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Partition Laplacian Energy of a Graph

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Abstract

The partition energy of a graph was introduced by E. Sampathkumar et al. in [19] in 2015. In this paper, by the motivation of this new energy, the partition Laplacian energy $LE_p(G)$ of a graph is introduced and the $LE_p(G)$ of some important graph classes is discussed. Also, we obtain some bounds for the partition Laplacian energy.

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1 Introduction

For standard definitions and terminology regarding graph theory, we refer [14]. Throughout this paper, we consider simple, undirected, signless graphs without loops and multiple edges. The concept of graph energy was introduced by

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Gutman [12] as the sum of the absolute values of the eigenvalues of the adjacency matrix of the given graph G. To estimate the total π-electron energy of a molecule has great importance in Chemistry. One can find other types of energy such as distance energy, maximum degree energy, color energy, covering energy, etc. in [1, 4, 5, 15].

2 Partition Laplacian Energy of a Graph

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Let the number of edges of G be m. The partition matrix $P(G) = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} 2, & \text{if there is an edge between } v_i \text{ and } v_j, \text{ where } v_i, v_j \in V_r \\ -1, & \text{if there is no edge between } v_i \text{ and } v_j, \text{ where } v_i, v_j \in V_r \\ 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets } V_r \text{ and } V_s \\ & \text{for } r \neq s, \text{ where } v_i \in V_r \text{ and } v_j \in V_s \\ 0, & \text{otherwise.} \end{cases}$$

Partition energy of a graph is the sum of the absolute values of the eigenvalues of its partition matrix. This concept was introduced by E. Sampathkumar et al., [19].

Motivated by the partition energy of a graph, in this section, we define the partition Laplacian energy of a graph. Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then $L_P(G) = D(G) - P(G)$ is called the partition Laplacian matrix of G. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of $L_P(G)$, arranged in non-increasing order. These eigenvalues are called partition Laplacian eigenvalues of G. The partition Laplacian energy of the graph G is defined as

$$LE_P(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}| \tag{1}$$

where m is the number of edges of G and $\frac{2m}{n}$ is the average degree of G.

In this paper, we study partition Laplacian energy of a graph with respect to a given partition of a graph. Further, we determine partition Laplacian energy of two types of complements of a partition graph called k-complement and $k(i)$-complement of a graph (see [18]).

Definition 2.1. *The complement of a graph G is a graph $\overline{G}$ on the same vertex set such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.*

Definition 2.2. *[18] Let G be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set V. The k-complement of G is denoted by $\overline{(G)_k}$ and is obtained by removing the edges between V_i and V_j and adding the edges between the vertices in V_i and V_j which are not in G, for all V_i and V_j in P_k where $i \neq j$.*

Definition 2.3. *[18] Let G be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set V. Then the $k(i)$-complement of G is denoted by $\overline{(G)_{k(i)}}$ and is obtained by removing the edges of G which are joining the vertices within V_r and adding the edges of $\overline{G}$ which are joining the vertices of V_r for each component set V_r in P_k.*

Definition 2.4. *The spectrum of a graph G is the arrangement of distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, with their multiplicities being $m_1, m_2, \ldots, m_r$, and we write it as*

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

3 Some Basic Properties of Partition Laplacian Energy of a Graph

Let $G = (V, E)$ be a graph with n vertices and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of V. For each i such that $1 \leq i \leq k$, let b_i denote the total number of edges joining the vertices in V_i, c_i be the total number of edges joining the vertices in V_i to the ones in V_j for $i \neq j$, $1 \leq j \leq k$, and d_i be the number of non-adjacent pairs of vertices within V_i. Let

$$m_1 = \sum_{i=1}^{k} b_i, \; m_2 = \sum_{i=1}^{k} c_i \;\text{ and }\; m_3 = \sum_{i=1}^{k} d_i.$$

Let $L_P(G)$ be the partition Laplacian matrix of G. If the characteristic polynomial of $L_P(G)$ is $\Phi_L^P(G, \lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$, then the coefficients a_i can be interpreted using the principal minors of $L_P(G)$.

Proposition 3.1. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are partition Laplacian eigenvalues of $P_k(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = \sum_{i=1}^{n} d_i^2 + 2[4m_1 + m_2 + m_3].$$

Proof. We know that

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j} a_{ij}^2 \\ &= \sum_{i=1}^{n} d_i^2 + 2[4m_1 + m_2 + m_3].\end{aligned}$$

□

The following general results follow easily:

Theorem 3.2. *Let G be a graph with n vertices and P_k be a partition of G. Then*

$$\sqrt{2K + n(n-1)D^{\frac{2}{n}}} \leq LE_{P_k}(G) \leq \sqrt{2K(n-1) + nD^{\frac{2}{n}}}$$

where $D = |det(L_P(G)) - \frac{2m}{n}I|$ and $K = 4m_1 + m_2 + m_3 + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$.

