

THE CONVOLUTION SUMS $\sum_{m < \frac{n}{8}} \sigma_1(m)\sigma_3(n - 8m)$ AND $\sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1(n - 8m)$

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ABSTRACT. In this paper we evaluate the convolution sum formulae of

$$\sum_{m < \frac{n}{8}} \sigma_1(m)\sigma_3(n - 8m) \quad \text{and} \quad \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1(n - 8m)$$

for all $n \in \mathbb{N}$.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 11A05.

KEYWORDS AND PHRASES. Divisor functions, Convolution sums.

1. INTRODUCTION

For $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define the divisor function and the infinite product sums :

$$(1) \quad \begin{aligned} \sigma_k(n) &= \sum_{d|n} d^k, & \Delta(q) &:= \sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \\ A(q) &:= \sum_{n=1}^{\infty} a(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^{12}, \\ I(q) &:= \sum_{n=1}^{\infty} i(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1 - q^n)^{12}(1 + q^n)^{20}}{(1 + q^{2n})^4}. \end{aligned}$$

Theorem 1.1. Let $n \in \mathbb{N}$ be an even integer. Then

$$i(n) = 128a\left(\frac{n}{2}\right).$$

For $q \in \mathbb{C}$ satisfying $|q| < 1$, the Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are

$$(2) \quad L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$(3) \quad M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n,$$

see [1, p. 318]. Now we need the following identities which can be found in Lahiri [8, p. 149]

$$\begin{aligned}
L^2(q) &= 1 - 288 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \\
M^2(q) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n, \\
(4) \quad M^3(q) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{432000}{691} \sum_{n=1}^{\infty} \tau(n)q^n, \\
(5) \quad L(q)M(q) &= 1 + 720 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \\
L(q)M^2(q) &= 1 + 720 \sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n, \\
(6) \quad N^2(q) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n - \frac{762048}{691} \sum_{n=1}^{\infty} \tau(n)q^n.
\end{aligned}$$

Then it was shown that

$$(7) \quad \Delta(q) = \frac{1}{1728} (M(q)^3 - N(q)^2)$$

by Ramanujan. And he gave in his notebook the following formulae, which are proved in [2, p. 126–129] :

$$(8) \quad L(q) = (1 - 5x)w^2 + 12x(1 - x)w \frac{dw}{dx},$$

$$(9) \quad M(q) = (1 + 14x + x^2)w^4,$$

$$(10) \quad N(q) = (1 + x)(1 - 34x + x^2)w^6,$$

$$(11) \quad L(q^2) = (1 - 2x)w^2 + 6x(1 - x)w \frac{dw}{dx},$$

$$M(q^2) = (1 - x + x^2)w^4,$$

$$N(q^2) = (1 + x)(1 - \frac{1}{2}x)(1 - 2x)w^6,$$

$$(12) \quad L(q^4) = (1 - \frac{5}{4}x)w^2 + 3x(1 - x)w \frac{dw}{dx},$$

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$$(13) \quad M(q^4) = (1 - x + \frac{1}{16}x^2)w^4,$$

$$N(q^4) = (1 - \frac{1}{2}x)(1 - x - \frac{1}{32}x^2)w^6,$$

where for $0 < x < 1$, w is defined by

$$w = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n}^2 x^n.$$

From (7), (9), and (10), we obtain

$$(14) \quad \Delta(q) = \frac{x(1-x)^4 w^{12}}{2^4}.$$

Applying the principle of duplication (see [2, p. 125])

$$q \rightarrow q^2, \quad x \rightarrow \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2, \quad w \rightarrow \left(\frac{1 + \sqrt{1-x}}{2} \right) w$$

to (14), we deduce that

$$(15) \quad \Delta(q^2) = \frac{x^2(1-x)^2 w^{12}}{2^8}.$$

Again applying the principle of duplication to (15) and (13) respectively, we have

$$(16) \quad \Delta(q^4) = \frac{x^4(1-x)w^{12}}{2^{16}},$$

$$(17) \quad L(q^8) = \left(\frac{5}{8} - \frac{11}{16}x + \frac{3}{8}\sqrt{1-x} \right) w^2 + \frac{3}{2}x(1-x)w \frac{dw}{dx},$$

and

$$(18) \quad \begin{aligned} M(q^8) &= \left(\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1-x} - \frac{15}{64}x\sqrt{1-x} \right) w^4 \\ &= -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4. \end{aligned}$$

