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POSET WEIGHT ENUMERATORS

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Abstract. Jurrius et al. give the explicit identities for relations among various weight enumerators with respect to Hamming weight [4]. These identities show the connections between coding theory and graph theory. In this paper, we generalize the weight enumerators of a code with respect to poset weight: (i) poset weight enumerator, (ii) r-th higher poset weight enumerator, (iii) extension poset weight enumerator and (iv) poset Tutte polynomial. And we give the relations among these enumerators that are the connection between poset codes and graph theory.

1. Introduction

Weight distribution of a code is one of main research topics in coding theory. In general, the weight distribution is represented by various weight enumerators(or weight generating functions). Jurrius et al. give the explicit identities for relations among various weight enumerators with respect to Hamming weight [4]. These identities show the connections between coding theory and graph theory. In this paper, we generalize the weight enumerators of a code with respect to poset weight: (i) poset weight enumerator, (ii) r-th higher poset weight enumerator, (iii) extension poset weight enumerator and (iv) poset Tutte polynomial. And we give the relations among these enumerators that are the connection between poset codes and graph theory.

Section 2 shows the definition of poset codes and properties of puncturing and shortening codes. Section 3 gives some useful identities used in proofs of propositions and lemmas in Section 4 and Section 5. We introduce poset weight enumerators in Section 4, and their relations in Section 5.

2. Poset codes

Poset and ideals. Let $\mathbb{P} = ([n], \ \leq_{\mathbb{P}})$ be a poset on $[n] = \{1, 2, \ldots, n\}$. The dual poset $\mathbb{P}^{\perp} = ([n], \ \leq_{\mathbb{P}^{\perp}})$ of $\mathbb{P}$ is defined by the poset on $[n]$ such that $i \leq_{\mathbb{P}^{\perp}} j$ if and only if $j \leq_{\mathbb{P}} i$. A subset $J \subseteq [n]$ is an ideal in $\mathbb{P}$ if $j \in J$ and $i \leq_{\mathbb{P}} j$ then $i \in J$. Let $\mathcal{I}(\mathbb{P})$ be the set of ideals in $\mathbb{P}$. Then $\mathcal{I}(\mathbb{P}^{\perp})$ equals $\{J \subseteq [n] : \overline{J} \in \mathcal{I}(\mathbb{P})\}$ where $\overline{J} := [n] \setminus J$. For $J \subseteq [n]$, let $\langle J\rangle_{\mathbb{P}}$ be the smallest ideal of $\mathbb{P}$ containing J and let $[J]_{\mathbb{P}}$ be the largest ideal of $\mathbb{P}$ contained in J [5], i.e., $\langle J\rangle_{\mathbb{P}} := \bigcap_{\substack{I \in \mathcal{I}(\mathbb{P}) \\ J \subseteq I}} I$ and $[J]_{\mathbb{P}} := \bigcup_{\substack{I \in \mathcal{I}(\mathbb{P}) \\ I \subseteq J}} I$.

Proposition 2.1 ([5]). *Let $I, J \subseteq [n]$.*

(i) *If $I \subseteq J$, then $I \subseteq \langle I\rangle_{\mathbb{P}} \subseteq \langle J\rangle_{\mathbb{P}}$ and $[I]_{\mathbb{P}} \subseteq [J]_{\mathbb{P}} \subseteq J$.*
(ii) $\langle I \cup J\rangle_{\mathbb{P}} = \langle I\rangle_{\mathbb{P}} \cup \langle J\rangle_{\mathbb{P}}$.
(iii) *$I \subseteq [J]_{\mathbb{P}}$, $\langle I\rangle_{\mathbb{P}} \subseteq J$ and $\langle I\rangle_{\mathbb{P}} \subseteq [J]_{\mathbb{P}}$ are equivalent.*

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(*iv*) *$J \in \mathcal{I}(\mathbb{P})$ if and only if $\overline{J} \in \mathcal{I}(\mathbb{P}^{\perp})$.*
(*v*) $\overline{\langle J \rangle_{\mathbb{P}}} = [\ \overline{J}\ ]_{\mathbb{P}^{\perp}}$.

