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GENERALIZATIONS OF SHERMAN'S THEOREM BY MONTGOMERY IDENTITY AND NEW GREEN FUNCTIONS

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Abstract. In this paper, we give generalization of Sherman inequality by using Green functions and Montgomery identity. We present Grüss and Ostrowski-type inequalities related to generalized Sherman inequality. We give mean value theorems and n-exponential convexity for the functional associated to generalized inequality. We also give a family of functions which support our results for exponentially convex functions and construct a class of means.

1. Introduction

We start with the concept of majorization which is exactly a partial ordering of vectors and determines the degree of similarity between the vector elements.
For fixed $m \geq 2$, let $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ denote two m-tuples. Let $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[m]}$ and $y_{[1]} \geq y_{[2]} \geq ... \geq y_{[m]}$ be their ordered components. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ or $\mathbf{y}$ is *majorized* by $\mathbf{x}$ and write $\mathbf{y} \prec \mathbf{x}$ if

$$\sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]}, \quad k = 1, ..., m-1, \text{ and } \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} x_i. \tag{1}$$

A notation from real vector space may be extended to real matrices. Let

$\mathcal{M}_{ml}(\mathbb{R})$ denotes the space of $m \times l$ real matrices. A matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ is called *row stochastic* if all of its entries are greater or equal to zero and the sum of the entries in each row is equal to 1. A square matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ is called *double stochastic* if all of its entries are greater or equal to zero and the sum of the entries in each column and each row is equal to 1.
The *Majorization theorem* due to Hardy et al (1929 [12]), gives connections with matrix theory (see also [17, p. 333]). For more detail see [3], [4], [5], [6] and [14].

Theorem 1.1. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Then the following statements are equivalent.*

(i) $\mathbf{y} \prec \mathbf{x}$*;*

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(ii) *There is a doubly stochastic matrix* $\mathbf{A}$ *such that* $\mathbf{y} = \mathbf{xA}$*;*

(iii) *The inequality* $\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i)$ *holds for each convex function* $\phi : \mathbb{R} \to \mathbb{R}$.

S. Sherman ([15], [20]) obtained the following general result.

Theorem 1.2 (Sherman's theorem)**.** *Let* $[\alpha, \beta] \subset \mathbb{R}$ *and for fixed* $l, m \in \mathbb{N}$ $l, m \geq 2$*, let* $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{u} \in [0, \infty)^l$, $\mathbf{v} \in [0, \infty)^m$ *and*

$$\mathbf{y} = \mathbf{x}\mathbf{A}^T \text{ and } \mathbf{u} = \mathbf{v}\mathbf{A} \tag{2}$$

for some row stochastic matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$*. Then for every convex function* $\phi : [\alpha, \beta] \to \mathbb{R}$ *we have*

$$\sum_{q=1}^{m} v_q \phi(y_q) \leq \sum_{p=1}^{l} u_p \phi(x_p). \tag{3}$$

Sherman obtained this useful generalization replacing the classical concept of majorization $\mathbf{y} \prec \mathbf{x}$ by the notion of weighted majorization (2) for two pairs $(\mathbf{x}, \mathbf{u})$ and $(\mathbf{y}, \mathbf{v})$, where $\mathbf{x} = (x_1, ..., x_l)$ and $\mathbf{y} = (y_1, ..., y_m)$ are real vectors and $\mathbf{u} = (u_1, ..., u_l)$ and $\mathbf{v} = (v_1, ..., v_m)$ are corresponding nonnegative weights. Here $\mathbf{A^T}$ denotes the transpose of of a matrix $\mathbf{A}$. In particular for $m = l$ and $u_p = v_q$ for $p, q = 1, ..., m$, the condition $\mathbf{u} = \mathbf{vA}$ assure the stochasticity on columns, so in that case we deal with doubly stochastic matrices. Then, as a special case of Sherman's inequality, we get the weighted version of majorization's inequality:

$$\sum_{p=1}^{m} u_p \phi(y_p) \leq \sum_{p=1}^{m} u_p \phi(x_p).$$

Denoting $U_m = \sum_{p=1}^{m} u_p$ and putting $y_1 = y_2 = ... = y_m = \frac{1}{U_m} \sum_{p=1}^{m} u_p x_p$, we obtain Jensen's inequality in the form

$$\phi\left(\frac{1}{U_m} \sum_{p=1}^{m} u_p x_p\right) \leq \frac{1}{U_m} \sum_{p=1}^{m} u_p \phi(x_p)$$

Now we recall the definition of $n-$convex function which we will use in the rest of paper.

Definition 1. *The divided difference of order* n*,* $n \in \mathbb{N}$*, of the function* $\phi : [\alpha, \beta] \to \mathbb{R}$ *at mutually different points* $x_0, x_1, ..., x_n \in [\alpha, \beta]$ *is defined recursively by*

$$[x_i; \phi] = \phi(x_i), \quad i = 0, ..., n$$

$$[x_0, ..., x_n; \phi] = \frac{[x_1, ..., x_n; \phi] - [x_0, ..., x_{n-1}; \phi]}{x_n - x_0}.$$

The value $[x_0, ..., x_n; \phi]$ *is independent of the order of the points* $x_0, ..., x_n$*.*

This definition may be extended to include the case in which some or all the points coincide. Assuming that $\phi^{(j-1)}(x)$ exists, we define

$$[\underbrace{x, ..., x}_{j\text{-times}}; \phi] = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \tag{4}$$

The notion of $n-$convexity was defined in terms of divided differences by Popoviciu [18]. A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is $n-$convex, $n \geq 0$, if its nth order divided differences $[x_0, x_1, ..., x_n; \phi]$ are nonnegative for all choices of $(n+1)$ distinct points $x_i \in [\alpha, \beta]$, $i = 0, ..., n$. Thus, a 0-convex functions is nonnegative, a 1-convex functions is nondecreasing and 2-convex functions is convex in the usual sense. If $\phi^{(n)}$ exists then ϕ is $n-$convex if and only if $\phi^{(n)} \geq 0$ (see [17]).