In [10, Theorem 1] for all $n \in \mathbb{N}$ we can see that

$$\begin{aligned} & \sum_{m < \frac{n}{8}} \sigma_1(m) \sigma_1(n - 8m) \\ &= \frac{1}{192} \sigma_3(n) + \frac{1}{64} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{16} \sigma_3\left(\frac{n}{4}\right) + \frac{1}{3} \sigma_8\left(\frac{n}{8}\right) + \left(\frac{1}{24} - \frac{n}{32}\right) \sigma_1(n) \\ &+ \left(\frac{1}{24} - \frac{n}{4}\right) \sigma_1\left(\frac{n}{8}\right) - \frac{1}{64} k(n) \end{aligned}$$

with

$$(19) \quad K(q) := \sum_{n=1}^{\infty} k(n) q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \frac{x\sqrt{1-x}w^4}{2^4}.$$

Moreover in [6, Theorem 3.2],[4, Theorem 1.2] we can find that

$$\begin{aligned} & \sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_3(n - 8m) \\ &= \frac{1}{8355840} \left\{ 16\sigma_7(n) + 240\sigma_7\left(\frac{n}{2}\right) + 3840\sigma_7\left(\frac{n}{4}\right) + 65536\sigma_7\left(\frac{n}{8}\right) - 34816\sigma_3(n) \right. \\ &\quad - 34816\sigma_3\left(\frac{n}{8}\right) + 18480b(n) + 197760b\left(\frac{n}{2}\right) - 3624960b\left(\frac{n}{4}\right) + 1020g(n) \\ &\quad \left. - 255h(n) \right\}, \end{aligned}$$

$$\begin{aligned} & \sum_{m < \frac{n}{8}} \sigma_1(m) \sigma_5(n - 8m) \\ &= \frac{1}{2193408} \left\{ 1344\sigma_7(n) + 4032\sigma_7\left(\frac{n}{2}\right) + 16128\sigma_7\left(\frac{n}{4}\right) + 65536\sigma_7\left(\frac{n}{8}\right) \right. \\ &\quad - 22848(n-4)\sigma_5(n) + 4352\sigma_1\left(\frac{n}{8}\right) - 1071h(n) - 2142g(n) \\ &\quad \left. - 35616b(n) - 1421952b\left(\frac{n}{2}\right) - 18665472b\left(\frac{n}{4}\right) \right\}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{m < \frac{n}{8}} \sigma_5(m) \sigma_1(n - 8m) \\ &= \frac{1}{280756224} \left\{ 32\sigma_7(n) + 2016\sigma_7\left(\frac{n}{2}\right) + 129024\sigma_7\left(\frac{n}{4}\right) + 11010048\sigma_7\left(\frac{n}{8}\right) \right. \\ &\quad - 11698176(2n-1)\sigma_5\left(\frac{n}{8}\right) + 557056\sigma_1(n) + 7497h(n) - 17136g(n) \\ &\quad \left. - 282912b(n) + 446208b\left(\frac{n}{2}\right) + 122142720b\left(\frac{n}{4}\right) \right\}, \end{aligned}$$

where

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$$(20) \quad \begin{aligned} B(q) &:= \sum_{n=1}^{\infty} b(n)q^n = q \prod_{n=1}^{\infty} (1-q^n)^8(1-q^{2n})^8 = \frac{x(1-x)^2 w^8}{2^4}, \\ G(q) &:= \sum_{n=1}^{\infty} g(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}} = x\sqrt{1-xw^8}, \\ H(q) &:= \sum_{n=1}^{\infty} h(n)q^n = 2^{12}q^3 \prod_{n=1}^{\infty} (1+q^{2n})^4(1-q^{4n})^{16} = x^3\sqrt{1-xw^8}. \end{aligned}$$

In succession we consider the similar forms like the above results and obtain as follows :

Theorem 1.2. Let $q \in \mathbb{C}$ satisfy $|q| < 1$. Then we have

(a)

$$\begin{aligned} L(q)M(q^8) &= \frac{1}{5376}N(q) + \frac{5}{1792}N(q^2) + \frac{5}{112}N(q^4) - \frac{148}{21}N(q^8) \\ &\quad + 8L(q)M(q) - \frac{45}{64}I(q) - \frac{405}{32}A(q) + \frac{45}{2}A(q^2), \end{aligned}$$

(b)

$$\begin{aligned} L(q^8)M(q) &= -\frac{37}{336}N(q) + \frac{5}{112}N(q^2) + \frac{5}{28}N(q^4) + \frac{16}{21}N(q^8) + \frac{1}{8}L(q)M(q) \\ &\quad + \frac{45}{8}I(q) + \frac{135}{2}A(q) + 90A(q^2). \end{aligned}$$

