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A NOTE ON CARLITZ'S TYPE q-CHANGHEE NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we consider the Carlitz's type q-analogue of Changhee numbers and polynomials and we give some explicit formulae for these numbers and polynomials.

1. Introduction

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of number x is defined as $[x]_q = \frac{q^x-1}{q-1}$. As is well known, the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see } [1-14]). \tag{1.1}$$

When $x = 0$, $E_n = E_n(0)$, $(n \geq 0)$, are called the Euler numbers. In [1,2,3] L. Carlitz considered the q-analogue of Euler numbers which are given by the recurrence relation as follows:

$$\mathcal{E}_{0,q} = 1,\ q(q\mathcal{E}_q + 1)^n + \mathcal{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing $\mathcal{E}_q^n$ by $\mathcal{E}_{n,q}$.

He also considered q-Euler polynomials which are defined by

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$$\mathcal{E}_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\mathcal{E}_{l,q},\quad \text{(see [2, 3])}. \tag{1.2}$$

In [8,9,10], Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(f)=\int_{\mathbb{Z}_p}f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1}f(x)(-q)^x, \tag{1.3}$$

where $f(x)$ is continuous function on $\mathbb{Z}_p$ and $[x]_{-q}=\frac{1-(-q)^x}{1+q}$.

From (1.3), he derived the following formula for the Carlitz's q-Euler polynomials:

$$\int_{\mathbb{Z}_p}[x+y]_q^n d\mu_{-q}(y)=\mathcal{E}_{n,q}(x),\ (n\geq 0),\quad \text{(see [7, 10])}. \tag{1.4}$$

When $x=0$, $\mathcal{E}_{n,q}=\int_{\mathbb{Z}_p}[x]_q^n d\mu_{-q}(x)$ are Carlitz's q-Euler numbers.

The Changhee polynomials are defined by the generating function to be

$$\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!},\quad \text{(see [5, 6])}. \tag{1.5}$$

Thus, by (1.5), we get

$$E_n(x)=\sum_{k=0}^{n}S_2(n,k)Ch_k(x),\ Ch_n(x)=\sum_{k=0}^{n}S_1(n,k)E_k(x),\ (n\geq 0), \tag{1.6}$$

where $S_2(n,k)$ is Stirling number of the second kind and $S_1(n,k)$ is the Stirling number of the first kind. In [10], the higher-order Carlitz's q-Euler polynomials are written by the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(r)}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[x_1+\cdots+x_r+x]_q t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r),\ (n\geq 0). \tag{1.7}$$

In this paper, we consider the Carlitz's type q-Changhee polynomials and numbers and we give explicit formulas for these numbers and polynomials.

2. Carlitz's type q-Changhee polynomials

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. From (1.3) and (1.5), we note that

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \quad \text{(see [4, 5, 6])}. \quad (2.1)$$

Thus, by (2.1), we get

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = Ch_n(x), \ (n \geq 0), \quad (2.2)$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

In the viewpoint of (1.4), we consider the Carlitz's type q-Changhee polynomials which are derived from the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}. \quad (2.3)$$

Thus, by (2.3), we get

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!} &= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) \frac{1}{k!}\Big(\log(1+t)\Big)^k \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{E}_{k,q}(x) S_1(n,k) \right) \frac{t^n}{n!}.\end{aligned} \quad (2.4)$$

Indeed,

$$\begin{aligned}\sum_{k=0}^{n} S_1(n,k) \int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) &= \sum_{k=0}^{n} S_1(n,k) \frac{1}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l} q^{lx}(-1)^l \frac{[2]_q}{1+q^{l+1}} \\ &= [2]_q \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{1}{(1-q)^k}\binom{k}{l} q^{lx}(-1)^l \frac{S_1(n,k)}{1+q^{l+1}}.\end{aligned}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\begin{aligned} Ch_{n,q}(x) &= [2]_q \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{1}{(1-q)^k} \binom{k}{l} q^{lx} (-1)^l \frac{S_1(n,k)}{1+q^{l+1}} \\ &= \sum_{k=0}^{n} S_1(n,k) \mathcal{E}_{n,q}(x). \end{aligned}$$

From (1.4), we note that

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y). \tag{2.5}$$

By (2.5), we get

$$\sum_{k=0}^{\infty} Ch_{k,q}(x) \frac{1}{k!} \left(e^t - 1\right)^k = \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{n!}. \tag{2.6}$$