In our main results of this paper, we will use the following generalized Montgomery identity.

Theorem 1.3 ([8]). *Let $n \in \mathbb{N}$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$ and $\alpha < \beta$. Then the following identity holds*

$$\phi(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(t)\,dt + \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)} \frac{(x-\alpha)^{k+2}}{\beta-\alpha} -$$

$$\sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)} \frac{(x-\beta)^{k+2}}{\beta-\alpha} + \frac{1}{(n-1)!}\int_\alpha^\beta T_n(x,s)\,\phi^{(n)}(s)\,ds, \tag{5}$$

where

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha}(x-s)^{n-1}, & \alpha \leq s \leq x, \\ \\ -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha}(x-s)^{n-1}, & x < s \leq \beta. \end{cases} \tag{6}$$

In case $n = 1$ the sum $\sum_{k=0}^{n-2} ...$ is empty, so the identity (5) reduces to the well-known Montgomery identity

$$\phi(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(t)\,dt + \int_\alpha^\beta P(x,s)\,\phi'(s)\,ds,$$

where $P(x,s)$ is the Peano kernel, defined by

$$P(x,s) = \begin{cases} \frac{s-\alpha}{\beta-\alpha}, & \alpha \leq s \leq x, \\ \\ \frac{s-\beta}{\beta-\alpha}, & x < s \leq \beta. \end{cases}$$

2. Main results

In this paper we will use the following Green functions defined on $[\alpha, \beta] \times [\alpha, \beta]$ by

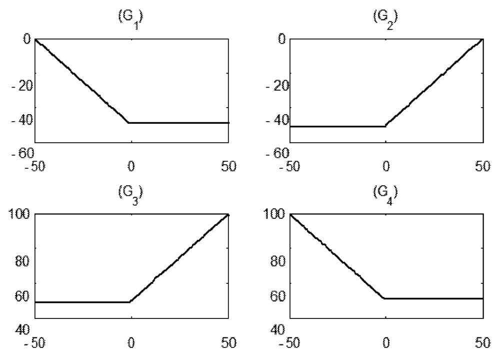
$$G_1(t,s) = \begin{cases} \alpha - s\ , & \alpha \leq s \leq t; \\ \alpha - t, & t \leq s \leq \beta. \end{cases} \tag{7}$$

$$G_2(t,s)=\begin{cases} t-\beta\ , & \alpha \le s \le t; \\ s-\beta, & t \le s \le \beta. \end{cases} \tag{8}$$

$$G_3(t,s)=\begin{cases} t-\alpha\ , & \alpha \le s \le t; \\ s-\alpha, & t \le s \le \beta. \end{cases} \tag{9}$$

$$G_4(t,s)=\begin{cases} \beta-s\ , & \alpha \le s \le t; \\ \beta-t, & t \le s \le \beta. \end{cases} \tag{10}$$

All these four functions are continuous convex with respect to t. The graph of these functions for fix "s" are given below.



Lemma 2.1. *For every function $\phi \in C^2([\alpha,\beta])$, the following identities are valid*

$$\phi(t)=\phi(\alpha)+(t-\alpha)\phi'(\beta)+\int_\alpha^\beta G_1(t,s)\phi''(s)ds, \tag{11}$$

$$\phi(t)=\phi(\beta)+(t-\beta)\phi'(\alpha)+\int_\alpha^\beta G_2(t,s)\phi''(s)ds, \tag{12}$$

$$\phi(t)=\phi(\beta)+(t-\alpha)\phi'(\alpha)-(\beta-\alpha)\phi'(\beta)+\int_\alpha^\beta G_3(t,s)\phi''(s)ds, \tag{13}$$

$$\phi(t)=\phi(\alpha)-(\beta-t)\phi'(\beta)+(\beta-\alpha)\phi'(\alpha)+\int_\alpha^\beta G_4(t,s)\phi''(s)ds, \tag{14}$$

where the functions G_w, $w \in \{1,2,3,4\}$ are defined as above in (7), (8), (9) and (10) respectively.

Proof. Using the techniques of integration we have

$$\begin{aligned}
\int_\alpha^\beta G_3(t,s)\phi''(s)ds &= \int_\alpha^t G_3(t,s)\phi''(s)ds + \int_t^\beta G_3(t,s)\phi''(s)ds \\
&= \int_\alpha^t (t-\alpha)\phi''(s)ds + \int_t^\beta (s-\alpha)\phi''(s)ds \\
&= (t-\alpha)\phi'(s)\big|_\alpha^t + (s-\alpha)\phi'(s)\big|_t^\beta - \int_t^\beta \phi'(s)ds \\
\Rightarrow \int_\alpha^\beta G_3(t,s)\phi''(s)ds &= (t-\alpha)\phi'(t) - (t-\alpha)\phi'(\alpha) + (\beta-\alpha)\,\phi'(\beta) \\
&\quad - (t-\alpha)\phi'(t) - \phi(\beta) + \phi(t),
\end{aligned}$$

which is equivalent to (13).

Analogously, we can prove other three identities. □

We sate our first main result in the following theorem.

Theorem 2.2. *Let $[\alpha,\beta] \subset \mathbb{R}$ and for fixed $l,m \in \mathbb{N}$, $l,m \geq 2$, let $\mathbf{x} = (x_1,...,x_l) \in [\alpha,\beta]^l$, $\mathbf{y} = (y_1,...,y_m) \in [\alpha,\beta]^m$, $\mathbf{u} = (u_1,...,u_l) \in \mathbb{R}^l$, $\mathbf{v} = (v_1,...,v_m) \in \mathbb{R}^m$ be such that (2) holds for some matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ satisfying the condition $\sum_{j=1}^{l} a_{ij} = 1$, $i = 1,2,...,m$. Then the following statements are equivalent:*

(i) *For every continuous convex function $\phi : [\alpha,\beta] \to \mathbb{R}$, we have*

$$\sum_{q=1}^{m} v_q\phi(y_q) \leq \sum_{p=1}^{l} u_p\phi(x_p). \tag{15}$$

(ii) *For all $s \in [\alpha,\beta]$, we have*

$$\sum_{q=1}^{m} v_q G_w(y_q,s) \leq \sum_{p=1}^{l} u_p G_w(x_p,s), \quad \textit{where } w = 1,2,3,4. \tag{16}$$

Proof. We give the proof only for $w = 3$, for other they are similar.

