Advanced Studies in Contemporary Mathematics
27 (2017), No. 4, pp. 431 - 440

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2017.27.4.431

A NOTE ON DEGENERATE EULERIAN NUMBERS AND POLYNOMIALS

DAE SAN KIM AND TAEKYUN KIM

Abstract. In this paper, we study the degenerate Eulerian polynomials and numbers and give some new and interesting identities associated with several special numbers and polynomials.

1. Introduction

In combinatorics, the Eulerian number $\left\langle {n \atop m} \right\rangle$, is the number of permutations of the numbers 1 to n in which exactly m elements are greater than the previous element.

Indeed, the generating function of Eulerian numbers is given by

$$\left(\sum_{k=0}^{\infty}(k+1)^n x^k\right)(1-x)^{n+1}=\sum_{m=1}^{\infty}\left\langle {n \atop m-1} \right\rangle x^m, \quad \text{(see [7, 10])}. \tag{1.1}$$

Thus, by (1.1), we get

$$\left\langle {n \atop m} \right\rangle=\sum_{l=0}^{m+1}\binom{n+1}{l}(-1)^l(m+1-l)^n,\ (n\in\mathbb{N}, m\geq 0). \tag{1.2}$$

2010 *Mathematics Subject Classification.* 11B83; 11S80.
Key words and phrases. Degenerate Eulerian numbers and polynomials.

From (1.2), we note that

$$
\begin{aligned}
\left\langle {n \atop m} \right\rangle &= \sum_{k=0}^{m+1} \binom{n+1}{k} (-1)^k (m+1-k)^n \\
&= \sum_{k=0}^{m+1} \binom{n+1}{k} (-1)^k (m+1-k)^{n-1} (m+1-k) \\
&= (m+1) \sum_{k=0}^{m+1} \binom{n+1}{k} (-1)^k (m+1-k)^{n-1} \\
&\quad - \sum_{k=1}^{m+1} \binom{n+1}{k} k (-1)^k (m+1-k)^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&= (m+1) \sum_{k=0}^{m+1} \binom{n+1}{k} (-1)^k (m+1-k)^{n-1} \\
&\quad - (n+1) \sum_{k=1}^{m+1} \binom{n}{k-1} (-1)^k (m+1-k)^{n-1} \\
&= (m+1) \sum_{k=0}^{m+1} \binom{n+1}{k} (-1)^k (m+1-k)^{n-1} + (n+1) \sum_{k=0}^{m} \binom{n}{k} (-1)^k (m-k)^{n-1} \\
&= (m+1) \sum_{k=0}^{m+1} \left(\binom{n}{k} + \binom{n}{k-1} \right) (-1)^k (m+1-k)^{n-1} + (n+1) \left\langle {n-1 \atop m-1} \right\rangle \\
&= (m+1) \left\langle {n-1 \atop m} \right\rangle \\
&\quad + (m+1) \sum_{k=1}^{m+1} \binom{n}{k-1} (-1)^k (m+1-k)^{n-1} + (n+1) \left\langle {n-1 \atop m-1} \right\rangle \\
&= (m+1) \left\langle {n-1 \atop m} \right\rangle + (m+1) \sum_{k=0}^{m} \binom{n}{k} (-1)^{k-1} (m-k)^{n-1} + (n+1) \left\langle {n-1 \atop m-1} \right\rangle \\
&= (m+1) \left\langle {n-1 \atop m} \right\rangle - (m+1) \left\langle {n-1 \atop m-1} \right\rangle + (n+1) \left\langle {n-1 \atop m-1} \right\rangle \\
&= (n-m) \left\langle {n-1 \atop m-1} \right\rangle + (m+1) \left\langle {n-1 \atop m} \right\rangle .
\end{aligned} \tag{1.3}
$$

By (1.3), we obtain the recurrence relation for Eulerian numbers as follows:

$$\left\langle {n \atop m} \right\rangle = (n-m)\left\langle {n-1 \atop m-1} \right\rangle + (m+1)\left\langle {n-1 \atop m} \right\rangle, \quad \text{(see [3, 7, 10])}. \tag{1.4}$$

As is well known, the Eulerian polynomials, $A_n(t)$, $(n \geq 0)$, are defined by the generating function

$$\frac{1-t}{e^{x(t-1)}-t} = e^{A(t)x} = \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}, \tag{1.5}$$

with the usual convention about replacing $A^n(t)$ by $A_n(t)$. From (1.4), we note that

$$\big(A(t)+(t-1)\big)^n - tA_n(t) = (1-t)\delta_{0,n}, \ (n \geq 0), \tag{1.6}$$

where $\delta_{n,k}$ is the Kronecker's symbol (see [7]). From (1.3), (1.4), (1.5) and (1.6), we note that

$$A_n(t) = \sum_{l=0}^{n} \left\langle {n \atop l} \right\rangle t^l, \ (n \geq 0), \quad \text{(see [3, 7, 10])}. \tag{1.7}$$