Theorem 1.3. Let $n \in \mathbb{N}$. Then we have

(a)

$$\begin{aligned} \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1(n-8m) &= \frac{1}{61440}\sigma_5(n) + \frac{1}{4096}\sigma_5(\frac{n}{2}) + \frac{1}{256}\sigma_5(\frac{n}{4}) + \frac{1}{12}\sigma_5(\frac{n}{8}) \\ &\quad - \frac{1}{24}(3n-1)\sigma_3(\frac{n}{8}) - \frac{1}{240}\sigma_1(n) + \frac{1}{8192}i(n) \\ &\quad + \frac{9}{4096}a(n) - \frac{1}{256}a(\frac{n}{2}), \end{aligned}$$

(b)

$$\begin{aligned} \sum_{m < \frac{n}{8}} \sigma_1(m)\sigma_3(n-8m) &= \frac{1}{768}\sigma_5(n) + \frac{1}{256}\sigma_5(\frac{n}{2}) + \frac{1}{64}\sigma_5(\frac{n}{4}) + \frac{1}{15}\sigma_5(\frac{n}{8}) \\ &\quad - \frac{1}{192}(3n-8)\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{8}) - \frac{1}{1024}i(n) \\ &\quad - \frac{3}{256}a(n) - \frac{1}{64}a(\frac{n}{2}). \end{aligned}$$

2. PROOFS THEOREM 1.1, THEOREM 1.2, AND THEOREM 1.3

Using (1) and (15) we have

$$(21) \quad A(q) = \frac{x(1-x)w^6}{2^4}$$

(refer to [3, (4.1)]). Applying the principle of duplication to (21) we obtain

$$\begin{aligned}
(22) \quad A(q^2) &= \frac{1}{2^4} \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 \left\{ 1 - \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 \right\} \left(\frac{1 + \sqrt{1-x}}{2} \right)^6 w^6 \\
&= \frac{1}{2^{10}} \cdot \frac{(1 - \sqrt{1-x})^2}{(1 + \sqrt{1-x})^2} \cdot \frac{(1 + \sqrt{1-x})^2 - (1 - \sqrt{1-x})^2}{(1 + \sqrt{1-x})^2} \\
&\quad \times (1 + \sqrt{1-x})^6 w^6 \\
&= \frac{1}{2^{10}} (1 - \sqrt{1-x})^2 \cdot 4\sqrt{1-x} \cdot (1 + \sqrt{1-x})^2 w^6 \\
&= \frac{1}{2^{10}} \cdot x^2 \cdot 4\sqrt{1-x} w^6 \\
&= \frac{x^2 \sqrt{1-x} w^6}{2^8}.
\end{aligned}$$

From (19) and (20) we deduce that

$$\begin{aligned}
I(q) &= \sqrt{2^4 K(q) G(q)} \\
&= \left\{ 2^4 q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 \cdot 2^4 q \prod_{n=1}^{\infty} \frac{(1 + q^n)^{32} (1 - q^n)^{16}}{(1 + q^{2n})^{12}} \right\}^{\frac{1}{2}} \\
&= 2^4 q \prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 - q^{2n})^2 (1 + q^{2n})^2 \cdot \frac{(1 + q^n)^{16} (1 - q^n)^8}{(1 + q^{2n})^6} \\
&= 2^4 q \prod_{n=1}^{\infty} (1 - q^n)^4 (1 + q^n)^4 \cdot \frac{(1 + q^n)^{16} (1 - q^n)^8}{(1 + q^{2n})^4} \\
&= 2^4 q \prod_{n=1}^{\infty} \frac{(1 - q^n)^{12} (1 + q^n)^{20}}{(1 + q^{2n})^4}
\end{aligned}$$

and

$$\begin{aligned}
(23) \quad I(q) &= \sqrt{2^4 K(q) G(q)} = \left\{ 2^4 \cdot \frac{x \sqrt{1-x} w^4}{2^4} \cdot x \sqrt{1-x} w^8 \right\}^{\frac{1}{2}} \\
&= x \sqrt{1-x} w^6.
\end{aligned}$$

Proposition 2.1. (See [3]) For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$L(q)M(q^2) = 2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2),$$

(b)

$$M(q)L(q^2) = \frac{1}{2}L(q)M(q) - \frac{11}{42}N(q) + \frac{16}{21}N(q^2),$$

(c)

$$N(q)N(q^2) = -\frac{147}{520}M^3(q) - \frac{1176}{65}M^3(q^2) + \frac{31}{104}N^2(q) + \frac{248}{13}N^2(q^2),$$