(i) ⇒ (ii): Let (i) holds. Let us consider the Green function G_3 defined by (9). Since the function $G_3(.,s)$, $s \in [\alpha,\beta]$, is continuous and convex on $[\alpha,\beta]$, therefore (15) holds for $G_3(.,s)$.

(ii) ⇒ (i): Let (ii) holds. Since every function $\phi : [\alpha,\beta] \to \mathbb{R}$, $\phi \in C^2([\alpha,\beta])$, can be written in the form (13). Therefore by some simple calculations, we deduce

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G_3(x_p,s) - \sum_{q=1}^{m} v_q G_3(y_q,s)\right] \phi''(s)ds. \tag{17}$$

Since ϕ is convex, therefore $\phi''(s) \geq 0$ for $s \in [\alpha,\beta]$. Furthermore, if for every $s \in [\alpha,\beta]$ the inequality (16) holds, then we have the right hand side of (17) is non negative and hence (15) holds. □

In the following theorem we give general identities for Sherman's inequality.

Theorem 2.3. *Let $n \in \mathbb{N}, n \geq 4$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$, $\alpha < \beta$. Suppose that $\mathbf{x} = (x_1, ..., x_l) \in [\alpha, \beta]^l$, $\mathbf{y} = (y_1, ..., y_m) \in [\alpha, \beta]^m$, $\mathbf{u} = (u_1, ..., u_l) \in \mathbb{R}^l$, $\mathbf{v} = (v_1, ..., v_m) \in \mathbb{R}^m$ be such that (2) holds for some matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ satisfying the condition $\sum\limits_{j=1}^{l} a_{ij} = 1$, $i = 1, 2, ..., m$. Let $G_w, w = 1, 2, 3, 4$, and T_n be as defined in (7), (8), (9), (10) and (6) respectively. Then we have the following identities.*

(i)

$$\begin{aligned}\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) &= \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t)\right] \\ &\times \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha}\right) dt \\ &+ \frac{1}{(n-3)!} \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t)\right] \tilde{T}_{n-2}(t, s)\, \phi^{(n)}(s)\, ds dt,\end{aligned} \tag{18}$$

where

$$\tilde{T}_{n-2}(t, s) = \begin{cases} \frac{1}{\beta-\alpha}\left[\frac{(t-s)^{n-2}}{n-2} + (t-\alpha)(t-s)^{n-3}\right], & \alpha \leq s \leq t, \\ \frac{1}{\beta-\alpha}\left[\frac{(t-s)^{n-2}}{n-2} + (t-\beta)(t-s)^{n-3}\right], & t < s \leq \beta. \end{cases} \tag{19}$$

(ii)

$$\begin{aligned}\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) &= \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha}\left[\frac{\sum_{p=1}^{l} u_p x_p^2 - \sum_{q=1}^{m} v_q y_q^2}{2}\right] \\ &+ \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t)\right] \times \\ &\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha}\right) dt \\ &+ \frac{1}{(n-3)!} \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t)\right] T_{n-2}(t, s)\, \phi^{(n)}(s)\, ds dt.\end{aligned} \tag{20}$$

Proof. Fix $w = 1,2,3,4$. Using (11), (12), (13) and (14) in $\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q)$ and applying (2), we obtain

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t)\right]\phi''(t)dt, \tag{21}$$

(i) Differentiating (5) twice with respect to t and rearranging the terms, we get

$$\phi''(t) = \sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)$$
$$+\frac{1}{(n-3)!}\int_\alpha^\beta \tilde{T}_{n-2}(t,s)\,\phi^{(n)}(s)\,ds. \tag{22}$$

Substituting (22) in (21) we obtain (18).

(ii) Replacing ϕ by ϕ'' and then n by $n-2$ in (5), we have

$$\phi''(t) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi''(t)\,dt + \sum_{k=0}^{n-4}\frac{\phi^{(k+3)}(\alpha)}{k!\,(k+2)}\frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4}\frac{\phi^{(k+3)}(\beta)}{k!\,(k+2)}\frac{(t-\beta)^{k+2}}{\beta-\alpha}$$
$$+\frac{1}{(n-3)!}\int_\alpha^\beta T_{n-2}(t,s)\,\phi^{(n)}(s)\,ds,$$

this implies that

$$\phi''(t) = \frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha} + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)$$
$$+\frac{1}{(n-3)!}\int_\alpha^\beta T_{n-2}(t,s)\,\phi^{(n)}(s)\,ds. \tag{23}$$

Using (23) in (21), we get (20).

□

Now we present the generalization of the Sherman theorem by using the above obtained identities.

Theorem 2.4. *Suppose that all the assumptions of Theorem 2.3 hold. Let for any even n the function $\phi : I \to \mathbb{R}$ is n-convex and*

$$\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t) \geq 0, \quad \text{for } w = 1,2,3,4. \tag{24}$$

Then the following inequalities hold:

(i)

$$\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \geq \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t) \right] \times$$

$$\sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt. \tag{25}$$

(ii)

$$\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \geq \frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha} \left[\frac{\sum_{p=1}^{l} u_p x_p^2 - \sum_{q=1}^{m} v_q y_q^2}{2} \right]$$

$$+ \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t) \right] \times$$

$$\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt. \tag{26}$$

Proof. (i) Since the function ϕ is n-convex so we have $\phi^{(n)} \geq 0$. Also it is obvious that if n is even then $\tilde{T}_{n-2} \geq 0$ because

Case I: If $\alpha \leq s \leq t$, then $t-s \geq 0$ and hence $\frac{(t-s)^{n-2}}{n-2} \geq 0$. Also $(t-\alpha) \geq 0$ and $(t-s)^{n-3} \geq 0$. So in this case from (19) we have $\tilde{T}_{n-2} \geq 0$.