The first few Eulerian polynomials are given by

$$\begin{aligned} 1+t+t^2+t^3+\cdots &= \frac{1}{1-t} = \frac{A_0(t)}{1-t}, \\ 1+2t+3t^2+4t^3+\cdots &= \frac{1}{(1-t)^2} = \frac{A_1(t)}{(1-t)^2}, \\ 1+2^2t+3^2t^2+4^2t^3+\cdots &= \frac{1+t}{(1-t)^3} = \frac{A_2(t)}{(1-t)^3}, \\ 1+2^3t+3^3t^2+4^3t^3+\cdots &= \frac{1+4t+t^2}{(1-t)^4} = \frac{A_3(t)}{(1-t)^4}. \end{aligned} \tag{1.8}$$

The Worpitzky's identity expresses x^n as the linear combination of Eulerian numbers with binomial coefficients as follows:

$$x^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}, \quad \text{(see [3, 4, 5, 6, 7, 9, 10])}. \tag{1.9}$$

From (1.6), we note that

$$A_0(t) = 1, A_n(t) = \frac{1}{t-1}\sum_{l=0}^{n-1}\binom{n}{l} A_l(t)(t-1)^{n-l}, \ (n \geq 1), \tag{1.10}$$

and

$$\sum_{k=1}^{m} k^m t^k = \sum_{i=1}^{n} (-1)^{n+i}\binom{n}{i}\frac{t^{m+1}A_{n-i}(t)}{(t-1)^{n-i+1}} m^i + (-1)^n\frac{t(t^m-1)}{(t-1)^{n+1}}A_n(t), \tag{1.11}$$

where $m \geq 1$ and $n \geq 0$ (see [7,10]).

In [6], the degenerate ordered Bell polynomials are defined by the generating function

$$\frac{1}{2-(1+\lambda t)^{\frac{1}{\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.12}$$

When $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ are called the degenerate ordered Bell numbers. It is well known that the Frobenius-Euler polynomials are given by the generating function

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(x|u)\frac{t^n}{n!}, \tag{1.13}$$

where $u \neq 1$. (see [8]). When $x = 0$, $H_n(u) = H_n(0|u)$ are called the Frobenius-Euler numbers. Recently, several authors have studied some interesting extensions and modifications of Eulerian polynomials and numbers (see [1-12]).

In this paper, we study the degenerate Eulerian polynomials and numbers, which are due to Carlitz (see [1]), and give some new and interesting identities for these numbers and polynomials associated with several special numbers and polynomials.

2. Degenerate Eulerian polynomials and numbers

We recall that the Stirling numbers of the first kind and of the second kind are defined by the generating function as follows:

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \tag{2.1}$$

and

$$\frac{1}{k!}\big(e^t-1\big)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [11])}. \tag{2.2}$$

For $\lambda \in \mathbb{R}$, we consider the degenerate Eulerian polynomials given by the generating function

$$\frac{1-t}{(1+\lambda x)^{\frac{t-1}{\lambda}}-t} = \sum_{n=0}^{\infty} A_{n,\lambda}(t)\frac{x^n}{n!}. \tag{2.3}$$

Note that $\lim_{\lambda\to 0} A_{n,\lambda}(t) = A_n(t)$, $(n \geq 0)$. From (2.3), we have

$$\begin{aligned}
1-t &= \left(\sum_{n=0}^{\infty} A_{n,\lambda}(t)\frac{x^n}{n!}\right)\cdot\left((1+\lambda x)^{\frac{t-1}{\lambda}} - t\right)\\
&= \left(\sum_{k=0}^{\infty} A_{k,\lambda}(t)\frac{x^k}{k!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{t-1}{\lambda}}{m}\lambda^m x^m - t\right)\\
&= \left(\sum_{k=0}^{\infty} A_{k,\lambda}(t)\frac{x^k}{k!}\right)\left(\sum_{m=0}^{\infty}(t-1)_{m,\lambda}\frac{x^m}{m!} - t\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}A_{k,\lambda}(t)(t-1)_{n-k,\lambda} - tA_{n,\lambda}(t)\right)\frac{x^n}{n!},
\end{aligned} \tag{2.4}$$

where $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$, $(x)_{0,\lambda} = 1$.