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(d)

$$N(q)N(q^4) = -\frac{4851}{16640}M^3(q) - \frac{69237}{4160}M^3(q^2) - \frac{77616}{65}M^3(q^4) + \frac{971}{3328}N^2(q) + \frac{3465}{208}N^2(q^2) + \frac{15536}{13}N^2(q^4),$$

(e)

$$L(q)M(q^4) = 4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q),$$

(f)

$$L(q^4)M(q) = \frac{1}{4}L(q)M(q) - \frac{4}{21}N(q) + \frac{5}{28}N(q^2) + \frac{16}{21}N(q^4) + 90A(q).$$

Proposition 2.2. (See [10, p. 392]) Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have

(a)

$$w^4 = \frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4),$$

(b)

$$xw^4 = \frac{1}{15}M(q) - \frac{1}{15}M(q^2),$$

(c)

$$x^2w^4 = \frac{16}{15}M(q^2) - \frac{16}{15}M(q^4).$$

Lemma 2.3. Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have

(a)

$$w^6 = -\frac{1}{63}N(q) + \frac{64}{63}N(q^4) + 16A(q),$$

(b)

$$xw^6 = -\frac{1}{63}N(q) + \frac{1}{63}N(q^2) + 8A(q),$$

(c)

$$x^2w^6 = -\frac{1}{63}N(q) + \frac{1}{63}N(q^2) - 8A(q),$$

(d)

$$x^3w^6 = -\frac{2}{63}N(q) + \frac{22}{21}N(q^2) - \frac{64}{63}N(q^4) - 16A(q).$$

Proof. (a) First by (8) and (12) we know that

$$\begin{aligned} & 4L(q^4) - L(q) \\ &= 4 \left\{ \left(1 - \frac{5}{4}x\right)w^2 + 3x(1-x)w \frac{dw}{dx} \right\} - \left\{ (1-5x)w^2 + 12x(1-x)w \frac{dw}{dx} \right\} \\ &= 3w^2, \end{aligned}$$

which shows that

$$(24) \quad w^2 = \frac{4}{3}L(q^4) - \frac{1}{3}L(q).$$

Then from Proposition 2.2 (a) and (24) we deduce that

$$\begin{aligned} w^6 &= w^4 \cdot w^2 \\ &= \left(\frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4) \right) \left(\frac{4}{3}L(q^4) - \frac{1}{3}L(q) \right) \\ &= \frac{4}{45}M(q)L(q^4) - \frac{1}{45}M(q)L(q) - \frac{8}{45}M(q^2)L(q^4) + \frac{2}{45}M(q^2)L(q) \\ &\quad + \frac{64}{45}M(q^4)L(q^4) - \frac{16}{45}M(q^4)L(q) \end{aligned}$$

and so we use (5), Proposition 2.1 (a), (b), (e), and (f).

- (b) It is obvious by Proposition 2.2 (b) and (24).
- (c) It is definite by Proposition 2.2 (c) and (24).
- (d) Let us consider (10) as follows :

$$\begin{aligned} N(q) &= (1+x)(1-34x+x^2)w^6 = (1-33x-33x^2+x^3)w^6 \\ &= w^6 - 33xw^6 - 33x^2w^6 + x^3w^6 \end{aligned}$$

and so we refer to Lemma 2.3 (a), (b), and (c). \square

Proof of Theorem 1.2. (a) By (8), (12), (18), and (23) let us expand

$$\begin{aligned} (25) \quad M(q^8)L(q) - 4M(q^8)L(q^4) &= \left(-\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4 \right) L(q) \\ &\quad - 4 \left(-\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4 \right) L(q^4) \\ &= -\frac{1}{32}M(q^2)L(q) + \frac{9}{16}M(q^4)L(q) + \frac{1}{8}M(q^2)L(q^4) - \frac{9}{4}M(q^4)L(q^4) \\ &\quad + \frac{15}{32}\sqrt{1-x}w^4(L(q) - 4L(q^4)) - \frac{15}{64}x\sqrt{1-x}w^4(L(q) - 4L(q^4)) \\ &= -\frac{1}{32}M(q^2)L(q) + \frac{9}{16}M(q^4)L(q) + \frac{1}{8}M(q^2)L(q^4) - \frac{9}{4}M(q^4)L(q^4) \\ &\quad + \frac{15}{32}\sqrt{1-x}w^4(-3w^2) - \frac{15}{64}x\sqrt{1-x}w^4(-3w^2) \\ &= -\frac{1}{32}M(q^2)L(q) + \frac{9}{16}M(q^4)L(q) + \frac{1}{8}M(q^2)L(q^4) - \frac{9}{4}M(q^4)L(q^4) \\ &\quad - \frac{45}{32}\sqrt{1-x}w^6 + \frac{45}{64}I(q). \end{aligned}$$

