M - f^{(n)}(x) \geq 0$, *i.e.*, F is $n-$convex function. Hence $A_k(F) \geq 0$ and we conclude

$$A_k(f) \leq MA_k(f_0).$$

Similarly, we have

$$LA_k(f_0) \leq A_k(f).$$

Combining these two inequalities we get

$$LA_k(f_0) \leq A_k(f) \leq MA_k(f_0).$$

If $A_k(f_0) = 0$, then $A_k(f) = 0$ and the statement (48) obviously holds.

If $A_k(f_0) \neq 0$, then $\dfrac{A_k(f)}{A_k(f_0)} \in [L, M]$. Hence there exists $\xi_k \in [a, b]$ such that $\dfrac{A_k(f)}{A_k(f_0)} = f^{(n)}(\xi_k)$, i.e. the statement of the theorem is proved. □

Theorem 6.2. *Let $k \in \{1, \ldots, 8\}$. Let $f, h \in C^n[a, b]$. If (U_k) holds, then there exists $\xi_k \in [a, b]$ such that*

$$\frac{A_k(f)}{A_k(h)} = \frac{f^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

assuming that both denominators are non-zero.

Proof. Fix $k \in \{1, \ldots, 8\}$. Let ω be defined as

$$\omega = A_k(h)f - A_k(f)h.$$

Since $\omega \in C^n[a,b]$, using Theorem 6.1 there exists ξ_k such that

$$A_k(\omega) = \omega^{(n)}(\xi_k)A_k(f_0).$$

Obviously, $A_k(\omega) = 0$ and $\omega^{(n)}(\xi_k) = A_k(h)f^{(n)}(\xi_k) - A_k(f)h^{(n)}(\xi_k)$. So

$$A_k(h)f^{(n)}(\xi_k) - A_k(f)h^{(n)}(\xi_k) = 0$$

which gives us the required result. □

Remark 6.3. If the inverse of $\frac{f^{(n)}}{h^{(n)}}$ exists, then for $k \in \{1, \ldots, 8\}$ from the above mean value theorem we can define generalized mean

$$\xi_k = \left(\frac{f^{(n)}}{h^{(n)}}\right)^{-1} \left(\frac{A_k(f)}{A_k(h)}\right). \tag{49}$$

6.1. **Exponentially Convex Functions.** In the present subsection, we pay attention to the concept of exponential convexity and how our results generate new classes of exponentially convex functions. Let us recall related definitions and some important results from [1] and [5].

Definition 6.4. A function $f : I \to \mathbb{R}$ is $n-$*exponentially convex* in the $J-$sense if the inequality

$$\sum_{i,j=1}^{n} u_i u_j f\left(\frac{t_i + t_j}{2}\right) \geq 0$$

holds for each $t_i, t_j \in I$ and $u_i, u_j \in \mathbb{R}$, $i, j \in \{1, \ldots, n\}$.

A function $f : I \to \mathbb{R}$ is $n-$*exponentially convex* if it is $n-$exponentially convex in the $J-$sense and continuous on I.

Remark 6.5. We can see from the definition that $1-$exponentially convex functions in the $J-$sense are in fact nonnegative functions. Also, $n-$exponentially convex functions in the $J-$sense are $k-$exponentially convex in the $J-$sense for every $k \in \mathbb{N}$ such that $k \leq n$.

Definition 6.6. A function $f : I \to \mathbb{R}$ is *exponentially convex in the $J-$sense*, if it is $n-$exponentially convex in the $J-$sense for each $n \in \mathbb{N}$.

A function $f : I \to \mathbb{R}$ is *exponentially convex* if it is exponentially convex in the $J-$sense and continuous on I.

Here, we get new results concerning the $n-$exponential convexity and exponential convexity for functionals A_k, $k \in \{1, \ldots, 8\}$ defined in the second and third sections.

Theorem 6.7. *Let $D_1 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is $r-$exponentially convex in the $J-$sense on I for any mutually distinct points $z_0, z_1, \ldots, z_n \in [a,b]$, $n \geq 2$. Let $k \in \{1, \ldots, 8\}$.*

If condition (U_k) holds, then the following statements are valid:

(a) *The function* $t \mapsto A_k(f_t)$ *is* $r-$*exponentially convex function in the* $J-$*sense on* I.

(b) *If the function* $t \mapsto A_k(f_t)$ *is continuous on* I, *then the function* $t \mapsto A_k(f_t)$ *is* $r-$*exponentially convex on* I.