Case II: If $t < s \leq \beta$, then $(t-s)^{n-3}$ and $(s-\beta)$ are non positive. As n is even so we have $(s-\beta)(t-s)^{n-3} \geq 0$, also $\frac{(t-s)^{n-2}}{n-2} \geq 0$. So in this case from (19) we have $\tilde{T}_{n-2} \geq 0$.

Now using (24) and the positivity of $\tilde{T}_{n-2}$ and $\phi^{(n)}$ in (18) we get (26).

(ii) The proof is similar to the proof of part (i).

□

In the following theorem we prove generalization of Sherman's theorem for positive weights.

Theorem 2.5. *Let $n \in \mathbb{N}, n \geq 4$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$, $\alpha < \beta$. Let $\mathbf{x} = (x_1, x_2, ..., x_l) \in [\alpha,\beta]^l$, $\mathbf{y} = (y_1, y_2, ..., y_m) \in [\alpha,\beta]^m$, $\mathbf{u} = (u_1, u_2, ..., u_l) \in [0,\infty]^l$ and $\mathbf{v} = (v_1, v_2, ..., v_m) \in [0,\infty]^m$ be such that (2) holds for some row stochastic matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$. If n is even and ϕ is n-convex function, then (25) and (26) hold. Moreover, if (25) and (26) hold and the* *functions define by*

$$L_1(.) = \int_\alpha^\beta G_w(.,t) \times \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt, \tag{27}$$

$$\begin{aligned} L_2(.) = {} & \frac{\phi'(\beta) - \phi'(\alpha)}{\beta-\alpha} \frac{(.)^2}{2} + \int_\alpha^\beta G_w(.,t) \times \\ & \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt, \quad \textit{where}, \ w = 1,2,3,4, \end{aligned} \tag{28}$$

are convex on $[\alpha, \beta]$*, then (3) holds.*

Proof. Since the function $G_w(.,t)$, $w \in \{1,2,3,4\}$, $t \in [\alpha,\beta]$, are convex, so by Sherman's theorem it holds that

$$\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t) \geq 0, \quad t \in [\alpha, \beta].$$

Applying Theorem 2.4 we obtain (25) and (26).

Since (25) holds, the right hand side of (25) can be rewritten in the form

$$\sum_{p=1}^{l} u_p L_1(x_p) - \sum_{q=1}^{m} v_q L_1(y_q),$$

where L_1 is defined by (27). Since L_1 is convex, therefore by Sherman's theorem we have

$$\sum_{p=1}^{l} u_p L_1(x_p) - \sum_{q=1}^{m} v_q L_1(y_q) \geq 0,$$

i.e. the right hand side of (25) is nonnegative, so the inequality (3) immediately follows.
Similarly we may get (3) by using the convexity of L_2. □

3. Grüss and Ostrowski type inequalities related to generalized Sherman's inequality

P. Cerone and S. S. Dragomir [10], considered Čebyšev functional

$$T(f,g) := \frac{1}{\beta-\alpha} \int_\alpha^\beta f(t)g(t)dt - \frac{1}{\beta-\alpha} \int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha} \int_\alpha^\beta g(t)dt$$

for Lebesgue integrable functions $f, g : [\alpha, \beta] \to \mathbb{R}$, proved the following two results which contain the Grüss and Ostrowski type inequalities [2].

Theorem 3.1. *Let $f : [\alpha, \beta] \to \mathbb{R}$ be Lebesgue integrable function and $g : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(g')^2 \in L[\alpha, \beta]$. Then*

$$|T(f,g)| \leq \frac{1}{\sqrt{2}} |T(f,f)|^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left(\int_\alpha^\beta (x-\alpha)(\beta - x)[g'(x)]^2 dx \right)^{\frac{1}{2}}. \tag{29}$$

The constant $\frac{1}{\sqrt{2}}$ in (29) is the best possible.

Theorem 3.2. *Let $g : [\alpha, \beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then*

$$|T(f,g)| \leq \frac{1}{2(\beta - \alpha)} \left\| f' \right\|_\infty \int_\alpha^\beta (x - \alpha)(\beta - x) dg(x). \tag{30}$$

The constant $\frac{1}{2}$ in (30) is the best possible.

Using previous two theorems we obtain upper bounds for the identities related to generalizations of Sherman's inequality.

To avoid many notations, under the assumptions of Theorem 2.3, we define functions P_1 and P_2 from $[\alpha, \beta]$ to $\mathbb{R}$ by

$$P_{1,w}(s) = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t) \right] \tilde{T}_{n-2}(t, s) dt, \quad \text{for, } w = 1,2,3,4. \tag{31}$$

$$P_{2,w}(s) = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t) \right] T_{n-2}(t, s) dt, \quad \text{for, } w = 1,2,3,4. \tag{32}$$

Theorem 3.3. *Let $n \in \mathbb{N}, n \geq 4$, $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L[\alpha, \beta]$, $P_{1,w}$, $P_{2,w}$, $w = 1,2,3,4$, be defined as in (31), (32) respectively. Then*

(i) *the remainders $\kappa^1(\phi; \alpha, \beta)$ define by*

$$\begin{aligned} \kappa^1(\phi; \alpha, \beta) = & \sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) - \\ & \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t) \right] \times \\ & \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt \\ & - \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!(\beta - \alpha)} \int_\alpha^\beta P_{1,w}(s) ds, \end{aligned} \tag{33}$$

satisfies the estimation

(34)
$$\left|\kappa^1(\phi;\alpha,\beta)\right| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}\left|T(P_{1,w},P_{1,w})\right|^{\frac{1}{2}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2ds\right)^{\frac{1}{2}}.$$