In a similar manner from (8), (11), (18), (22), and (23) we can obtain

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$$\begin{aligned}
(26) \quad & M(q^8)L(q) - 2M(q^8)L(q^2) \\
&= \left(-\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4 \right) L(q) \\
&\quad - 2 \left(-\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4 \right) L(q^2) \\
&= -\frac{1}{32}M(q^2)L(q) + \frac{9}{16}M(q^4)L(q) + \frac{1}{16}M(q^2)L(q^2) - \frac{9}{8}M(q^4)L(q^2) \\
&\quad + \frac{15}{32}\sqrt{1-x}w^4(L(q) - 2L(q^2)) - \frac{15}{64}x\sqrt{1-x}w^4(L(q) - 2L(q^2)) \\
&= -\frac{1}{32}M(q^2)L(q) + \frac{9}{16}M(q^4)L(q) + \frac{1}{16}M(q^2)L(q^2) - \frac{9}{8}M(q^4)L(q^2) \\
&\quad - \frac{15}{32}\sqrt{1-x}w^4(1+x)w^2 + \frac{15}{64}x\sqrt{1-x}w^4(1+x)w^2 \\
&= -\frac{1}{32}M(q^2)L(q) + \frac{9}{16}M(q^4)L(q) + \frac{1}{16}M(q^2)L(q^2) - \frac{9}{8}M(q^4)L(q^2) \\
&\quad - \frac{15}{32}\sqrt{1-x}w^6 - \frac{15}{64}x\sqrt{1-x}w^6 + \frac{15}{64}\sqrt{1-x}x^2w^6 \\
&= -\frac{1}{32}M(q^2)L(q) + \frac{9}{16}M(q^4)L(q) + \frac{1}{16}M(q^2)L(q^2) - \frac{9}{8}M(q^4)L(q^2) \\
&\quad - \frac{15}{32}\sqrt{1-x}w^6 - \frac{15}{64}I(q) + \frac{15}{64} \cdot 2^8 A(q^2).
\end{aligned}$$

Then by (25) and (26) we have

$$\begin{aligned}
& 2M(q^8)L(q) - 6M(q^8)L(q^2) + 4M(q^8)L(q^4) \\
&= 3 \{ M(q^8)L(q) - 2M(q^8)L(q^2) \} - \{ M(q^8)L(q) - 4M(q^8)L(q^4) \} \\
&= -\frac{1}{16}M(q^2)L(q) + \frac{9}{8}M(q^4)L(q) + \frac{3}{16}M(q^2)L(q^2) - \frac{27}{8}M(q^4)L(q^2) \\
&\quad - \frac{1}{8}M(q^2)L(q^4) + \frac{9}{4}M(q^4)L(q^4) + 180A(q^2) - \frac{45}{32}I(q)
\end{aligned}$$

and so we refer to (5), Proposition 2.1 (a), (b), and (e).

(b) First by (25) we deduce that

$$\begin{aligned}
\sqrt{1-x}w^6 &= \frac{32}{45} \left\{ -M(q^8)L(q) + 4M(q^8)L(q^4) - \frac{1}{32}M(q^2)L(q) \right. \\
&\quad \left. + \frac{9}{16}M(q^4)L(q) + \frac{1}{8}M(q^2)L(q^4) - \frac{9}{4}M(q^4)L(q^4) + \frac{45}{64}I(q) \right\}
\end{aligned}$$

and so appealing to (5), Proposition 2.1 (a), (b), (e), and Theorem 1.2 (a), we write the above identity as

$$(27) \quad \sqrt{1-x}w^6 = \frac{1}{63}N(q^2) - \frac{22}{21}N(q^4) + \frac{128}{63}N(q^8) + I(q) - 16A(q^2).$$

Second from (8), (9), and (17) we observe that

$$\begin{aligned}
& (L(q) - 8L(q^8)) M(q) \\
&= L(q)M(q) - 8L(q^8)M(q) \\
&= \left(-4 + \frac{1}{2}x - 3\sqrt{1-x} \right) w^2 \cdot (1 + 14x + x^2)w^4 \\
&= -4w^6 - \frac{111}{2}xw^6 + 3x^2w^6 + \frac{1}{2}x^3w^6 - 3\sqrt{1-x}w^6 - 42x\sqrt{1-x}w^6 \\
&\quad - 3x^2\sqrt{1-x}w^6
\end{aligned}$$

and so we apply (22), (23), (5), Lemma 2.3, and (27). \square

Proof of Theorem 1.3. (a) By (2) and (3) we note that

$$\begin{aligned}
& 24 \cdot 240 \sum_{N=1}^{\infty} \left(\sum_{m < \frac{N}{8}} \sigma_3(m)\sigma_1(N-8m) \right) q^N \\
&= \left(24 \sum_{n=1}^{\infty} \sigma_1(n)q^n \right) \left(240 \sum_{m=1}^{\infty} \sigma_3(m)q^{8m} \right) \\
&= (1 - L(q)) (M(q^8) - 1) \\
&= M(q^8) - 1 - L(q)M(q^8) + L(q)
\end{aligned}$$

and so we use Theorem 1.2 (a).