Theorem 3.3. *If the partition Laplacian energy of a graph is a rational number, then it must be a positive even number.*

4 Partition Laplacian Energy of Some Standard Graphs

Theorem 4.1. *If K_n is the complete graph of order n, then*

$$LE_{P_1}(K_n) = 4(n-1).$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Consider that all the vertices are in one component.

$$P_{P_1}(K_n) = \begin{bmatrix} n-1 & -2 & -2 & \dots & -2 & -2 \\ -2 & n-1 & -2 & \dots & -2 & -2 \\ -2 & -2 & n-1 & \dots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & -2 & \dots & n-1 & -2 \\ -2 & -2 & -2 & \dots & -2 & n-1 \end{bmatrix}.$$

The characteristic equation is

$$[\lambda - (n+1)]^{n-1}[\lambda + (n-1)] = 0$$

and the partition Laplacian eigenvalues are

$$Lspec_{P_1}(K_n) = \begin{pmatrix} -(n-1) & n+1 \\ 1 & n-1 \end{pmatrix}.$$

As the number of vertices is n, the number of edges is $\frac{n(n-1)}{2}$ and the average vertex degree is $n-1$ in K_n, the partition Laplacian energy is

$$\begin{aligned} LE_{P_1}(K_n) &= |-(n-1)-(n-1)| + |(n+1)-(n-1)|(n-1) \\ &= 4(n-1). \end{aligned}$$

□

Theorem 4.2. *The* 1*-partition Laplacian energy of the cycle graph* C_n *is*

$$LE_{P_1}(C_n) = |n-7| + \sum_{m=1}^{n-1} |1 + 6\cos\frac{2\pi m}{n}|.$$

Proof. Consider that all the vertices are in one component. Then the 1-partition Laplacian matrix is

$$P_1(C_n) = \begin{bmatrix} 2 & -2 & 1 & 1 & 1 & \dots & 1 & -2 \\ -2 & 2 & -2 & 1 & 1 & \dots & 1 & 1 \\ 1 & -2 & 2 & -2 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \dots & 2 & -2 \\ -2 & 1 & 1 & 1 & 1 & \dots & -2 & 2 \end{bmatrix}.$$

This is a circullant matrix of order n.

Its eigenvalues are

$$\lambda_m = \begin{cases} n-5, & \text{for } m = 0 \\ 1 - 6\cos\frac{2\pi m}{n}, & \text{for } 0 < m \leq n \end{cases}$$

As the average vertex degree is 2 in the cycle graph C_n, the 1-partition Laplacian energy is

$$LE_{P_1}(C_n) = |n-5-2| + \sum_{m=1}^{n-1} |1 - 6\cos\frac{2\pi m}{n} - 2|.$$

Therefore we get

$$LE_{P_1}(C_n) = |n-7| + \sum_{m=1}^{n-1} |1 + 6\cos\frac{2\pi m}{n}|.$$

□

Theorem 4.3. *The* 1*-partition Laplacian energy of the star graph* $K_{1,n-1}$ *is*

$$LE_{P_1}(K_{1,n-1}) = \frac{2(n-1)(n-2)}{n} + 4\sqrt{n-1}.$$

Proof. Consider once more that all the vertices are in one component. Then the 1-partition Laplacian matrix is

$$P_1(K_{1,n-1}) = \begin{bmatrix} n-1 & -2 & -2 & \dots & -2 & -2 \\ -2 & 1 & 1 & \dots & 1 & 1 \\ -2 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & 1 & 1 & \dots & 1 & 1 \\ -2 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

Hence, the characteristic equation is

$$(\lambda)^{n-2}[\lambda^2 - (2n-2)\lambda + (n^2 - 6n + 5)] = 0.$$

Therefore the spectrum is

$$Spec_{P_1}(K_{1,n-1}) = \begin{pmatrix} 0 & n-1+2\sqrt{n-1} & n-1-2\sqrt{n-1} \\ n-2 & 1 & 1 \end{pmatrix}.$$