If the phrase "$r-$*exponentially convex" is replaced with "exponentially convex", then statements also hold.*

Proof. (a) Fix $k \in \{1,2\}$. Let us define the function ω for $t_i, t_j \in I,\ u_i u_j \in \mathbb{R},\ i,j \in \{1,\dots,r\}$ as follows

$$\omega = \sum_{i,j=1}^{r} u_i u_j f_{\frac{t_i+t_j}{2}},$$

Since the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is $r-$exponentially convex in the $J-$sense, therefore

$$[z_0, z_1, \dots, z_n; \omega] = \sum_{i,j=1}^{r} u_i u_j [z_0, z_1, \dots, z_n; f_{\frac{t_i+t_j}{2}}] \geq 0$$

which implies that ω is $n-$convex function on I and using Theorem 2.3 we get $A_k(\omega) \geq 0$. Hence

$$\sum_{i,j=1}^{r} u_i u_j A_k(f_{\frac{t_i+t_j}{2}}) \geq 0.$$

We conclude that the function $t \mapsto A_k(f_t)$ is an $r-$exponentially convex function on I in $J-$sense. Other cases are proved in a similar manner.

(b) This part easily follows from definition of $n-$exponentially convex function. □

Remark 6.8. Condition "$D_1 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is $r-$exponentially convex" can be replaced with "$D_1 = \{f_t : t \in I\}$ be a class of n-time differentiable functions such that the function $t \mapsto f_t^{(n)}$ is $r-$exponentially convex".

As a consequence of the above theorem we give the following theorem which connects A_k with log-convexity.

Theorem 6.9. *Let* $D_2 = \{f_t : t \in I\}$ *be a class of functions such that the function* $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ *is* $2-$*exponentially convex in the* $J-$*sense on* I *for any mutually distinct points* $z_0, z_1, \dots, z_n \in [a,b]$, $n \geq 2$. *Let* $k \in \{1,\dots,8\}$.

If condition (U_k) *holds, then the following statements are valid:*

(a) *If the function* $t \mapsto A_k(f_t)$ *is positive continuous, then it is log-convex on* I. *Moreover, the following Lyapunov type inequality holds for* $r < s < t,\ r,\ s,\ t \in I$

$$[A_k(f_s)]^{t-r} \leq [A_k(f_r)]^{t-s} [A_k(f_t)]^{s-r}. \tag{50}$$

(b) *If the function* $t \mapsto A_k(f_t)$ *is positive and differentiable on* I, *then for every* $s,t,u,v \in I$ *such that* $s \leq u$ *and* $t \leq v$, *we have*

$$\mu_{s,t}(A_k, D_2) \leq \mu_{u,v}(A_k, D_2) \tag{51}$$

where $\mu_{s,t}$ is defined as

$$\mu_{s,t}(A_k, D_2) = \begin{cases} \left(\frac{A_k(f_s)}{A_k(f_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}A_k(f_s)}{A_k(f_s)}\right), & s = t \end{cases} \tag{52}$$

for $f_s, f_t \in D_2$.

Furthermore, if $r, r_1, \ldots, r_l, r+r_1, \ldots, r+r_l, r+r_1+\ldots+r_l \in I$, then

$$A_k(f_r)^{n-1}A_k(f_{r+r_1+\ldots+r_l}) \geq A_k(f_{r+r_1}) \cdot \ldots \cdot A_k(f_{r+r_l}). \tag{53}$$

Particularly, if $0 \in I$, then we get the Čebyšev type inequality

$$A_k(f_0)^{n-1}A_k(f_{r_1+\ldots+r_l}) \geq A_k(f_{r_1}) \cdot \ldots \cdot A_k(f_{r_l}).$$

Proof. (a) Applying Theorem 6.7 for $r = 2$ we get that $t \mapsto A_k(f_t)$ is 2-exponentially convex in J−sense i.e. for any $t_1, t_2 \in I,\ u_1, u_2 \in \mathbb{R}$

$$u_1^2A_k(f_{t_1}) + 2u_1u_2A_k(f_{\frac{t_1+t_2}{2}}) + u_2^2A_k(f_{t_2}) \geq 0.$$

If we consider the left-hand side as a nonnegative quadratic polinomial, then its discriminant is nonpositive, i.e.