(b) From (2) and (3) we know that

$$\begin{aligned}
& 24 \cdot 240 \sum_{N=1}^{\infty} \left(\sum_{m < \frac{N}{8}} \sigma_1(m)\sigma_3(N-8m) \right) q^N \\
&= \left(240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right) \left(24 \sum_{m=1}^{\infty} \sigma_1(m)q^{8m} \right) \\
&= (M(q) - 1) (1 - L(q^8)) \\
&= M(q) - M(q)L(q^8) - 1 + L(q^8)
\end{aligned}$$

and so we refer to Theorem 1.2 (b). \square

The following relation involving divisor functions

$$(28) \quad \sigma_k(pn) - \left(p^k + 1 \right) \sigma_k(n) + p^k \sigma_k\left(\frac{n}{p}\right) = 0$$

for a prime p and $k, n \in \mathbb{N}$ is given in [9, Theorem 3.1(ii)].

Proof of Theorem 1.1. We put an even positive integer $n = 2L$ for $L \in \mathbb{N}$ in the following identity and so by (28) we have

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$$\begin{aligned}
& \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1(n-8m) \\
&= \sum_{m < \frac{L}{4}} \sigma_3(m)\sigma_1(2L-8m) \\
(29) \quad &= \sum_{m < \frac{L}{4}} \sigma_3(m) \left\{ 3\sigma_1(L-4m) - 2\sigma_1\left(\frac{L-4m}{2}\right) \right\} \\
&= 3 \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1\left(\frac{n}{2}-4m\right) - 2 \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1\left(\frac{n}{4}-2m\right).
\end{aligned}$$

Then applying

$$\begin{aligned}
& \sum_{m < \frac{n}{2}} \sigma_3(m)\sigma_1(n-2m) \\
&= \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5\left(\frac{n}{2}\right) + \frac{(1-3n)}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_1(n)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m < \frac{n}{4}} \sigma_3(m)\sigma_1(n-4m) &= \frac{1}{3840}\sigma_5(n) + \frac{1}{256}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{12}\sigma_5\left(\frac{n}{4}\right) \\
&\quad + \frac{(1-3n)}{24}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{240}\sigma_1(n) + \frac{1}{256}a(n)
\end{aligned}$$

in [3] into Eq. (29) we obtain

$$\begin{aligned}
(30) \quad & \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1(n-8m) \\
&= \frac{1}{1280}\sigma_5\left(\frac{n}{2}\right) + \frac{13}{3840}\sigma_5\left(\frac{n}{4}\right) + \frac{1}{12}\sigma_5\left(\frac{n}{8}\right) - \frac{1}{24}(3n-1)\sigma_3\left(\frac{n}{8}\right) - \frac{1}{80}\sigma_1\left(\frac{n}{2}\right) \\
&\quad + \frac{1}{120}\sigma_1\left(\frac{n}{4}\right) + \frac{3}{256}a\left(\frac{n}{2}\right).
\end{aligned}$$

Since Eq. (30) is the same as Theorem 1.3 (a) for even n , therefore we equate them and have

$$\begin{aligned}
(31) \quad & -\frac{1}{61440}\sigma_5(n) + \frac{11}{20480}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{1920}\sigma_5\left(\frac{n}{4}\right) + \frac{1}{240}\sigma_1(n) - \frac{1}{80}\sigma_1\left(\frac{n}{2}\right) \\
& + \frac{1}{120}\sigma_1\left(\frac{n}{4}\right) - \frac{1}{8192}i(n) - \frac{9}{4096}a(n) + \frac{1}{64}a\left(\frac{n}{2}\right) = 0.
\end{aligned}$$

Finally by (1) and (28) we obtain

$$a(n) = 0, \quad \sigma_5(n) = 33\sigma_5\left(\frac{n}{2}\right) - 32\sigma_5\left(\frac{n}{4}\right), \quad \text{and} \quad \sigma_1(n) = 3\sigma_1\left(\frac{n}{2}\right) - 2\sigma_1\left(\frac{n}{4}\right)$$

for even n and so applying these facts into (31) we conclude the proof.