Proof. (i) It is trivial by definition. (ii) $(\Rightarrow)$ For any $a \in \langle I \cup J \rangle_{\mathbb{P}}$, there exists $b \in I$ or $b \in J$ such that $a \leq_{\mathbb{P}} b$. So $a \in \langle I \rangle_{\mathbb{P}}$ or $a \in \langle J \rangle_{\mathbb{P}}$. $(\Leftarrow)$ It is derived from (i) directly. (iii) Suppose $I \subseteq [J]_{\mathbb{P}}$. By (i), $I \subseteq \langle I \rangle_{\mathbb{P}} \subseteq \langle [J]_{\mathbb{P}} \rangle_{\mathbb{P}} = [J]_{\mathbb{P}} \subseteq J$. Suppose $\langle I \rangle_{\mathbb{P}} \subseteq J$. By (i), $I \subseteq \langle I \rangle_{\mathbb{P}} = [\langle I \rangle_{\mathbb{P}}]_{\mathbb{P}} \subseteq [J]_{\mathbb{P}} \subseteq J$. Suppose $\langle I \rangle_{\mathbb{P}} \subseteq [J]_{\mathbb{P}}$. By (i), $I \subseteq \langle I \rangle_{\mathbb{P}} \subseteq [J]_{\mathbb{P}} \subseteq J$. (iv) Suppose $\overline{J} \notin \mathcal{I}(\mathbb{P}^{\perp})$. By definition of an ideal, there exists $(x, y) \in \overline{J} \times J$ such that $x \geq_{\mathbb{P}^{\perp}} y$. Since $x \geq_{\mathbb{P}^{\perp}} y$ is equivalent to $x \leq_{\mathbb{P}} y$, we have $J \notin \mathcal{I}(\mathbb{P})$. (v) By (iv) we have $\overline{\langle J \rangle_{\mathbb{P}}} = \overline{\bigcap_{\substack{I \in \mathcal{I}(\mathbb{P}) \\ J \subset I}} I} = \bigcup_{\substack{I \in \mathcal{I}(\mathbb{P}) \\ J \subset I}} \overline{I} = \bigcup_{\substack{\overline{I} \in \mathcal{I}(\mathbb{P}^{\perp}) \\ \overline{I} \subset \overline{J}}} \overline{I} = \bigcup_{\substack{I \in \mathcal{I}(\mathbb{P}^{\perp}) \\ I \subset \overline{J}}} I = [\ \overline{J}\ ]_{\mathbb{P}^{\perp}}$. □

Poset weight and distance. Let $\mathbb{F}_q$ be a finite field with q elements. In a vector space $\mathbb{F}_q^n$, we define $\mathbb{P}$-weight $w_{\mathbb{P}}$ by $w_{\mathbb{P}}(v) := |\langle \mathrm{supp}(v) \rangle_{\mathbb{P}}|$ for $v = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$ where $\mathrm{supp}(v) := \{i : v_i \neq 0\}$. The $\mathbb{P}$-distance of two vectors $u, v \in \mathbb{F}_q^n$ is defined by $\mathrm{dist}_{\mathbb{P}}(u, v) := |w_{\mathbb{P}}(u - v)|$. It is easy to prove that $\mathrm{dist}_{\mathbb{P}}$ is metric [1]. We say the space equipped with the $\mathbb{P}$-distance a $\mathbb{P}$-metric space.

Poset codes were introduced in [1]. They are just nonempty subsets in $\mathbb{F}_q^n$, equipped with any poset weight $w_{\mathbb{P}}$ instead of the usual Hamming weight w_H. In this paper, given a poset $\mathbb{P}$ on $[n]$, we consider a linear $\mathbb{P}$-code $\mathcal{C}_{\mathbb{P}}$ over $\mathbb{F}_q$ with length n and dimension k. The dual code of $\mathcal{C}_{\mathbb{P}}$ is defined by $\{u \in \mathbb{F}_q^n : u \cdot c = 0 \text{ for } c \in \mathcal{C}_{\mathbb{P}}\}$ where $\cdot$ is usual inner product and weights of dual codewords are computed with respect to $w_{\mathbb{P}^{\perp}}$. The $(n-k)$-dimensional dual poset code of $\mathcal{C}_{\mathbb{P}}$ is denoted by ${\mathcal{C}^{\perp}}_{\mathbb{P}^{\perp}}$.