(ii) *The remainder* $\kappa^2(\phi;\alpha,\beta)$ *define by*

$$\kappa^2(\phi;\alpha,\beta) = \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) - \frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha}\left[\frac{\sum_{p=1}^{l} u_p x_p^2 - \sum_{q=1}^{m} v_q y_q^2}{2}\right]$$
$$-\int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t)\right]\times$$
$$\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1}-\phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt$$

(35)
$$-\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(n-3)!(\beta-\alpha)}\int_\alpha^\beta P_{2,w}(s)ds,$$

satisfies the estimation

(36)
$$\left|\kappa^2(\phi;\alpha,\beta)\right| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}\left|T(P_{2,w},P_{2,w})\right|^{\frac{1}{2}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2ds\right)^{\frac{1}{2}}.$$

Proof. (i) Comparing (18) and (33) we get

(37)
$$\kappa^1(\phi;\alpha,\beta) = \frac{1}{(n-3)!}\int_\alpha^\beta P_{1,w}(s)\phi^{(n)}(s)ds - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(n-3)!\,(\beta-\alpha)}\int_\alpha^\beta P_{1,w}(s)ds.$$

Applying Theorem 3.1 for $f \to P_{1,w}$, $g \to \phi^{(n)}$ and using Čebyšev functional, we get

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta P_{1,w}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta P_{1,w}(s)ds\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(s)ds\right|$$

(38)
$$\leq \frac{1}{\sqrt{2}}\left|T(P_{1,w},P_{1,w})\right|^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2ds\right)^{\frac{1}{2}}.$$

Therefore from (37) and (38) we get (34).

(ii) Proceeding similarly as in part (i) we obtain (36).

□

Theorem 3.4. *Let* $n \in \mathbb{N}, n \geq 4$, $\phi : [\alpha,\beta] \to \mathbb{R}$ *be such that* $\phi^{(n)}$ *is monotonic nondecreasing on* $[\alpha,\beta]$ *and* $P_{1,w}$, $P_{2,w}$, $w = 1,2,3,4$, *be defined as in (31) and (32) respectively. Then*

(i) *The remainder $\kappa^1(\phi;\alpha,\beta)$ defined by (33) satisfies the estimation*

(39)
$$\left|\kappa^1(\phi;\alpha,\beta)\right| \leq \frac{\|P'_{1,w}\|_\infty}{(n-3)!}\left[\frac{(\beta-\alpha)\left(\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right)}{2} - \left\{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right\}\right].$$

(ii) *The remainder $\kappa^2(\phi;\alpha,\beta)$ defined by (35) satisfies the estimation*

(40)
$$\left|\kappa^2(\phi;\alpha,\beta)\right| \leq \frac{\|P'_{2,w}\|_\infty}{(n-3)!}\left[\frac{(\beta-\alpha)\left(\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right)}{2} - \left\{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right\}\right].$$

Proof. (i) Since

(41)
$$\kappa^1(\phi;\alpha,\beta) = \frac{1}{(n-3)!}\int_\alpha^\beta P_{1,w}(s)\phi^{(n)}(s)ds - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(n-3)!\,(\beta-\alpha)}\int_\alpha^\beta P_{1,w}(s)ds.$$

Applying Theorem 3.2 for $f \to P_{1,w}$, $g \to \phi^{(n)}$ and using Čebyšev functional, we get

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta P_{1,w}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta P_{1,w}(s)ds\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(s)ds\right|$$

(42)
$$\leq \frac{1}{2(\beta-\alpha)}\left\|P'_{1,w}\right\|_\infty \int_\alpha^\beta (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds.$$

Since

$$\int_\alpha^\beta (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds = \int_\alpha^\beta \left[2s-(\alpha+\beta)\right]\phi^{(n)}(s)ds$$
$$= (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left[\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right].$$

Therefore, from (41) and (42), we deduce (39).

(i) Similarly we can prove (40). □

Here, the symbol $L_p[a,b]$ $(1 \leq p < \infty)$ denotes the space of $p-$power integrable functions defined on the interval $[a,b]$ equipped with the norm

$$\| \phi \|_p = \left(\int_a^b |\phi(t)|^p dt\right)^{\frac{1}{p}} \text{ for all } \phi \in L_p[a,b],$$

and space of essentially bounded functions on $[a,b]$, denoted by $L_\infty[a,b]$, with the norm

$$\| \phi \|_\infty = ess \sup_{t\in[a,b]} |\phi(t)|.$$

In the following theorem we present Ostrowski type inequality related to generalizations of Sherman's inequality.

Theorem 3.5. *Let $n \in \mathbb{N}, n \geq 4$, (p,q) be a pair of conjugate exponents, i.e. $1 \leq p,q \leq \infty$ and $1/p + 1/q = 1$. Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\left|\phi^{(n)}\right|^p \in L[\alpha,\beta]$. Let $P_{1,w}$ and $P_{2,w}$, $w = 1,2,3,4$, be defined as in (31), (32) respectively. Then the following inequalities hold.*

(i)

$$\left| \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) - \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t) \right] \times \right.$$
$$\left. \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt \right|$$
$$\leq \frac{1}{(n-3)!} \left\| \phi^{(n)} \right\|_p \left\| P_{1,w} \right\|_q .$$

The constant $\|P_{1,w}\|_q$ *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

(ii)

$$\left| \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) - \frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha} \left[\frac{\sum_{p=1}^{l} u_p x_p^2 - \sum_{q=1}^{m} v_q y_q^2}{2} \right] \right.$$
$$- \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t) \right] \times$$
$$\left. \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt \right|$$
$$\leq \frac{1}{(n-3)!} \left\| \phi^{(n)} \right\|_p \left\| P_{2,w} \right\|_q$$