On the other hand,

$$\begin{aligned} \sum_{k=0}^{\infty} Ch_{k,q}(x) \frac{1}{k!} \left(e^t - 1\right)^k &= \sum_{k=0}^{\infty} Ch_{k,q}(x) \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} Ch_{k,q}(x) S_2(n,k) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

Thus, by (2.6) and (2.7), we get the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,q}(x) = \sum_{k=0}^{n} Ch_{k,q}(x) S_2(n,k).$$

From Theorem 2.1, we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}\\
&=[2]_q\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} S_1(n,k)\frac{1}{(1-q)^k}\sum_{l=0}^{k}\binom{k}{l}q^{lx}(-1)^l\sum_{m=0}^{\infty}\Big(-q^{l+1}\Big)^m\right)\frac{t^n}{n!}\\
&=[2]_q\sum_{m=0}^{\infty}(-q)^m\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_1(n,k)[m+x]_q^k\right)\frac{t^n}{n!} \qquad (2.8)\\
&=[2]_q\sum_{m=0}^{\infty}(-q)^m\sum_{n=0}^{\infty}\binom{[m+x]_q}{n}t^n\\
&=[2]_q\sum_{m=0}^{\infty}(-q)^m(1+t)^{[m+x]_q}.
\end{aligned}$$

Therefore, by (2.8), we obtain the following theorem.

Theorem 2.3. *The generating function of the Carlitz's type q-Changhee polynomials is given by*

$$[2]_q\sum_{m=0}^{\infty}(-q)^m(1+t)^{[m+x]_q}=\sum_{n=0}^{\infty}Ch_{n,q}(x)\frac{t^n}{n!}.$$

In particular, for $x=0$, we have

$$[2]_q\sum_{m=0}^{\infty}(-q)^m(1+t)^{[m]_q}=\sum_{n=0}^{\infty}Ch_{n,q}\frac{t^n}{n!}.$$

From (1.3), we easily note that

$$qI_{-q}(f_1)+I_{-q}(f)=[2]_qf(0),\ \text{where}\ f_1(x)=f(x+1). \qquad (2.9)$$

Thus, by (2.9), we get

$$q\int_{\mathbb{Z}_p}(1+t)^{[x+1+y]_q}d\mu_{-q}(y)+\int_{\mathbb{Z}_p}(1+t)^{[x+y]_q}d\mu_{-q}(y)=[2]_q(1+t)^{[x]_q}. \qquad (2.10)$$

By (2.3) and (2.10), we get

$$\sum_{n=0}^{\infty}\Big(qCh_{n,q}(x+1)+Ch_{n,q}(x)\Big)\frac{t^n}{n!}=[2]_q\sum_{n=0}^{\infty}\Big([x]_q\Big)_n\frac{t^n}{n!}. \qquad (2.11)$$

Comparing the coefficients on the both sides of (2.11), we get

$$qCh_{n,q}(x+1)+Ch_{n,q}(x)=[2]_q\Big([x]_q\Big)_n=[2]_q\sum_{l=0}^{n}S_1(n,l)[x]_q^l,\ (n\geq 0). \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $n\geq 0$, we have*

$$qCh_{n,q}(x+1)+Ch_{n,q}(x)=[2]_q\sum_{l=0}^{n}S_1(n,l)[x]_q^l.$$

From (2.3), we have

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{[x+y]_q}{n}d\mu_{-q}(y)t^n=\sum_{n=0}^{\infty}\frac{Ch_{n,q}(x)}{n!}t^n. \tag{2.13}$$

Thus, by (2.13), we get

$$\int_{\mathbb{Z}_p}\binom{[x+y]_q}{n}d\mu_{-q}(y)=\frac{Ch_{n,q}(x)}{n!},\ (n\geq 0).$$

Now, we observe that

$$(1+t)^{[x+y]_q}=(1+t)^{[x]_q+q^x[y]_q}=(1+t)^{[x]_q}\cdot(1+t)^{q^x[y]_q}. \tag{2.14}$$

Thus, by (2.14), we get

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_{n,q}(x)\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}(1+t)^{[x+y]_q}d\mu_{-q}(y)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}S_1(n,k)\int_{\mathbb{Z}_p}\Big([x]_q+q^x[y]_q\Big)^k d\mu_{-q}(y)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}S_1(n,k)[x]_q^{k-l}q^{lx}\mathcal{E}_{l,q}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.15}$$

Therefore, by (2.15), we obtain the following theorem.