For simplicity of notation, from now on, we let $\mathcal{C}$ and $\mathcal{C}^{\perp}$ stand for $\mathcal{C}_{\mathbb{P}}$ and ${\mathcal{C}^{\perp}}_{\mathbb{P}^{\perp}}$ respectively.

Puncturing and shortening codes. For $J \subset [n]$, the puncturing code $\mathcal{C}^J$ on J of $\mathcal{C}$ is defined by $\mathcal{C}^J := \{c^J \in \mathbb{F}_q^{|J|} : c = (c_1, \ldots, c_n) \in \mathcal{C}\}$ where $c^J = (c_i)_{i \in J} \in \mathbb{F}_q^{|J|}$ and the shortening code $\mathcal{C}_J$ on J of $\mathcal{C}$ is defined by $\mathcal{C}_J := \{c^J \in \mathbb{F}_q^{|J|} : c \in \mathcal{C},\ \mathrm{supp}(c) \subseteq J\}$. It is well known that $(\mathcal{C}^{\perp})_J = (\mathcal{C}^J)^{\perp}$, and note that this is derived without poset duality [6].

Rank function. Let $\mathcal{G}$ be the $(k \times n)$-generator matrix of $\mathcal{C}$ over $\mathbb{F}_q$ with length n and dimension k. The rank functions $\rho : [n] \to \mathbb{Z}$ of $\mathcal{C}$ is defined by $\rho(J) := \mathrm{rank}(\mathcal{G}^J)$ for $J \subseteq [n]$ where $\mathcal{G}^J$ is the sub-matrix of the generator matrix $\mathcal{G}$ of $\mathcal{C}$ consisting of the columns indexed by J [7]. Note that $\mathcal{G}^J$ is the generator matrix of the puncturing code $\mathcal{C}^J$, so $\rho(J) = \dim(\mathcal{C}^J)$. Let $\rho^{\perp}$ denote the rank function of $\mathcal{C}^{\perp}$ and it is called the corank function of $\mathcal{C}$.

Proposition 2.2. *Let $J \subseteq [n]$.*

(*i*) $\rho(\emptyset) = 0$.
(*ii*) $0 \leq \rho(J) \leq \min(k, |J|)$.
(*iii*) *If $J_1 \subseteq J_2 \subseteq [n]$, then $\rho(J_1) \leq \rho(J_2)$.*
(*iv*) *If $\rho(J) = |J|$, then $\rho(I) = |I|$ for all $I \subseteq J$.*

Proof. $(i) - (iv)$ are obtained from the definition directly. □

Lemma 2.3. *Let $J \subseteq [n]$.*

(*i*) *If* $\rho(J) = k$, *then* $\rho^{\perp}(\overline{J}) = |\overline{J}|$.
(*ii*) $\dim(\mathcal{C}_J) = |J| - \rho^{\perp}(J) = k - \rho(\overline{J})$.

Proof. (*i*) If $\rho(J) = \dim(\mathcal{C}^J) = k$, then $c^J \neq 0$ for all nonzero $c \in \mathcal{C}$, so $\mathcal{C}_{\overline{J}} = \{0\}$. Therefore $0 = \dim(\mathcal{C}_{\overline{J}}) = |\overline{J}| - \dim((\mathcal{C}^{\perp})^{\overline{J}}) = |\overline{J}| - \rho^{\perp}(\overline{J})$. (*ii*) For $X \subseteq J$ with $\rho(X) = |X| = \rho(J)$, there exists $Y \subseteq \overline{J}$ such that $k = \rho(X \cup Y) = |X \cup Y| = |X| + |Y|$. By (*i*), we know $\rho^{\perp}(\overline{X \cup Y}) = |\overline{X \cup Y}| = n - k$, and $\rho^{\perp}(\overline{J}) \geq \rho^{\perp}(\overline{J} \setminus Y) = |\overline{J} \setminus Y| = |\overline{J}| - |Y| = |\overline{J}| - (k - |X|) = |\overline{J}| - (k - \rho(J))$. By same way, we have $\rho(\overline{J}) \geq |\overline{J}| - (n - k - \rho^{\perp}(J))$. Replacing $\overline{J}$ by J, $\rho(J) \geq |J| - (n - k - \rho^{\perp}(\overline{J}))$. Thus we have $\rho^{\perp}(\overline{J}) \geq |\overline{J}| - k + \rho(J) \geq |\overline{J}| - k + |J| - (n - k - \rho^{\perp}(\overline{J})) = \rho^{\perp}(\overline{J})$. Hence it is proved. □

Corollary 2.4 is derived from Lemma 2.3 (*ii*).