The constant $\|P_{2,w}\|_q$ *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. The proof is similar to the proof of Theorem 12 in [1]. □

4. Mean value theorems and Exponential convexity with applications

Motivated by the inequalities (25) and (26) under the assumptions of Theorem 2.4, we define the linear functionals $\Lambda_{1,w}$ and $\Lambda_{2,w}$, $w = 1, 2, 3, 4$, from $C^n([\alpha, \beta])$ to $\mathbb{R}$ by

$$\Lambda_{1,w}(\phi) = \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) -$$
$$\int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t) \right] \times$$
$$\sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt. \tag{43}$$

$$\Lambda_{2,w}(\phi) = \sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q)$$
$$- \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \left[\frac{\sum_{p=1}^{l} u_p x_p^2 - \sum_{q=1}^{m} v_q y_q^2}{2} \right]$$
$$- \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G_w(x_p, t) - \sum_{q=1}^{m} v_q G_w(y_q, t) \right] \times$$
$$\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt. \tag{44}$$

Using the linearity of these functionals we derive mean-value theorems of Lagrange and Cauchy type.

Theorem 4.1. *Let $\phi \in C^n([\alpha, \beta])$ and $\Lambda_{\rho,w} : C^n([\alpha, \beta]) \to \mathbb{R}$, $\rho \in \{1, 2\}$, be the linear functionals defined by (43) and (44). Then there exists $\xi_{\rho,w} \in [\alpha, \beta]$, $w = 1, 2, 3, 4, \rho = 1, 2$, such that*

$$\Lambda_\rho(\phi) = \phi^{(n)}(\xi_{\rho,w}) \Lambda_{\rho,w}(\varphi),$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Similar to the proof of Theorem 4.1 in [13]. □

Theorem 4.2. *Let $\phi, \psi \in C^n([\alpha, \beta])$ and $\Lambda_{\rho,w} : C^n([\alpha, \beta]) \to \mathbb{R}$, $w = 1, 2, 3, 4, \rho = 1, 2$, be the linear functionals defined by (43) and (44). Then there exists $\xi_{\rho,w} \in [\alpha, \beta]$, $w = 1, 2, 3, 4, \rho = 1, 2$, such that*

$$\frac{\Lambda_{\rho,w}(\phi)}{\Lambda_{\rho,w}(\psi)} = \frac{\phi^{(n)}(\xi_{\rho,w})}{\psi^{(n)}(\xi_{\rho,w})},,$$

provided that the denominators are non-zero.

Proof. Similar to the proof of Corollary 4.2 in [13]. □

Remark 1. *If the function $\frac{\phi^{(n)}}{\psi^{(n)}}$ is invertible then by previous theorem we can write*

$$\xi_{\rho,w} = \left(\frac{\phi^{(n)}}{\psi^{(n)}} \right)^{-1} \left(\frac{\Lambda_{\rho,w}(\phi)}{\Lambda_{\rho,w}(\psi)} \right), \quad \rho = 1, 2, w = 1, 2, 3, 4.$$

Applying Exponential convexity method [13], we construct some new families of exponentially convex functions or in the special case logarithmically convex functions. The outcome are some new classes of two-parameter Cauchy-type means.

Through the rest of paper, I denotes an open interval in $\mathbb{R}$.

Definition 2. *For fixed $n \in \mathbb{N}$, a function $\phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} p_i p_j \phi\left(\frac{x_i + x_j}{2} \right) \geq 0$$

holds for all choices $p_i \in \mathbb{R}$ *and* $x_i \in I$, $i = 1, ..., n$.
A function $\phi : I \to \mathbb{R}$ *is* n*-exponentially convex on* I *if it is* n*-exponentially convex in the Jensen sense and continuous on* I.

The notation of n-exponential convexity is introduced in [16].

Remark 2. *From Definition 2 it follows that* 1*-exponentially convex functions in the Jensen sense are exactly nonnegative functions. Moreover,* n*-exponentially convex functions in the Jensen sense are* k*-exponentially convex in the Jensen sense for every* $k \in \mathbb{N}$, $k \leq n$.

Definition 3. *A function* $\phi : I \to \mathbb{R}$ *is exponentially convex in the Jensen sense on* I *if it is* n*-exponentially convex in the Jensen sense for all* $n \in \mathbb{N}$.

One of the most important properties of exponentially convex functions is their integral representation (see [9, p. 211]).

Theorem 4.3. *The function* $\phi : I \to \mathbb{R}$ *is exponentially convex on* I *if and only if*

$$\phi(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(s), \tag{45}$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

The next example is deduced using integral representation (45) and some results of the Laplace transform (see [19, p. 214]).

Example 1. *The function* $\phi : (0, \infty) \to (0, \infty)$ *defined by* $\phi(x) = e^{-k\sqrt{x}}$ *is exponentially convex on* $(0, \infty)$ *for every* $k > 0$ *as* $e^{-k\sqrt{x}} = \int_0^\infty e^{-xt} e^{-k^2/4t} \frac{k}{2\sqrt{\pi t^3}} dt$.

Definition 4. *A function* $\phi : I \to (0, \infty)$ *is said to be logarithmically convex in the Jensen sense if*

$$\phi\left(\frac{x+y}{2}\right) \leq \sqrt{\phi(x)\phi(y)}$$

holds for all $x, y \in I$.

Definition 5. *A function* $\phi : I \to (0, \infty)$ *is said to be logarithmically convex or log-convex if*

$$\phi\left((1-\lambda)s + \lambda t\right) \leq \phi(s)^{1-\lambda}\phi(t)^{\lambda}$$

holds for all $s, t \in I, \lambda \in [0, 1]$.

Remark 3. *If a function is continuous and log-convex in the Jensen sense then it is also log-convex. We can also easily see that for positive functions exponential convexity implies log-convexity (consider the Definition 2 for* $n = 2$*).*

The following lemmas are equivalent to definition of convexity (see [17]).