Comparing the coefficients on both sides of (2.4), we get

$$\sum_{k=0}^{n}\binom{n}{k}A_{k,\lambda}(t)(t-1)_{n-k,\lambda} - tA_{n,\lambda}(t) = (1-t)\delta_{0,n}. \tag{2.5}$$

Thus, from (2.5), we have

$$\sum_{k=0}^{n-1}\binom{n}{k}A_{k,\lambda}(t)(t-1)_{n-k,\lambda} = (t-1)A_{n,\lambda}(t),\ \ (n \geq 1),\ \ A_{0,\lambda}(t) = 1. \tag{2.6}$$

For $n \geq 1$, we have

$$A_{n,\lambda}(t) = \frac{1}{t-1}\sum_{k=0}^{n-1}\binom{n}{k}A_{k,\lambda}(t)(t-1)_{n-k,\lambda}. \tag{2.7}$$

From (2.3), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} A_{n,\lambda}(t)\frac{x^n}{n!} &= \frac{1-t}{(1+\lambda x)^{\frac{t-1}{\lambda}} - t} = \frac{1-t}{e^{\frac{t-1}{\lambda}\log(1+\lambda x)} - t}\\
&= \sum_{k=0}^{\infty} A_k(t)\frac{\lambda^{-k}}{k!}\big(\log(1+\lambda x)\big)^k\\
&= \sum_{k=0}^{\infty} A_k(t)\lambda^{-k}\sum_{n=k}^{\infty} S_1(n,k)\frac{\lambda^n x^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_k(t)\lambda^{n-k}S_1(n,k)\right)\frac{x^n}{n!}.
\end{aligned} \tag{2.8}$$

Thus, by comparing the coefficients on both sides of (2.8), we get

$$A_{n,\lambda}(t)=\sum_{k=0}^{n}A_k(t)\lambda^{n-k}S_1(n,k),\ \ (n\geq 0). \tag{2.9}$$

In view of (1.7), we define the degenerate Eulerian polynomials by

$$A_{n,\lambda}(t)=\sum_{l=0}^{n}\left\langle {n \atop l}\right\rangle_{\lambda} t^l. \tag{2.10}$$

Thus, we easily get $\lim_{\lambda\to 0}\left\langle {n \atop l}\right\rangle_{\lambda}=\left\langle {n \atop l}\right\rangle$, $(n\geq 0)$. From (1.7), (2.9) and (2.10), we have

$$\begin{aligned}\sum_{l=0}^{n}\left\langle {n \atop l}\right\rangle_{\lambda} t^l=A_{n,\lambda}(t)&=\sum_{k=0}^{n}A_k(t)\lambda^{n-k}S_1(n,k)\\&=\sum_{k=0}^{n}\sum_{l=0}^{k}\left\langle {k \atop l}\right\rangle t^l\lambda^{n-k}S_1(n,k)\\&=\sum_{l=0}^{n}\left(\sum_{k=l}^{n}\left\langle {k \atop l}\right\rangle\lambda^{n-k}S_1(n,k)\right)t^l.\end{aligned} \tag{2.11}$$

Comparing the coefficients on both sides of (2.11), we obtain

$$\left\langle {n \atop l}\right\rangle_{\lambda}=\sum_{k=l}^{n}\left\langle {k \atop l}\right\rangle\lambda^{n-k}S_1(n,k),\ \ (0\leq l\leq n). \tag{2.12}$$

By (1.12) and (2.3), we get

$$\sum_{n=0}^{\infty}b_{n,\lambda}\frac{x^n}{n!}=\frac{1}{2-(1+\lambda x)^{\frac{1}{\lambda}}}=\sum_{n=0}^{\infty}A_{n,\lambda}(2)\frac{x^n}{n!}. \tag{2.13}$$

Thus, by (2.11), we have

$$\begin{aligned}b_{n,\lambda}=A_{n,\lambda}(2)&=\sum_{l=0}^{n}\left\langle {n \atop l}\right\rangle_{\lambda}2^l\\&=\sum_{l=0}^{n}\sum_{k=l}^{n}\left\langle {k \atop l}\right\rangle\lambda^{n-k}S_1(n,k)2^l,\end{aligned} \tag{2.14}$$

where $0\leq l\leq n$. For $0\leq l\leq n$, we have

$$\left\langle {n \atop l}\right\rangle_{\lambda}=\sum_{k=l}^{n}\sum_{m=0}^{l+1}\binom{k+1}{m}(-1)^m(l+1-m)^k\lambda^{n-k}S_1(n,k). \tag{2.15}$$