Corollary 2.4. *Let* $\mathcal{C}$ *be an* $[n,k]$ *linear code over* $\mathbb{F}_q$ *and let* ρ *be the rank function of* $\mathcal{C}$. *If* $J \subseteq [n]$, *then* $|\mathcal{C}_J| = q^{k-\rho(\overline{J})}$.

3. Useful identities

Propositions below are very useful to derive several weight polynomial identities.

Proposition 3.1. *For* $0 \leq j \leq i \leq n$,
(*i*) $\binom{n}{i}\binom{i}{j} = \binom{n}{j}\binom{n-j}{n-i}$.
(*ii*) $a_i = \sum_{j=i}^{n} \binom{j}{i} b_j = \sum_{j=0}^{n-i} \binom{n-j}{i} b_{n-j}$ *if and only if* $b_i = \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{i}\, a_j$.

Proof. (*i*) It is easy to prove. (*ii*) is derived from (*i*) as follows.

$$\begin{aligned}
\sum_{j=i}^{n} \binom{j}{i} b_j &= \sum_{j=0}^{n} \binom{j}{i} \sum_{m=j}^{n} (-1)^{m-j} \binom{m}{j} a_m = \sum_{j=i}^{n} \sum_{m=j}^{n} (-1)^{m-j} \binom{m}{j}\binom{j}{i} a_m \\
&= \sum_{j=i}^{n} \sum_{m=0}^{n} (-1)^{m-j} \binom{m}{i}\binom{m-i}{m-j} a_m = \sum_{m=0}^{n} \binom{m}{i} a_m \sum_{j=i}^{n} (-1)^{m-j} \binom{m-i}{m-j} \\
&= \sum_{m=0}^{n} \binom{m}{i} a_m \sum_{j=i}^{m} (-1)^{m-j} \binom{m-i}{m-j} = \sum_{m=0}^{n} \binom{m}{i} a_m (1-1)^{m-i} = a_i.
\end{aligned}$$

□

Proposition 3.2. *We have*
(*i*) $\sum_{J \subseteq [n]} (x-y)^{|J|} y^{|\overline{J}|} = x^n$.
(*ii*) $\sum_{J \subseteq [n]} (-1)^{i-|J|} \binom{|\overline{J}|}{n-i} = \delta_{i,0}$ *where* $\delta_{a,b}$ *is Kronecker delta.*
(*iii*) $\sum_{J \subseteq [n]} (-1)^{|J|-i} \binom{|J|}{i} a_J = \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{i} \sum_{\substack{J \subseteq [n] \\ |J|=j}} a_J$.
(*iv*) $\sum_{i=0}^{n} \left(\sum_{J \subseteq [n]} (-1)^{i-|J|} \binom{|\overline{J}|}{n-i} a_J \right) x^{n-i} y^i = \sum_{J \subseteq [n]} a_J (x-y)^{|\overline{J}|} y^{|J|}$.

Proof. (i) $\sum_{J\subseteq[n]}(x-y)^{|J|}y^{|\overline{J}|} = \sum_{i=0}^{n}\sum_{\substack{J\subseteq[n]\\|J|=i}}(x-y)^i y^{n-i} = \sum_{i=0}^{n}\binom{n}{i}(x-y)^i y^{n-i} = ((x-y)+y)^n = x^n$.