Lemma 4.4. *Let* $\phi : I \to \mathbb{R}$ *be a convex function. Then for any* $x_1, x_2, x_3 \in I$ *such that* $x_1 < x_2 < x_3$ *the following is valid*

$$(x_3 - x_2)\,\phi(x_1) + (x_1 - x_3)\,\phi(x_2) + (x_2 - x_1)\,\phi(x_3) \geq 0.$$

Lemma 4.5. *Let $\phi : I \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, y_1, y_2, \in I$ such that $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$ the following is valid*

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \le \frac{\phi(y_2) - \phi(y_1)}{y_2 - y_1}.$$

In order to obtain results regarding the exponential convexity, we define the families of functions as follows.

For every choice of $l+1$ mutually different points $x_0, x_1, ..., x_l \in [\alpha, \beta]$ we define

- $\mathcal{F}_1 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is n-exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_2 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_3 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is 2-exponentially convex in the Jensen sense on $I\}$

Theorem 4.6. *Let $\Lambda_{1,w}$ and $\Lambda_{2,w}$ be the linear functional defined as in (43) and (44) respectively. Let $\mathcal{F}_1$ be family of functions associated with $\Lambda_{\rho,w}$ for $\rho = 1, 2, w = 1, 2, 3, 4$. Then the following statements hold:*

(i) *The function $t \mapsto \Lambda_{\rho,w}(\phi_t)$, is n-exponentially convex in the Jensen sense on I.*
(ii) *If the function $t \mapsto \Lambda_{\rho,w}(\phi_t)$, is continuous on I, then it is n-exponentially convex on I.*

Proof. The proof is similar to the proof of Theorem 16 in [1]. □

The following corollary is an easy consequence of the previous theorem.

Corollary 4.7. *Let $\Lambda_{1,w}$ and $\Lambda_{2,w}$ be the linear functional defined as in (43) and (44) respectively. Let $\mathcal{F}_2$ be family of functions associated with $\Lambda_{\rho,w}$, $\rho = 1, 2, w = 1, 2, 3, 4$. Then the following statements hold:*

(i) *The function $t \mapsto \Lambda_{\rho,w}(\phi_t)$ is exponentially convex in the Jensen sense on I.*
(ii) *If the function $t \mapsto \Lambda_{\rho,w}(\phi_t)$ is continuous on I, then it is exponentially convex on I.*

Corollary 4.8. *Let $\Lambda_{1,w}$ and $\Lambda_{2,w}$ be the linear functional defined as in (43) and (44) respectively. Let $\mathcal{F}_3$ be family of functions associated with $\Lambda_{\rho,w}$, $\rho = 1, 2, w = 1, 2, 3, 4$. Then the following statements hold:*

(i) *If the function $t \mapsto \Lambda_{\rho,w}(\phi_t)$ is continuous on I, then it is 2-exponentially convex on I. If $t \mapsto \Lambda_{\rho,w}(\phi_t)$ is additionally positive, then it is also log-convex on I. Furthermore, for every choice $r, s, t \in I$ such that $r < s < t$, it holds*

$$[\Lambda_{\rho,w}(\phi_s)]^{t-r} \le [\Lambda_{\rho,w}(\phi_r)]^{t-s} [\Lambda_{\rho,w}(\phi_t)]^{s-r}.$$

(ii) *If the function $t \mapsto \Lambda_{\rho,w}(\phi_t)$ is positive and differentiable on I, then for all $r, s, u, v \in I$ such that $r \le u$, $s \le v$, we have*

$$M_{r,s}(\Lambda_{\rho,w}, \mathcal{F}_3) \le M_{u,v}(\Lambda_{\rho,w}, \mathcal{F}_3), \tag{46}$$

where

$$M_{r,s}\left(\Lambda_{\rho,w},\mathcal{F}_3\right)=\begin{cases}\left(\frac{\Lambda_{\rho,w}(\phi_r)}{\Lambda_{\rho,w}(\phi_s)}\right)^{\frac{1}{r-s}}, & r\neq s,\\ \exp\left(\frac{\frac{d}{dr}(\Lambda_{\rho,w}(\phi_r))}{\Lambda_{\rho,w}(\phi_r)}\right), & r=s.\end{cases}\tag{47}$$

Proof. The proof is similar to the proof of Corollary 2 in [1]. □

Remark 4. *Note that the results from Theorem 4.6, Corollary 4.7 and Corollary 4.8 still hold when two of the points $x_0,...,x_l\in[a,b]$ coincide, say $x_1=x_0$, for a family of differentiable functions ϕ_t such that the function $t\mapsto\phi_t[x_0,...,x_l]$ is an n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense,* log*-convex in the Jensen sense), and furthermore, they still hold when all $(l+1)$ points coincide for a family of l differentiable functions with the same property. The proofs are obtained by (4) and suitable characterization of convexity.*

As an example of application of the previous results, consider the family of functions

$$\Omega=\{\varphi_t:(0,\infty)\to(0,\infty):\ t\in(0,\infty)\}$$

defined by

$$\varphi_t(x)=\frac{e^{-x\sqrt{t}}}{\left(-\sqrt{t}\right)^n}.$$

Since $\frac{d^n\varphi_t}{dx^n}(x)=e^{-x\sqrt{t}}>0$, the function φ_t is n-convex function for every $t>0$. Moreover, the function $t\mapsto\frac{d^n\varphi_t}{dx^n}(x)$ is exponentially convex. Therefore, using the same arguments as in proof of Theorem 4.6, we conclude that the function $t\mapsto[x_0,x_1,...,x_l;\varphi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Then from Corollary 4.7 it follows that $t\mapsto\Lambda_{\rho,w}(\varphi_t)$, $\rho=1,2,w=1,2,3,4$ is exponentially convex in the Jensen sense. It is easy to verify that the function $t\mapsto\Lambda(\varphi_t)$ is continuous, so it is exponentially convex.