Note that

$$\lim_{\lambda\to 0}\left\langle {n \atop l} \right\rangle_{\lambda} = \sum_{m=0}^{l+1}\binom{n+1}{m}(-1)^m(l+1-m)^n$$
$$= \left\langle {n \atop l} \right\rangle, \ (0 \le l \le n).$$

From (2.8), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} A_{n,\lambda}(t)\frac{x^n}{n!} &= \frac{1-t}{(1+\lambda x)^{\frac{t-1}{\lambda}}-t} = \frac{1-t}{e^{\frac{t-1}{\lambda}\log(1+\lambda x)}-t} \\ &= \sum_{n=0}^{\infty} H_n(t)\frac{1}{n!}\left(\frac{t-1}{\lambda}\right)^n\big(\log(1+\lambda x)\big)^n \\ &= \sum_{k=0}^{\infty} H_k(t)\left(\frac{t-1}{\lambda}\right)^k\sum_{n=k}^{\infty} S_1(n,k)\frac{\lambda^n x^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(t-1)^k H_k(t)\lambda^{n-k}S_1(n,k)\right)\frac{x^n}{n!},\end{aligned} \tag{2.16}$$

where $H_n(t)$ is the Frobenius-Euler numbers. By (2.16), we get

$$A_{n,\lambda}(t) = \sum_{k=0}^{n}\lambda^{n-k}S_1(n,k)H_k(t)(t-1)^k, \ (n \ge 0). \tag{2.17}$$

Let us take $t=2$. Then we have

$$b_{n,\lambda} = \sum_{k=0}^{n}\lambda^{n-k}S_1(n,k)H_k(2), \ (n \ge 0). \tag{2.18}$$

3. Further remark

Let p be an odd prime number. Throughout this section, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is normalized so that $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. As notations, the q-numbers are defined by

$$[x]_q = \frac{1-q^x}{1-q}, \quad \text{and} \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

Let f be a continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{3.1}$$

From (3.1), we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1(x) = f(x+1). \tag{3.2}$$

By (3.2), we get

$$\left(\frac{q + (1+\lambda t)^{-\frac{1+q}{\lambda}}}{q} \right) \int_{\mathbb{Z}_p} (1+\lambda t)^{-\frac{x}{\lambda}(1+q)} d\mu_{-q^{-1}}(x) = [2]_{q^{-1}}, \tag{3.3}$$

where $\lambda \in \mathbb{Z}_p$ and $|t|_p < p^{-\frac{1}{p-1}}$. Thus, from (3.3), we have

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{-\frac{x}{\lambda}(1+q)} d\mu_{-q^{-1}}(x) = \frac{1+q}{q + (1+\lambda t)^{-\frac{1+q}{\lambda}}} \tag{3.4}$$

From (2.3) and (3.4), we note that

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{-\frac{x}{\lambda}(1+q)} d\mu_{-q^{-1}}(x) = \sum_{n=0}^{\infty} A_{n,\lambda}(-q) \frac{t^n}{n!}. \tag{3.5}$$

Now, we define the degenerate rising factorials as follows:

$$< x >_{0,\lambda} = 1, \; < x >_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda), \; (n \geq 1). \tag{3.6}$$

It is not difficult to show that

$$\begin{aligned} (1+\lambda t)^{-\frac{x}{\lambda}(1+q)} &= e^{-\frac{x}{\lambda}(1+q)\log(1+\lambda t)} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{\lambda}\right)^n (1+q)^n \frac{\left(\log(1+\lambda t)\right)^n}{n!} \\ &= \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{\lambda}\right)^k (1+q)^k \sum_{n=k}^{\infty} S_1(n,k) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k \lambda^{n-k} (1+q)^k S_1(n,k) x^k \right) \frac{t^n}{n!}. \end{aligned} \tag{3.7}$$

From (3.2), we note that

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q^{-1}}(x) = \frac{q+1}{e^t + q} = \sum_{n=0}^{\infty} H_n(-q) \frac{t^n}{n!}. \tag{3.8}$$

Thus, by (3.8), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_{-q^{-1}}(x) = H_n(-q), \ (n \geq 0),$$

where $H_n(-q)$ are the Frobenius-Euler numbers. From (3.7) and (3.8), we have

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{-\frac{x}{\lambda}(1+q)} d\mu_{-q^{-1}}(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k \lambda^{n-k} (1+q)^k S_1(n,k) H_k(-q) \right) \frac{t^k}{k!} \tag{3.9}$$

Comparing the coefficients on both sides of (3.5) and (3.9), we have

$$A_{n,\lambda}(-q) = \sum_{k=0}^{n} (-1)^k \lambda^{n-k} (1+q)^k S_1(n,k) H_k(-q), \ (n \geq 0). \tag{3.10}$$