As the number of vertices is n, the number of edges is $n-1$, and the average vertex degree is $\frac{2(n-1)}{n}$ in the star graph, the 1-partition Laplacian energy is

$$\begin{aligned} LE_{P_1}(K_{1,n-1}) &= |0 - \tfrac{2(n-1)}{n}|(n-2) + |n-1+2\sqrt{n-1} - \tfrac{2(n-1)}{n}| \\ &\quad + |n-1-2\sqrt{n-1} - \tfrac{2(n-1)}{n}|. \end{aligned}$$

Therefore we get

$$LE_{P_1}(K_{1,n-1}) = \frac{2(n-1)(n-2)}{n} + 4\sqrt{n-1}.$$

□

Definition 4.4. *The crown graph* S_n^0 *for an integer* $n \geq 3$ *is the graph with the vertex set* $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ *and the edge set*

$$\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}.$$

S_n^0 *is therefore equivalent to the complete bipartite graph* $K_{n,n}$ *with horizontal edges removed.*

Theorem 4.5. *The* 1*-partition Laplacian energy of the crown graph* S_n^0 *is*

$$LE_{P_1}(S_n^0) = 10n - 12.$$

Proof. Let S_n^0 be the crown graph of order $2n$ and consider that all of its vertices $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$ are in the same component. Then the 1-partition Laplacian matrix is

$$P_1(S_n^0)=\begin{bmatrix} n-1 & 1 & 1 & \dots & 1 & 1 & -2 & \dots & -2 & -2 \\ 1 & n-1 & 1 & \dots & 1 & -2 & 1 & \dots & -2 & -2 \\ 1 & 1 & n-1 & \dots & 1 & -2 & -2 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & n-1 & -2 & -2 & \dots & -2 & 1 \\ 1 & -2 & -2 & \dots & -2 & n-1 & 1 & \dots & 1 & 1 \\ -2 & 1 & -2 & \dots & -2 & 1 & n-1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & 1 & \dots & -2 & 1 & 1 & \dots & n-1 & 1 \\ -2 & -2 & -2 & \dots & 1 & 1 & 1 & \dots & 1 & n-1 \end{bmatrix}.$$

Hence the characteristic equation is

$$(\lambda-(n-5))^{n-1}(\lambda-(n+1))^{n-1}(\lambda-1)(\lambda-(4n-5))=0.$$

Therefore the spectrum is

$$Spec_{P_1}(S_n^0)=\begin{pmatrix} n-5 & n+1 & 1 & 4n-5 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

As the number of vertices is $2n$, the number of edges is $n(n-1)$ and the average vertex degree is $n-1$ in a crown graph, we obtain the 1-partition Laplacian energy of it as

$$\begin{aligned} LE_{P_1}(S_n^0) &= |n-5-(n-1)|(n-1)+|n+1-(n-1)|(n-1) \\ &\quad +|1-(n-1)|+|4n-5-(n-1)| \\ &= 10n-12. \end{aligned}$$ $\square$

Theorem 4.6. *The* 1*-partition Laplacian energy of the cocktail party graph* $K_{n\times 2}$ *is*

$$LE_{P_1}(K_{n\times 2})=10(n-1).$$

Proof. Consider that all the vertices are in one component. The 1-partition Laplacian matrix is

$$P_1(K_{n\times 2})=\begin{bmatrix} 2(n-1) & 1 & -2 & \dots & -2 & -2 & -2 \\ 1 & 2(n-1) & -2 & \dots & -2 & -2 & -2 \\ -2 & -2 & 2(n-1) & \dots & -2 & -2 & -2 \\ -2 & -2 & 1 & \dots & -2 & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -2 & -2 & -2 & \dots & 1 & -2 & -2 \\ -2 & -2 & -2 & \dots & 2(n-1) & -2 & -2 \\ -2 & -2 & -2 & \dots & -2 & 2(n-1) & 1 \\ -2 & -2 & -2 & \dots & -2 & 1 & 2(n-1) \end{bmatrix}.$$

Then the characteristic equation is

$$(\lambda + (2n-3))(\lambda - (2n+3))^{n-1}(\lambda - (2n-3))^{n} = 0$$

and therefore the spectrum is

$$Spec_{P_1}(K_{n\times 2}) = \begin{pmatrix} -2n+3 & 2n+3 & 2n-3 \\ 1 & n-1 & n \end{pmatrix}.$$

As the number of vertices is $2n$, the number of edges is $2n(n-1)$ and the average vertex degree is $2(n-1)$, the 1-partition Laplacian energy of a cocktail party graph is given by

$$\begin{aligned} LE_{P_1}(K_n \times 2) &= |-(2n-3) - 2(n-1)| + |(2n+3) - 2(n-1)|(n-1) \\ &\quad + |2n-3-2(n-1)|n \\ &= 10(n-1). \end{aligned}$$