(ii) By Proposition 3.1 (i), it is proved as follows.

$$\begin{aligned}\sum_{J\subseteq[n]}(-1)^{i-|J|}\binom{|\overline{J}|}{n-i} &= \sum_{j=0}^{n}\sum_{\substack{J\subseteq[n]\\|J|=j}}(-1)^{i-j}\binom{n-j}{n-i} = \sum_{j=0}^{n}\binom{n}{j}\binom{n-j}{n-i}(-1)^{i-j}\\ &= \sum_{j=0}^{n}\binom{n}{i}\binom{i}{j}(-1)^{i-j} = \binom{n}{i}\sum_{j=0}^{n}\binom{i}{j}(-1)^{i-j}\\ &= \binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j} = \binom{n}{i}(1-1)^i = \delta_{i,0}.\end{aligned}$$

(iii) $\sum_{j=i}^{n}(-1)^{j-i}\binom{j}{i}\sum_{\substack{J\subseteq[n]\\|J|=j}}a_J = \sum_{j=0}^{n}\sum_{\substack{J\subseteq[n]\\|J|=j}}(-1)^{j-i}\binom{j}{i}a_J = \sum_{J\subseteq[n]}(-1)^{|J|-i}\binom{|J|}{i}a_J$.

(iv)

$$\begin{aligned}\sum_{i=0}^{n}\left(\sum_{J\subseteq[n]}(-1)^{i-|J|}\binom{|\overline{J}|}{n-i}a_J\right)x^{n-i}y^i &= \sum_{J\subseteq[n]}a_J\sum_{i=0}^{n}(-1)^{i-|J|}\binom{|\overline{J}|}{n-i}x^{n-i}y^i\\ = \sum_{J\subseteq[n]}a_J\, y^{|J|}\sum_{i=0}^{n}(-1)^{|\overline{J}|-i}\binom{|\overline{J}|}{i}x^i y^{|\overline{J}|-i} &= \sum_{J\subseteq[n]}a_J(x-y)^{|\overline{J}|}y^{|J|}.\end{aligned}$$

□

Let $\begin{bmatrix}k\\r\end{bmatrix}_q$ denote the number of r-dimensional subspaces of k-dimensional vector space $\mathbb{F}_q^k$. We know that $\begin{bmatrix}k\\r\end{bmatrix}_q$ equals $\frac{[k,r]_q}{[r,r]_q}$ where $[k,r]_q := \prod_{j=0}^{r-1}(q^k-q^j)$. The number of $m\times n$ matrices over $\mathbb{F}_q$ having rank r is $[m,r]_q\begin{bmatrix}n\\r\end{bmatrix}_q$, and so $q^{mn} = |M_{m\times n}(\mathbb{F}_q)|$ can be represented by

$$q^{mn} = \sum_{r=0}^{m}|\{A\in M_{m\times n}(\mathbb{F}_q) : \mathrm{rank}(A)=r\}| = \sum_{r=0}^{m}[m,r]_q\begin{bmatrix}n\\r\end{bmatrix}_q, \tag{1}$$

where $M_{m\times n}(\mathbb{F}_q)$ is the set of $m\times n$ matrices having entries in $\mathbb{F}_q$.

Proposition 3.3 ([5]). *We have*

$$\prod_{r=0}^{k-1}(x-q^r) = \sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q(-1)^{k-r}q^{\frac{(k-r-1)(k-r)}{2}}x^r.$$

Proof. It is proved by induction on k. □

From Proposition 3.3, we have, for $m\geq r$,

$$[m,r]_q = \prod_{j=0}^{r-1}(q^m-q^j) = \sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}_q(-1)^{r-j}q^{\frac{(r-j-1)(r-j)}{2}}(q^j)^m. \tag{2}$$

4. Several weight enumerators for $\mathbb{P}$-codes

Let $\mathbb{P}$ be a poset on $[n]$. Let $\mathcal{C}$ be an $\mathbb{P}$-code with length n over $\mathbb{F}_q$. In this section, we introduce three weight enumerators and a polynomial of a $\mathbb{P}$-code $\mathcal{C}$ as follows:

(i) $\mathbb{P}$-weight enumerator $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y)$.
(ii) r-th higher $\mathbb{P}$-weight enumerator $\mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y)$ for $0 \le r \le k$.
(iii) Extension $\mathbb{P}$-weight enumerator $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,t)$.
(iv) $\mathbb{P}$-Tutte polynomial $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y)$.