For this family of functions, with assumption that $[\alpha,\beta]\subset(0,\infty)$ and $t\mapsto\Lambda_{\rho,w}(\varphi_t)$ is positive, (47), for $\rho=1$, $w=1,2,3,4$, becomes

$$M_{\eta,\zeta}\left(\Lambda_{1,w},\Omega\right)=\left(\frac{\sqrt{\zeta}^n}{\sqrt{\eta}^n}\cdot\frac{\sum\limits_{p=1}^{l}u_pe^{-x_p\sqrt{\eta}}-\sum\limits_{q=1}^{m}v_qe^{-y_q\sqrt{\eta}}-A_{1,w}}{\sum\limits_{p=1}^{l}u_pe^{-x_p\sqrt{\zeta}}-\sum\limits_{q=1}^{m}v_qe^{-y_q\sqrt{\zeta}}-B_{1,w}}\right)^{\frac{1}{\eta-\zeta}},\quad \eta\neq\zeta,$$

$$M_{\eta,\eta}\left(\Lambda_{1,w},\Omega\right)=\exp\left(\frac{\frac{1}{2\eta}\left(\sum\limits_{q=1}^{m}v_qy_qe^{-y_q\sqrt{\eta}}-\sum\limits_{p=1}^{l}u_px_pe^{-x_p\sqrt{\eta}}\right)-A_{2,w}}{\sum\limits_{p=1}^{l}u_pe^{-x_p\sqrt{\eta}}-\sum\limits_{q=1}^{m}v_qe^{-y_q\sqrt{\eta}}-A_{1,w}}-\frac{n}{2\eta}\right),$$

Similarly for $\rho = 2$, we have

$$M_{\eta,\zeta}\left(\Lambda_{2,w},\Omega\right) = \left(\frac{\sqrt{\zeta}^{n}}{\sqrt{\eta}^{n}}.\frac{\sum\limits_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum\limits_{q=1}^{m} v_q e^{-y_q\sqrt{\eta}} - C_{1,w}}{\sum\limits_{p=1}^{l} u_p e^{-x_p\sqrt{\zeta}} - \sum\limits_{q=1}^{m} v_q e^{-y_q\sqrt{\zeta}} - D_{1,w}}\right)^{\frac{1}{\eta-\zeta}}, \quad \eta \neq \zeta,$$

$$M_{\eta,\eta}\left(\Lambda_{2,w},\Omega\right) = \exp\left(\frac{\frac{1}{2\eta}\left(\sum\limits_{q=1}^{m} v_q y_q e^{-y_q\sqrt{\eta}} - \sum\limits_{p=1}^{l} u_p x_p e^{-x_p\sqrt{\eta}}\right) - C_{2,w}}{\sum\limits_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum\limits_{q=1}^{m} v_q e^{-y_q\sqrt{\eta}} - C_{1,w}} - \frac{n}{2\eta}\right),$$

where

$$A_{1,w} = \int_{\alpha}^{\beta}\left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t)\right]\times$$
$$\left[\sum_{k=2}^{n-1}\frac{(-\sqrt{\eta})^k k}{(k-1)!}\;\frac{e^{-\alpha\sqrt{\eta}}\,(t-\alpha)^{k-1} - e^{-\beta\sqrt{\eta}}\,(t-\beta)^{k-1}}{\beta-\alpha}\right]dt$$

$$B_{1,w} = \int_{\alpha}^{\beta}\left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t)\right]\times$$
$$\left[\sum_{k=2}^{n-1}\frac{(-\sqrt{\zeta})^k k}{(k-1)!}\;\frac{e^{-\alpha\sqrt{\zeta}}\,(t-\alpha)^{k-1} - e^{-\beta\sqrt{\zeta}}\,(t-\beta)^{k-1}}{\beta-\alpha}\right]dt$$

$$A_{2,w} = \frac{dA_{1,w}}{d\eta}$$

$$C_{1,w} = \int_{\alpha}^{\beta}\left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t)\right]\times$$
$$\left[\frac{\left(e^{-\alpha\sqrt{\eta}} - e^{-\beta\sqrt{\eta}}\right)\sqrt{\eta}}{\beta-\alpha} + \sum_{k=3}^{n-1}\frac{(-\sqrt{\eta})^k(k-2)}{(k-1)!}\frac{e^{-\alpha\sqrt{\eta}}\,(t-\alpha)^{k-1} - e^{-\beta\sqrt{\eta}}\,(t-\beta)^{k-1}}{\beta-\alpha}\right]dt$$

$$D_{1,w} = \int_{\alpha}^{\beta}\left[\sum_{p=1}^{l} u_p G_w(x_p,t) - \sum_{q=1}^{m} v_q G_w(y_q,t)\right]\times$$
$$\left[\frac{\left(e^{-\alpha\sqrt{\zeta}} - e^{-\beta\sqrt{\zeta}}\right)\sqrt{\zeta}}{\beta-\alpha} + \sum_{k=3}^{n-1}\frac{(-\sqrt{\zeta})^k(k-2)}{(k-1)!}\frac{e^{-\alpha\sqrt{\zeta}}\,(t-\alpha)^{k-1} - e^{-\beta\sqrt{\zeta}}\,(t-\beta)^{k-1}}{\beta-\alpha}\right]dt$$

$$C_{2,w} = \frac{dC_{1,w}}{d\eta}$$

Using Theorem 4.2 it follows that

$$M_{\eta,\zeta}\left(\Lambda_{\rho,w},\Omega\right) = -\left(\sqrt{\eta}+\sqrt{\zeta}\right)\log\mu_{\eta,\zeta}\left(\Lambda_{\rho,w},\Omega\right)$$

satisfies

$$\alpha \leq M_{\eta,\zeta}\left(\Lambda_{\rho,w},\Omega\right) \leq \beta,$$

i.e. $M_{\eta,\zeta}\left(\Lambda_{\rho,w},\Omega\right)$ are means. By Corollary 4.8, using (46), it follows that these means are monotonic.

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