Theorem 2.5. *For $n\geq 0$, we have*

$$Ch_{n,q}(x)=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}S_1(n,k)[x]_q^{k-l}q^{lx}\mathcal{E}_{l,q}.$$

From (1.3), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x \\ &= \lim_{N\to\infty}\frac{1}{[dp^N]_{-q}}\sum_{x=0}^{dp^N-1} f(x)(-q)^x,\end{aligned} \tag{2.16}$$

where $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. For $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, we have

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(y) = \lim_{N\to\infty}\frac{1}{[dp^N]_{-q}}\sum_{a=0}^{d-1}\sum_{x=0}^{p^N-1} f(a+dx)(-q)^{a+dx}. \tag{2.17}$$

By (2.17), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+t)^{[x+y]_q}d\mu_{-q}(y)\\ &= \frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}(-q)^a\int_{\mathbb{Z}_p}(1+t)^{[d]_q\left[\frac{a+x}{d}+y\right]_{q^d}}d\mu_{-q^d}(y)\\ &= \frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}(-q)^a\sum_{k=0}^{\infty}[d]_q^k\int_{\mathbb{Z}_p}\left[\tfrac{a+x}{d}+y\right]_{q^d}^k d\mu_{-q}(y)\frac{1}{k!}\left(\log(1+t)\right)^k\\ &= \frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}(-q)^a\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}[d]_q^k\mathcal{E}_{k,q^d}\left(\tfrac{a+x}{d}\right)S_1(n,k)\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}\sum_{k=0}^{n}(-q)^a[d]_q^k\mathcal{E}_{k,q^d}\left(\tfrac{a+x}{d}\right)S_1(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.18}$$

Therefore, by (2.3) and (2.18), we obtain the following theorem.

Theorem 2.6. *For $n\geq 0$, we have*

$$Ch_{n,q}(x) = \frac{[2]_q}{[2]_{q^d}}\sum_{k=0}^{n}[d]_q^k\left(\sum_{a=0}^{d-1}(-q)^a\mathcal{E}_{k,q^d}\left(\tfrac{a+x}{d}\right)S_1(n,k)\right).$$

For $r\in\mathbb{N}$, the higher-order Carlitz's type q-Changhee polynomials are also given by the multivariate fermionic p-adic q-integral as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{[x_1+x_2+\cdots+x_r+x]_q}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty}Ch_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.19}$$

Thus, we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{[x_1+x_2+\cdots+x_r+x]_q}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
&=\sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^k d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{1}{k!}\Big(\log(1+t)\Big)^k\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_1(n,k)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^k d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_1(n,k)\mathcal{E}_{k,q}^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.20}$$

By (2.19) and (2.20), we get

$$Ch_{n,q}^{(r)}(x)=\sum_{k=0}^{n}S_1(n,k)\mathcal{E}_{k,q}^{(r)}(x). \tag{2.21}$$

When $x=0$, $Ch_{n,q}^{(r)}=Ch_{n,q}^{(r)}(0)$ are called the Carlitz's type q-Changhee numbers.

By (1.7) and (2.19), we get

$$\begin{aligned}
\sum_{k=0}^{\infty}Ch_{k,q}^{(r)}(x)\frac{1}{k!}\Big(e^t-1\Big)^k&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[x_1+x_2+\cdots+x_r+x]_q t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
&=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(r)}(x).
\end{aligned} \tag{2.22}$$

On the other hand,

$$\begin{aligned}
\sum_{k=0}^{\infty}Ch_{k,q}^{(r)}(x)\frac{1}{k!}\Big(e^t-1\Big)^k&=\sum_{k=0}^{\infty}Ch_{k,q}^{(r)}(x)\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}Ch_{k,q}^{(r)}(x)S_2(n,k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.23}$$

Comparing the coefficients on the both sides of (2.22) and (2.23), we have

$$\mathcal{E}_{n,q}^{(r)}(x)=\sum_{k=0}^{n}Ch_{k,q}^{(r)}(x)S_2(n,k). \tag{2.24}$$

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