$$[A_k(f_{\frac{t_1+t_2}{2}})]^2 - A_k(f_{t_1}) \cdot A_k(f_{t_2}) \leq 0.$$

This means that $t \mapsto A_k(f_t)$ is log-convex in J−sense. From continuity we conclude that $t \mapsto A_k(f_t)$ is log-convex. Using the Jensen inequality for convex combination $s = \frac{t-s}{t-r}r + \frac{s-r}{t-r}t$ we get

$$\log A_k(f_s) \leq \frac{t-s}{t-r}\log A_k(f_r) + \frac{s-r}{t-r}\log A_k(f_t)$$

$$\log[A_k(f_s)]^{t-r} \leq \log[A_k(f_r)]^{t-s} + \log[A_k(f_t)]^{s-r},$$

which gives (50).

(b) For a convex function φ, the inequality

$$\frac{\varphi(s) - \varphi(t)}{s - t} \leq \frac{\varphi(u) - \varphi(v)}{u - v} \tag{54}$$

holds for all $s, t, u, v \in I$ such that $s \leq u,\ t \leq v,\ s \neq t,\ u \neq v$.
Since by (a), $A_k(f_t)$ is log-convex, so setting $\varphi(t) = \log A_k(f_t)$ in (54) we have

$$\frac{\log A_k(f_s) - \log A_k(f_t)}{s-t} \leq \frac{\log A_k(f_u) - \log A_k(f_v)}{u-v}, \tag{55}$$

for $s \leq u,\ t \leq v,\ s \neq t,\ u \neq v$, which is equivalent to (51) i.e. to

$$\left(\frac{A_k(f_s)}{A_k(f_t)}\right)^{\frac{1}{s-t}} \leq \left(\frac{A_k(f_u)}{A_k(f_v)}\right)^{\frac{1}{u-v}}. \tag{56}$$

The cases for $s = t$ and / or $u = v$ follow from taking respective limits.

Putting in (56) $t = v = r,\ s = r + r_1 + \ldots + r_l,\ u = r + r_i$ we get

$$\left(\frac{A_k(f_{r+r_1+\ldots+r_l})}{A_k(f_r)}\right)^{\frac{1}{r_1+\ldots+r_l}} \leq \left(\frac{A_k(f_{r+r_i})}{A_k(f_r)}\right)^{\frac{1}{r_i}}$$

$$\left(\frac{A_k(f_{r+r_1+\ldots+r_l})}{A_k(f_r)}\right)^{\frac{r_i}{r_1+\ldots+r_l}} \leq \frac{A_k(f_{r+r_i})}{A_k(f_r)}.$$

Multiplying all the inequalities for $i = 1, 2, \ldots, l$ we get (53). $\square$

Let us consider some examples:

Example 6.10. Let $F_1 = \{\psi_t : [a, b] \subset \mathbb{R} \to [0, \infty) : t \in \mathbb{R}\}$ be a family of functions defined by

$$\psi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & \text{t} \neq 0, \\ \frac{x^n}{n!}, & \text{t} = 0. \end{cases}$$

Since $\frac{d^n}{dx^n}\psi_t(x) = e^{tx}$, the function $t \mapsto \frac{d^n}{dx^n}\psi_t(x)$ is exponentially convex (see [1]). Using Theorem 6.7, we have that $t \mapsto A_k(\psi_t),\ k \in \{1, \ldots, 8\}$ are exponentially convex.

Assume that $t \mapsto A_k(\psi_t) > 0$ for $k \in \{1, \ldots, 8\}$. By introducing convex functions ψ_t in (49), we obtain the following means:

$$\mathfrak{M}_{s,t}(A_k, F_1) = \begin{cases} \frac{1}{s-t} \log\left(\frac{A_k(\psi_s)}{A_k(\psi_t)}\right) & , \quad s \neq t, \\ \frac{A_k(id \cdot \psi_s)}{A_k(\psi_s)} - \frac{n}{s} & , \quad s = t \neq 0, \\ \frac{A_k(id \cdot \psi_0)}{(n+1)A_k(\psi_0)} & , \quad s = t = 0. \end{cases}$$

where id stands for identity function on $[a, b] \subset \mathbb{R}$. In particular for $k = 1$ we have