□

Theorem 4.7. *The* 1*-partition Laplacian energy of the complete bipartite graph* $K_{n,n}$ *is*

$$LE_{P_1}(K_{n,n}) = 6n - 2.$$

Proof. Suppose that all of the vertices are in the same component. The 1-partition Laplacian matrix is

$$P_1(K_{n,n}) = \begin{bmatrix} n & 1 & 1 & 1 & \dots & -2 & -2 & -2 & -2 \\ 1 & n & 1 & 1 & \dots & -2 & -2 & -2 & -2 \\ 1 & 1 & n & 1 & \dots & -2 & -2 & -2 & -2 \\ 1 & 1 & 1 & n & \dots & -2 & -2 & -2 & -2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -2 & -2 & -2 & -2 & \dots & n & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & \dots & 1 & n & 1 & 1 \\ -2 & -2 & -2 & -2 & \dots & 1 & 1 & n & 1 \\ -2 & -2 & -2 & -2 & \dots & 1 & 1 & 1 & n \end{bmatrix}.$$

Hence the characteristic equation is

$$(\lambda + 1)(\lambda - (n-1))^{2n-2}(\lambda - (4n-1)) = 0$$

and the spectrum is

$$Spec_{P_1}(K_{n,n}) = \begin{pmatrix} -1 & n-1 & 4n-1 \\ 1 & 2n-2 & 1 \end{pmatrix}.$$

Here the number of vertices is $2n$, the number of edges is n^2 and the average vertex degree is n implying the 1-partition Laplacian energy is

$$\begin{aligned} LE_{P_1}(K_{n,n}) &= |-1-n| + |(n-1)-n|(2n-2) + |4n-1-n| \\ &= 6n-2. \end{aligned}$$

□

Theorem 4.8. *The* 1*-partition Laplacian energy of double star graph* $S_{n,n}$ *is*

$$LE_{P_1}(S_{n,n}) = \frac{(2n-1)(2n-4)}{n} + \sqrt{36n-11} + \sqrt{4n^2-8n+5}.$$

Proof. Suppose that all of the vertices stay in one component. The 1-partition Laplacian matrix is

$$P_1(S_{n,n}) = \begin{bmatrix} n & -2 & -2 & \dots & -2 & -2 & 1 & 1 & \dots & 1 \\ -2 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ -2 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ -2 & 1 & 1 & \dots & 1 & n & -2 & -2 & \dots & -2 \\ 1 & 1 & 1 & \dots & 1 & -2 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & -2 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & -2 & 1 & 1 & \dots & 1 \end{bmatrix}$$

and the characteristic equation becomes

$$(\lambda)^{2n-4}[\lambda^2 - (2n-1)\lambda + (n-1)][\lambda^2 - 5\lambda - (9n-9)] = 0.$$

Hence the spectrum is
$$\begin{pmatrix} 0 & \frac{(2n-1)+\sqrt{4n^2-8n+5}}{2} & \frac{(2n-1)-\sqrt{4n^2-8n+5}}{2} & \frac{5+\sqrt{36n-11}}{2} & \frac{5-\sqrt{36n-11}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$
Having $2n$ vertices, $2n-1$ edges and average vertex degree $\frac{2n-1}{n}$, the 1-partition Laplacian energy would be

$$\begin{aligned} LE_{P_1}(K_{n,n}) &= |0 - \tfrac{2n-1}{n}| + |\tfrac{(2n-1)+\sqrt{4n^2-8n+5}}{2} - \tfrac{2n-1}{n}| \\ &+ |\tfrac{(2n-1)-\sqrt{4n^2-8n+5}}{2} - \tfrac{2n-1}{n}| \\ &+ |\tfrac{5+\sqrt{36n-11}}{2} - \tfrac{2n-1}{n}| \\ &+ |\tfrac{5-\sqrt{36n-11}}{2} - \tfrac{2n-1}{n}| \end{aligned}$$

implying that

$$LE_{P_1}(S_{n,n}) = \tfrac{(2n-1)(2n-4)}{n} + \sqrt{36n-11} + \sqrt{4n^2-8n+5}. \qquad \square$$

Theorem 4.9. *The* 2*-partition Laplacian energy of the crown graph of order* $2n$ *is*

$$LE_{P_2}(S_n^0) = 4(n-1).$$

We omit the proof of this theorem, since the proof is same as the color Laplacian energy of S_n^0 with minimum number of colors as in [7].