□

Lemma 2.4. *Let $q \in \mathbb{C}$ with $|q| < 1$. We define $F(q)$ by*

$$F(q) := \sum_{n=1}^{\infty} f(n)q^n = q^5 \prod_{n=1}^{\infty} (1 - q^n)^{24}(1 + q^n)^{32}(1 + q^{2n})^{32} = \frac{x^5 w^{12}}{2^{20}}.$$

Then we have

(a)

$$x^5 w^{12} = 1048576 F(q),$$

(b)

$$x^4 w^{12} = 65536 \Delta(q^4) + 1048576 F(q),$$

(c)

$$\begin{aligned} x^3 w^{12} = & \frac{1}{7020} M^3(q) + \frac{16}{1755} M^3(q^2) + \frac{1}{9828} N^2(q) - \frac{23}{2457} N^2(q^2) \\ & - 64 \Delta(q^2), \end{aligned}$$

(d)

$$\begin{aligned} x^2 w^{12} = & \frac{1}{3510} M^3(q) + \frac{32}{1755} M^3(q^2) + \frac{1}{4914} N^2(q) - \frac{46}{2457} N^2(q^2) \\ & + 128 \Delta(q^2) - 65536 \Delta(q^4) - 1048576 F(q), \end{aligned}$$

(e)

$$\begin{aligned} x w^{12} = & \frac{1}{3510} M^3(q) + \frac{32}{1755} M^3(q^2) + \frac{1}{4914} N^2(q) - \frac{46}{2457} N^2(q^2) \\ & + 16 \Delta(q) + 896 \Delta(q^2) - 1048576 F(q), \end{aligned}$$

(f)

$$\begin{aligned} w^{12} = & \frac{11}{1170} M^3(q) + \frac{314}{585} M^3(q^2) + \frac{22528}{585} M^3(q^4) - \frac{5}{546} N^2(q) \\ & - \frac{440}{819} N^2(q^2) - \frac{10240}{273} N^2(q^4) + 32 \Delta(q) + 1280 \Delta(q^2) \\ & + 131072 \Delta(q^4). \end{aligned}$$

Proof. (a) Proof is obvious by the definition of $F(q)$.

(b) We expand $\Delta(q^4)$ in (16) as

$$\Delta(q^4) = \frac{x^4(1-x)w^{12}}{2^{16}} = \frac{x^4 w^{12}}{2^{16}} - \frac{x^5 w^{12}}{2^{16}}$$

then we use Lemma 2.4 (a).

(c) In [5, Theorem 2.6(e)] we see that

$$(32) \quad A^2(q) = \Delta(q^2).$$

And by Lemma 2.3 (b) and (c) we observe that

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$$\begin{aligned} x^3w^{12} &= x^2w^6 \cdot xw^6 \\ &= \left(-\frac{1}{63}N(q) + \frac{1}{63}N(q^2) - 8A(q)\right) \left(-\frac{1}{63}N(q) + \frac{1}{63}N(q^2) + 8A(q)\right) \\ &= \frac{1}{3969}N^2(q) - \frac{2}{3969}N(q)N(q^2) + \frac{1}{3969}N^2(q^2) - 64A^2(q) \end{aligned}$$

and so we use Proposition 2.1 (c) and (32).

(d) Let us consider Eq. (15),

$$\Delta(q^2) = \frac{x^2(1-x)^2w^{12}}{2^8} = \frac{x^2w^{12}}{2^8} - \frac{x^3w^{12}}{2^7} + \frac{x^4w^{12}}{2^8}$$

thus we apply Lemma 2.4 (b) and (c).

(e) From (14) we have

$$\Delta(q) = \frac{x(1-x)^4w^{12}}{2^4} = \frac{x^5w^{12}}{2^4} - \frac{x^4w^{12}}{2^2} + \frac{3x^3w^{12}}{2^3} - \frac{x^2w^{12}}{2^2} + \frac{xw^{12}}{2^4}$$

therefore we appeal to Lemma 2.4 (a), (b), (c) and (d).