In particular,

$$\begin{aligned}
(1+\lambda t)^{-\frac{x}{\lambda}(1+q)} &= \sum_{n=0}^{\infty} \binom{-\frac{x}{\lambda}(1+q)}{n} \lambda^n t^n \\
&= \sum_{n=0}^{\infty} \big(-\tfrac{x}{\lambda}(1+q) \big)_n \lambda^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n < (1+q)x >_{n,\lambda} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n < x >_{n,\frac{\lambda}{1+q}} (1+q)^n \frac{t^n}{n!}
\end{aligned} \tag{3.11}$$

From (3.11), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} A_{n,\lambda}(-q) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+\lambda t)^{-\frac{x}{\lambda}(1+q)} d\mu_{-q^{-1}}(x) \\
&= \sum_{n=0}^{\infty} (-1)^n (1+q)^n \int_{\mathbb{Z}_p} < x >_{n,\frac{\lambda}{1+q}} d\mu_{-q^{-1}}(x) \frac{t^n}{n!}.
\end{aligned} \tag{3.12}$$

Thus, by comparing the coefficients on the both sides of (3.12), we get

$$\int_{\mathbb{Z}_p} < x >_{n,\frac{\lambda}{1+q}} d\mu_{-q^{-1}}(x) = (-1)^n \frac{A_{n,\lambda}(-q)}{(1+q)^n}, \ (n \geq 0). \tag{3.13}$$

For any positive real number λ, the degenerate unsigned Stirling numbers of the first kind $|S_{1,\lambda}(n,l)|$ are defined by

$$< x >_{n,\lambda} = \sum_{l=0}^{n} |S_{1,\lambda}(n,l)| x^l, \ (n \geq 0). \tag{3.14}$$

From (3.14), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} < x >_{n,\frac{\lambda}{1+q}} d\mu_{-q^{-1}}(x) &= \sum_{l=0}^{n} |S_{1,\frac{\lambda}{1+q}}(n,l)| \int_{\mathbb{Z}_p} x^l d\mu_{-q^{-1}}(x) \\ &= \sum_{l=0}^{n} |S_{1,\frac{\lambda}{1+q}}(n,l)| H_l(-q).\end{aligned} \tag{3.15}$$

Hence, by (3.13) and (3.15), we get

$$\sum_{l=0}^{n} |S_{1,\frac{\lambda}{1+q}}(n,l)| H_l(-q) = (-1)^n \frac{A_{n,\lambda}(-q)}{(1+q)^n}, \ (n \geq 0).$$

References

1. L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51–88.
2. L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions,* D. Reidel Publishing Co., **1974**, p. 228.
3. D. S. Kim, T. Kim, W. J. Kim, D. V. Dolgy, *A note on Eulerian polynomials,* Abstr. Appl. Anal. **2012**, Art. ID 269640, 10 pp.
4. D. S. Kim, T. Kim, S.-H. Rim, *Frobenius-type Eulerian polynomials and umbral calculus,* Proc. Jangjeon Math. Soc., **16** (2013), no. 2, 285-292.
5. T. Kim, *Degenerate ordered Bell numbers and polynomials,* Proc. Jangjeon Math. Soc., **20** (2017), no. 2, (in press),
6. T. Kim, D. S. Kim, *Some identities of Eulerian polynomials arising from nonlinear differential equations,* Iran. J. Sci. Technol. Trans. Sci. **(2016)**. doi:10.1007/s40995-016-0073-0.
7. T. Kim, *A note on degenerate Stirling polynomials of the second kind,* Proc. Jangjeon Math. Soc. Vol. 20 (2017), No. 3, 319-331 **20** (2017), no. 3,319-331.
8. T. Kim, D. S. Kim, G.-W. Jang, *Extended Stirling polynomials of the second kind and and extended Bell polynomials,* Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 365–376.
9. A. Knopfmacher, N. Robbins, *Some arithmetic properties of Eulerian numbers,* J. Combin. Math. Combin. Comput. **36** (2001), 31-42.
10. I. Mező, *A kind of Eulerian numbers connected to Whitney numbers of Dowling lattices,* Discrete Math. **328** (2014), 88-95.
11. S. Roman, *The umbral calculus,* Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, **1984** x+193 pp.
12. S. Tanimoto, *A study of Eulerian numbers by means of an operator on permutations,* European J. Combin., **24** (2003), no. 1, 33-43.

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea

E-mail address: `dskim@sogang.ac.kr`

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

E-mail address: `tkkim@kw.ac.kr`