Theorem 4.10. *The* 2*-partition Laplacian energy of the double star graph* $S_{n,n}$ *is*

$$LE_{P_2}(S_{n,n}) = \frac{(2n-1)(2n-4)}{n} + \sqrt{n^2+8n} + \sqrt{n^2+4n-4}$$

for $n \leq 4$ *and*

$$LE_{P_2}(S_{n,n}) = \frac{(2n-1)(2n-4)}{n} + \sqrt{n^2+8n} + (3n-4)$$

for $n \geq 5$.

Proof. In the double star graph, we put the centers $\{u_0, v_0\}$ into one component and the remaining vertices to the other component of the partition. Then the 2-partition Laplacian matrix is

$$P_2(S_{n,n}) = \begin{bmatrix} n & -1 & -1 & \dots & -1 & -2 & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & \dots & 1 & 0 & 1 & 1 & \dots & 1 \\ -1 & 1 & 1 & \dots & 1 & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 1 & -1 & \dots & 1 & 0 & 1 & 1 & \dots & 1 \\ -2 & 0 & 0 & \dots & 0 & n & -1 & -1 & \dots & -1 \\ 0 & 1 & 1 & \dots & 1 & -1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 1 & \dots & 1 & -1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 & -1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

Now the characteristic equation is

$$\lambda^{2n-4}[\lambda^2 - (n+2)\lambda - (n-1)][\lambda^2 - (3n-4)\lambda + (2n^2 - 7n + 5)] = 0$$

giving the spectrum as

$$\begin{pmatrix} 0 & \frac{n+2+\sqrt{n^2+8n}}{2} & \frac{n+2-\sqrt{n^2+8n}}{2} & \frac{3n-4+\sqrt{n^2+4n-4}}{2} & \frac{3n-4-\sqrt{n^2+4n-4}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Here the graph has $2n$ vertices, $2n-1$ edges and has an average vertex degree $\frac{2n-1}{n}$ implying that the 2-partition Laplacian energy is

$$\begin{aligned} LE_{P_2}(S_{n,n}) &= |0 - \tfrac{2n-1}{n}|(2n-4) + |\tfrac{(n+2)+\sqrt{n^2+8n}}{2} - \tfrac{2n-1}{n}| \\ &+ |\tfrac{(n+2)-\sqrt{n^2+8n}}{2} - \tfrac{2n-1}{n}| \\ &+ |\tfrac{(3n-4)+\sqrt{n^2+4n-4}}{2} - \tfrac{2n-1}{n}| \\ &+ |\tfrac{(3n-4)-\sqrt{n^2+4n-4}}{2} - \tfrac{2n-1}{n}|. \end{aligned}$$

Therefore,

$$LE_{P_2}(S_{n,n}) = \frac{(2n-1)(2n-4)}{n} + \sqrt{n^2+8n} + \sqrt{n^2+4n-4}$$

for $n \leq 4$ and

$$LE_{P_2}(S_{n,n}) = \frac{(2n-1)(2n-4)}{n} + \sqrt{n^2+8n} + (3n-4)$$

for $n \geq 5$. $\square$

Theorem 4.11. *The* 2*-partition Laplacian energy of the cycle graph* C_{2n} *is*

$$LE_{P_2}(C_{2n}) = 2n - 2 + \sum_{m=1, m\neq n}^{2n-1} |1 + 2\cos\frac{\pi m}{n}|.$$

Proof. Consider the odd labeled vertices $v_1, v_3, v_5, \cdots, v_{2n-1}$ are in one component and the even labeled vertices $v_2, v_4, v_6, \cdots, v_{2n}$ are in the other component. Then the 2-partition Laplacian matrix is

$$P_1(C_{2n}) = \begin{bmatrix} 2 & -1 & 1 & 0 & 1 & \dots & 1 & -1 \\ -1 & 2 & -1 & 1 & 0 & \dots & 0 & 1 \\ 1 & -1 & 2 & -1 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & 0 & 1 & \dots & 2 & -1 \\ -1 & 1 & 0 & 1 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

This is a circullant matrix of order $2n$. Its eigenvalues are

$$\lambda_m = \begin{cases} n-1, & \text{for } m = 0 \\ 3+n, & \text{for } m = n \\ 2 - 2\cos\frac{\pi m}{n}, & \text{for } 0 < m \leq 2n-1 \end{cases}$$

As the average vertex degree is 2 in the cycle graph C_{2n}, the 2-partition Laplacian energy is

$$LE_{P_2}(C_{2n}) \quad = |n-1-2| + |3+n-2| + \sum_{m=1, m\neq n}^{2n-1} |1 - 2\cos\frac{\pi m}{n} - 2|.$$