$$\mathfrak{M}_{s,t}(A_1, F_1) = \frac{1}{s-t} \log\left(\frac{t^n}{s^n} \frac{\sum_{i=1}^m p_i e^{sx_i} - \sum_{k=0}^{n-1} \frac{s^k e^{sa}}{k!} \sum_{i=1}^m p_i(x_i-a)^k}{\sum_{i=1}^m p_i e^{tx_i} - \sum_{k=0}^{n-1} \frac{t^k e^{ta}}{k!} \sum_{i=1}^m p_i(x_i-a)^k}\right), \quad s \neq t; s, t \neq 0;$$

$$\mathfrak{M}_{s,s}(A_1, F_1) = \frac{\sum_{i=1}^m p_i x_i e^{sx_i} - \sum_{k=0}^{n-1} \frac{(ks^{k-1}+as^k)e^{sa}}{k!} \sum_{i=1}^m p_i(x_i-a)^k}{\sum_{i=1}^m p_i e^{sx_i} - \sum_{k=0}^{n-1} \frac{s^k e^{sa}}{k!} \sum_{i=1}^m p_i(x_i-a)^k} - \frac{n}{s}, \quad s \neq 0;$$

$$\mathfrak{M}_{0,0}(A_1, F_1) = \frac{\sum_{i=1}^m p_i \frac{x_i^{n+1}}{n!} - \sum_{k=0}^{n-1} \frac{(n+1)a^{n-k+1}}{(n-k+1)!k!} \sum_{i=1}^m p_i(x_i-a)^k}{(n+1)\left(\sum_{i=1}^m p_i \frac{x_i^n}{n!} - \sum_{k=0}^{n-1} \frac{a^{n-k}}{(n-k)!k!} \sum_{i=1}^m p_i(x_i-a)^k\right)}.$$

Here $\mathfrak{M}_{s,t}(A_k, F_1) = \log(\mu_{s,t}(A_k, F_1))$, $k \in \{1, \ldots, 8\}$ are in fact means.

Remark 6.11. We observe here that $\left(\frac{\frac{d^n \psi_s}{dx^n}}{\frac{d^n \psi_t}{dx^n}}\right)^{\frac{1}{s-t}} (\log \xi) = \xi$ is a mean for $\xi \in [a, b]$ where $a, b \in \mathbb{R}_+$.

Example 6.12. Let $n \in \mathbb{N}$, $F_2 = \{\varphi_t : [0, \infty) \to \mathbb{R} : t \in \mathbb{R}, t > n\}$ be a family of functions defined as

$$\varphi_t(x) = \frac{x^t}{t(t-1) \cdot \ldots \cdot (t-n+1)}.$$

Since $t \mapsto \frac{d^n}{dx^n}\varphi_t(x) = x^{t-n} = e^{(t-n)\log x}$ is exponentially convex, by Theorem 6.7 we conclude that $t \mapsto A_k(\varphi_t),\ k \in \{1, \ldots, 8\}$ are exponentially convex.

We assume that $A_k(\varphi_t) > 0$ for $k \in \{1, \ldots, 8\}$. For this family of convex functions we obtain the following means:

$$\mathfrak{M}_{s,t}(A_k, F_2) = \begin{cases} \left(\frac{A_k(\varphi_s)}{A_k(\varphi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left((-1)^{n-1}(n-1)! \frac{A_k(\varphi_0 \varphi_s)}{A_k(\varphi_s)} + \sum_{k=0}^{n-1} \frac{1}{k-s}\right), & s = t. \end{cases}$$

In particular for $k = 1$ we have

$$\mathfrak{M}_{s,t}(A_1, F_2) = \left(\frac{t(t-1)\cdots(t-n+1)}{s(s-1)\cdots(s-n+1)} \frac{\sum_{i=1}^m p_i x_i^s}{\sum_{i=1}^m p_i x_i^t} \right)^{\frac{1}{s-t}}, \; s \neq t$$
$$\mathfrak{M}_{s,s}(A_1, F_2) = \exp\left(\frac{\sum_{i=1}^m p_i {x_i}^s \log x_i}{\sum_{i=1}^m p_i \, {x_i}^s} + \sum_{k=0}^{n-1} \frac{1}{k-s} \right).$$

For other examples see paper [1].

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1 - University of Karachi, Faculty of Science, Department of Mathematical Sciences, University Road, Karachi-75270, Pakistan
E-mail address: asifrk@uok.edu.pk

2 - Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan

3- Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28, 10000 Zagreb, Croatia
E-mail address: pecaric@hazu.hr

4 - Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia
E-mail address: mpraljak@pbf.hr

5 - Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia
E-mail address: varosans@math.hr