(f) First owing to (4) and (6) we deduce that

$$(33) \quad \sum_{n=1}^{\infty} \sigma_{11}(n)q^n = -\frac{691}{65520} + \frac{7}{1040}M^3(q) + \frac{25}{6552}N^2(q).$$

Also in [7, Remark 4.1], [5, Theorem 2.5(l)] we can see that

$$\begin{aligned} N(q)A(q) &= -\frac{66}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{66}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} + \frac{757}{691} \sum_{n=1}^{\infty} \tau(n)q^n \\ &\quad - \frac{194928}{691} \sum_{n=1}^{\infty} \tau(n)q^{2n} + 278528 \sum_{n=1}^{\infty} \tau(n)q^{4n} \\ &\quad + 4325376 \sum_{n=1}^{\infty} f(n)q^n \end{aligned}$$

and

$$\begin{aligned} N(q^4)A(q) &= -\frac{33}{22112} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{33}{22112} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} \\ &\quad + \frac{22145}{22112} \sum_{n=1}^{\infty} \tau(n)q^n + \frac{74883}{2764} \sum_{n=1}^{\infty} \tau(n)q^{2n} \\ &\quad + 8384 \sum_{n=1}^{\infty} \tau(n)q^{4n} + 67584 \sum_{n=1}^{\infty} f(n)q^n. \end{aligned}$$

By applying (33) into the above two equations respectively, we have

$$\begin{aligned}
(34) \quad N(q)A(q) = & -\frac{231}{359320}M^3(q) + \frac{231}{359320}M^3(q^2) - \frac{275}{754572}N^2(q) \\
& + \frac{275}{754572}N^2(q^2) + \frac{757}{691}\Delta(q) - \frac{194928}{691}\Delta(q^2) \\
& + 278528\Delta(q^4) + 4325376F(q)
\end{aligned}$$

and

$$\begin{aligned}
(35) \quad N(q^4)A(q) = & -\frac{231}{22996480}M^3(q) + \frac{231}{22996480}M^3(q^2) - \frac{275}{48292608}N^2(q) \\
& + \frac{275}{48292608}N^2(q^2) + \frac{22145}{22112}\Delta(q) + \frac{74883}{2764}\Delta(q^2) \\
& + 8384\Delta(q^4) + 67584F(q).
\end{aligned}$$

Second from Lemma 2.3 (a) we obtain

$$\begin{aligned}
w^{12} &= w^6 \cdot w^6 \\
&= \left(-\frac{1}{63}N(q) + \frac{64}{63}N(q^4) + 16A(q) \right) \left(-\frac{1}{63}N(q) + \frac{64}{63}N(q^4) + 16A(q) \right) \\
&= \frac{1}{63^2}N^2(q) - \frac{2 \cdot 64}{63^2}N(q)N(q^4) - \frac{32}{63}N(q)A(q) + \frac{2 \cdot 16 \cdot 64}{63}N(q^4)A(q) \\
&\quad + \frac{64^2}{63^2}N^2(q^4) + 16^2A^2(q)
\end{aligned}$$

therefore we refer to Proposition 2.1 (d), (32), (34), and (35). \square

Corollary 2.5. Let $q \in \mathbb{C}$ with $|q| < 1$. Then

$$\begin{aligned}
I^2(q) = & \frac{1}{7020}M^3(q) + \frac{16}{1755}M^3(q^2) + \frac{1}{9828}N^2(q) - \frac{23}{2457}N^2(q^2) + 192\Delta(q^2) \\
& - 65536\Delta(q^4) - 1048576F(q).
\end{aligned}$$

Proof. From (23) we deduce that

$$I^2(q) = (x\sqrt{1-x}w^6)^2 = x^2w^{12} - x^3w^{12}$$

and so we use Lemma 2.4 (c) and (d). \square

3. APPENDIX

The first eighteen values of $\tau(n)$ are given in the Table 1,

The convolution sums $\sum_{m < \frac{n}{8}} \sigma_1(m)\sigma_3(n-8m)$ and $\sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_1(n-8m)$

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n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	7	-16744	13	-577738
2	-24	8	84480	14	401856
3	252	9	-113643	15	1217160
4	-1472	10	-115920	16	987136
5	4830	11	534612	17	-6905934
6	-6048	12	-370944	18	2727432

TABLE 1. $\tau(n)$ for n ($1 \leq n \leq 18$)

similarly the first eighteen values of $a(n)$ and $i(n)$ are listed in the following tables.

n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	7	-88	13	-418
2	0	8	0	14	0
3	-12	9	-99	15	-648
4	0	10	0	16	0
5	54	11	540	17	594
6	0	12	0	18	0

TABLE 2. $a(n)$ for n ($1 \leq n \leq 18$)

n	$i(n)$	n	$i(n)$	n	$i(n)$
1	16	7	-384	13	7648
2	128	8	0	14	-11264
3	320	9	2512	15	-23680
4	0	10	6912	16	0
5	-1184	11	1984	17	-19168
6	-1536	12	0	18	-12672

TABLE 3. $i(n)$ for n ($1 \leq n \leq 18$)

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