Therefore we get

$$LE_{P_2}(C_{2n}) = 2n - 2 + \sum_{m=1, m\neq n}^{2n-1} |1 + 2\cos\frac{\pi m}{n}|.$$

$\square$

Theorem 4.12. *The 2-partition energy of the* $2(i)$*-complement of the star graph* $K_{1,n-1}$ *in which the central vertex of degree* $n-1$ *is in one component and vertices of degree* 1 *are in the other component is*

$$2(n-2) + 2\sqrt{n^2 - 3n + 3}.$$

Proof. Consider $2(i)$-complement of star graph $K_{1,n-1}$ in which the vertex of degree $n-1$ is in one component and the remaining vertices are in the second component. Its partition Laplacian matrix is

$$P_2(\overline{(K_{1,n-1})_{2(i)}}) = \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 & -1 \\ -1 & n-1 & -2 & \dots & -2 & -2 \\ -1 & -2 & n-1 & \dots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -2 & \dots & n-1 & -2 \\ -1 & -2 & -2 & \dots & -2 & n-1 \end{bmatrix}.$$

Hence the characteristic equation would be

$$(\lambda - (n+1))^{n-2}[\lambda^2 - 2\lambda - (n^2 - 3n + 2)] = 0$$

implying the spectrum is

$$\begin{pmatrix} n+1 & 1+\sqrt{n^2-3n+3} & 1-\sqrt{n^2-3n+3} \\ n-2 & 1 & 1 \end{pmatrix}.$$

The number of vertices is n, the number of edges is $\frac{n(n-1)}{2}$ and the average vertex degree is $n-1$ together implying that the $2(i)$-partition Laplacian energy of the star graph is

$$\begin{aligned} LE_{P_2}(\overline{(K_{1,n-1})_{2(i)}}) &= |n+1-(n-1)|(n-2) \\ &\quad + |1+\sqrt{n^2-3n+3}-(n-1)| \\ &\quad + |1-\sqrt{n^2-3n+3}-(n-1)|. \end{aligned}$$

Therefore

$$LE_{P_2}(\overline{(K_{1,n-1})_{2(i)}}) = 2(n-2) + 2\sqrt{n^2-3n+3}.$$

□

Theorem 4.13. *The $2(i)$-partition Laplacian energy of the crown graph of order $2n$ is*

$$LE_{P_2}(\overline{(S_n^0)_{2(i)}}) = 6(n-1).$$

Proof. The $2(i)$-partition of the crown graph is the cocktail party graph. We omit the proof since it is similar to the one for the color Laplacian energy of $K_{n\times 2}$ with minimum number of colors as in [7]. □

Theorem 4.14. *The 2-partition Laplacian energy of 2-complement of the cocktail party graph $K_{n\times 2}$ is*

$$LE_{P_2}(\overline{(K_{n\times 2})_{(2)}}) = 8(n-1).$$

Proof. Consider the 2-complement of the cocktail party graph $\overline{(K_{n\times 2})_{(2)}}$ whose vertex set is partitioned into $U_n = \{u_1, u_2, \dots, u_n\}$ and $V_n = \{v_1, v_2, \dots, v_n\}$. The 2-partition Laplacian matrix is

$$P_2(\overline{(K_{n\times 2})_{(2)}}) = \begin{bmatrix} n & -2 & -2 & \dots & -2 & -1 & 0 & \dots & 0 & 0 \\ -2 & n & -2 & \dots & -2 & 0 & -1 & \dots & 0 & 0 \\ -2 & -2 & n & \dots & -2 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & -2 & \dots & n & 0 & 0 & \dots & & -1 \\ -1 & 0 & 0 & \dots & 0 & n & 2 & \dots & -2 & -2 \\ 0 & -1 & 0 & \dots & 0 & -2 & n & \dots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & -2 & -2 & \dots & n & -2 \\ 0 & 0 & 0 & \dots & -1 & -2 & -2 & \dots & -2 & n \end{bmatrix}.$$

Hence the characteristic polynomial is

$$[\lambda + (n-1)][\lambda + (n-3)][\lambda - (n+1)]^{n-1}[\lambda - (n+3)]^{n-1} = 0$$

implying that the 2-partition Laplacian spectra is

$$Spec_{P_2}(\overline{(K_{n\times 2})_{(2)}}) = \begin{pmatrix} -n+1 & -n+3 & n+1 & n+3 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

The number of vertices is $2n$, the number of edges is n^2 and the average vertex degree is n in the 2-complement of the cocktail party graph, the 2-partition Laplacian energy is

$$\begin{aligned} LE_{P_2}(\overline{(K_{n\times 2})_{(2)}}) &= |-(n-1)-n| + |-(n-3)-n| \\ &\quad +|n+1-n|(n-1) + |n+3-n|(n-1) \end{aligned}$$

and therefore we get

$$LE_{P_2}(\overline{(K_{n\times 2})_{(2)}}) = 8(n-1).$$

□

Theorem 4.15. *The* 2*-partition Laplacian energy of* $2(i)$*-complement of double star graph* $S_{n,n}$ *is*

$$LE_{P_2}(\overline{(S_{n,n})_{2(i)}}) = \begin{cases} \frac{(3n-1)(2n-4)}{n} + \sqrt{9n^2 - 20n + 12} + \sqrt{n^2 + 8n} & \textit{for } n = 3, \\ \frac{(3n-1)(2n-4)}{n} + \frac{2(2n^2 - 3n + 1)}{n} + \sqrt{n^2 + 8n} & \textit{for } n \geq 4. \end{cases}$$

Proof. In 2(i)-complement of the double star graph, the centers $\{u_0, v_0\}$ are put in one component and the remaining vertices are put into the second component. The minimum dominating 2-partition matrix is

$$P_2(\overline{(S_{n,n})_{2(i)}}) = \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 & 1 & 0 & \dots & 0 \\ -1 & 2n-2 & -2 & \dots & -2 & 0 & -2 & \dots & -2 \\ -1 & -2 & 2n-2 & \dots & -2 & 0 & -2 & \dots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -2 & \dots & 2n-2 & 0 & -2 & \dots & -2 \\ 1 & 0 & 0 & \dots & 0 & n-1 & -1 & \dots & -1 \\ 0 & -2 & -2 & \dots & -2 & -1 & 2n-2 & \dots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -2 & -2 & \dots & -2 & -1 & -2 & \dots & -2 \\ 0 & -2 & -2 & \dots & -2 & -1 & -2 & \dots & 2n-2 \end{bmatrix}.$$

Therefore the characteristic equation is

$$(\lambda - 2n)^{2n-4}[\lambda^2 - (3n-2)\lambda + (2n^2 - 5n + 1)][\lambda^2 + (n-4)\lambda - (2n^2 - 3n - 1)] = 0$$

giving the spectrum as

$$\begin{pmatrix} 2n & \frac{-n+4+\sqrt{9n^2-20n+12}}{2} & \frac{-n+4-\sqrt{9n^2-20n+12}}{2} & \frac{3n-2+\sqrt{n^2+8n}}{2} & \frac{3n-2-\sqrt{n^2+8n}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The graph has $2n$ vertices, $2n^2 - 3n + 1$ edges and an average vertex degree $\frac{2n^2-3n+1}{n}$ giving the 2-partition Laplacian energy as

$$\begin{aligned} LE_{P_2}(\overline{(S_{n,n})_{2(i)}}) &= |2n - \tfrac{2n^2-3n+1}{n}|(2n-4) \\ &\quad + |\tfrac{-n+4+\sqrt{9n^2-20n+12}}{2} - \tfrac{2n^2-3n+1}{n}| \\ &\quad + |\tfrac{-n+4-\sqrt{9n^2-20n+12}}{2} - \tfrac{2n^2-3n+1}{n}| \\ &\quad + |\tfrac{3n-2+\sqrt{n^2+8n}}{2} - \tfrac{2n^2-3n+1}{n}| \\ &\quad + |\tfrac{3n-2-\sqrt{n^2+8n}}{2} - \tfrac{2n^2-3n+1}{n}|. \end{aligned}$$

Therefore we obtain

$$LE_{P_2}(\overline{(S_{n,n})_{2(i)}}) = \begin{cases} \frac{(3n-1)(2n-4)}{n} + \sqrt{9n^2-20n+12} + \sqrt{n^2+8n} & \text{for } n = 3, \\ \frac{(3n-1)(2n-4)}{n} + \frac{2(2n^2-3n+1)}{n} + \sqrt{n^2+8n} & \text{for } n \geq 4. \end{cases}$$

□

5 Partition Laplacian energy of graphs with one edge deleted

In this section we obtain the partition Laplacian energy for certain graphs with one edge deleted.

Theorem 5.1. *Let e be the edge of complete graph K_n.*

$$LE_{P_1}(K_n - e) = 2\left(\frac{n^2 - n - 4}{n} + \sqrt{n^2 + 2n - 7}\right).$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Consider that all the vertices are in one component.

$$P_{P_1}(K_n - e) = \begin{bmatrix} n-2 & -2 & -2 & \dots & -2 & -2 \\ -2 & n-2 & -2 & \dots & -2 & -2 \\ -2 & -2 & n-1 & \dots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & -2 & \dots & n-1 & -2 \\ -2 & -2 & -2 & \dots & -2 & n-1 \end{bmatrix}.$$

The characteristic equation is

$$[\lambda - (n-3)][\lambda - (n+1)]^{n-1}[\lambda + (n-1)][\lambda^2 - 4\lambda - (n^2 + 2n - 11)] = 0$$

and the partition Laplacian eigenvalues are

$$Lspec_{P_1}(K_n - e) = \begin{pmatrix} n-3 & n+1 & 2+\sqrt{n^2+2n-7} & 2-\sqrt{n^2+2n-7} \\ 1 & n-3 & 1 & 1 \end{pmatrix}.$$

As the number of vertices is n, the number of edges is $\frac{n^2-n-2}{2}$ and the average vertex degree is $\frac{n^2-n-2}{n}$ in K_n, the partition Laplacian energy is

$$\begin{aligned} LE_{P_1}(K_n - e) &= |n - 3 - \tfrac{n^2-n-2}{n}| \\ &\quad + |n + 1 - \tfrac{n^2-n-2}{n}|(n-3) \\ &\quad + |2 + \sqrt{n^2 + 2n - 7} - \tfrac{n^2-n-2}{n}| \\ &\quad + |2 + \sqrt{n^2 + 2n - 7} - \tfrac{n^2-n-2}{n}| \\ &= 2\left(\tfrac{n^2-n-4}{n} + \sqrt{n^2 + 2n - 7}\right). \end{aligned}$$ □

Theorem 5.2. *Let e be an edge of the complete bipartite graph $K_{n,n}$. The 1-partition Laplacian energy of $K_{n,n} - e$ is*

$$LE_{P_1}(K_{n,n} - e) = \frac{4 - 2n}{n} + \sqrt{9n^2 + 24n - 32} + \sqrt{n^2 + 4n - 4}.$$

Proof. Suppose that all of the vertices are in one component. The 1-partition

Laplacian matrix is

$$P_1(K_{n,n}-e)=\begin{bmatrix} n-1 & 1 & 1 & 1 & \ldots & 1 & -2 & -2 & -2 \\ 1 & n & 1 & 1 & \ldots & -2 & -2 & -2 & -2 \\ 1 & 1 & n & 1 & \ldots & -2 & -2 & -2 & -2 \\ 1 & 1 & 1 & n & \ldots & -2 & -2 & -2 & -2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & -2 & -2 & -2 & \ldots & n-1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & \ldots & 1 & n & 1 & 1 \\ -2 & -2 & -2 & -2 & \ldots & 1 & 1 & n & 1 \\ -2 & -2 & -2 & -2 & \ldots & 1 & 1 & 1 & n \end{bmatrix}.$$

Hence the characteristic equation is

$$(\lambda^2-n\lambda-(n-1))(\lambda-(n-1))^{2n-4}(\lambda^2+(6-5n)\lambda+(4n^2-21n+17))=0$$

and the spectrum is

$$Spec_{P_1}(K_{n,n})-e=\begin{pmatrix} n-1 & \frac{n+\sqrt{n^2+4n-4}}{2} & \frac{n-\sqrt{n^2+4n-4}}{2} & \frac{5n-6-\sqrt{9n^2+24n-32}}{2} & \frac{5n-6+\sqrt{9n^2+24n-32}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Here the number of vertices is $2n$, the number of edges is n^2-1 and the average vertex degree is $\frac{n^2-1}{n}$ implying the 1-partition Laplacian energy is

$$\begin{aligned} LE_{P_1}(K_{n,n}-e) &= |n-1-\tfrac{n^2-1}{n}|(2n-4) \\ &\quad +|\tfrac{n+\sqrt{n^2+4n-4}}{2}-\tfrac{n^2-1}{n}| \\ &\quad +|\tfrac{n-\sqrt{n^2+4n-4}}{2}-\tfrac{n^2-1}{n}| \\ &\quad +|\tfrac{5n-6+\sqrt{9n^2+24n-32}}{2}-\tfrac{n^2-1}{n}| \\ &\quad +|\tfrac{5n-6-\sqrt{9n^2+24n-32}}{2}-\tfrac{n^2-1}{n}| \\ &= \tfrac{4-2n}{n}+\sqrt{9n^2+24n-32}+\sqrt{n^2+4n-4}. \end{aligned}$$

□

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