4.1. **$\mathbb{P}$-weight enumerator $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y)$.** The $\mathbb{P}$-weight enumerator $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y)$ of $\mathcal{C}$ is defined by

$$\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y) := \sum_{c\in\mathcal{C}} x^{n-w_{\mathbb{P}}(c)} y^{w_{\mathbb{P}}(c)} = \sum_{i=0}^{n} \mathcal{A}_{i,\mathbb{P}}(\mathcal{C})\ x^{n-i}y^i,$$

where $\mathcal{A}_{i,\mathbb{P}}(\mathcal{C}) = |\{c \in \mathcal{C} : w_{\mathbb{P}}(c) = i\}|$.

Lemma 4.1. *For $0 \le i \le n$,*

(i) $\sum_{j=0}^{n} \binom{n-j}{n-i} \mathcal{A}_{j,\mathbb{P}}(\mathcal{C}) = \sum_{\substack{J\subseteq[n]\\|J|=i}} |\mathcal{C}_{[J]_{\mathbb{P}}}|.$

(ii) $\mathcal{A}_{n-i,\mathbb{P}}(\mathcal{C}) = \sum_{j=i}^{n} (-1)^{j-i}\binom{j}{i} \sum_{\substack{J\subseteq[n]\\|J|=n-j}} |\mathcal{C}_{[J]_{\mathbb{P}}}| = \sum_{J\subseteq[n]} (-1)^{|\overline{J}|-i}\binom{|\overline{J}|}{i} |\mathcal{C}_{[J]_{\mathbb{P}}}|.$

Proof. (i) By Proposition 2.1 (iii), we have

$$\begin{aligned}
\sum_{\substack{J\subseteq[n]\\|J|=i}} |\mathcal{C}_{[J]_{\mathbb{P}}}| &= \sum_{\substack{J\subseteq[n]\\|J|=i}} \sum_{\substack{c\in\mathcal{C}\\ \mathrm{supp}(c)\subseteq[J]_{\mathbb{P}}}} 1 = \sum_{\substack{J\subseteq[n]\\|J|=i}} \sum_{\substack{c\in\mathcal{C}\\ \langle\mathrm{supp}(c)\rangle_{\mathbb{P}}\subseteq J}} 1 = \sum_{c\in\mathcal{C}} \sum_{\substack{J\subseteq[n]\\|J|=i\\ \langle\mathrm{supp}(c)\rangle_{\mathbb{P}}\subseteq J}} 1 \\
&= \sum_{c\in\mathcal{C}} \binom{n-|\langle\mathrm{supp}(c)\rangle_{\mathbb{P}}|}{i-|\langle\mathrm{supp}(c)\rangle_{\mathbb{P}}|} = \sum_{j=0}^{n} \sum_{\substack{c\in\mathcal{C}\\ w_{\mathbb{P}}(c)=j}} \binom{n-j}{i-j} \\
&= \sum_{j=0}^{n} \binom{n-i}{n-j} \mathcal{A}_{j,\mathbb{P}}(\mathcal{C}).
\end{aligned}$$

(ii) (i) and Proposition 3.1 (ii), we have

$$\begin{aligned}
\mathcal{A}_{n-i,\mathbb{P}}(\mathcal{C}) &= \sum_{j=i}^{n} (-1)^{j-i}\binom{j}{i} \sum_{\substack{J\subseteq[n]\\|J|=j}} |\mathcal{C}_{[\overline{J}]_{\mathbb{P}}}| = \sum_{j=i}^{n} \sum_{\substack{J\subseteq[n]\\|J|=j}} (-1)^{j-i}\binom{j}{i} |\mathcal{C}_{[\overline{J}]_{\mathbb{P}}}| \\
&= \sum_{j=0}^{n} \sum_{\substack{J\subseteq[n]\\|J|=j}} (-1)^{j-i}\binom{j}{i} |\mathcal{C}_{[\overline{J}]_{\mathbb{P}}}| = \sum_{J\subseteq[n]} (-1)^{|J|-i}\binom{|J|}{i} |\mathcal{C}_{[\overline{J}]_{\mathbb{P}}}|.
\end{aligned}$$

□

Theorem 4.2. *Let $\mathbb{P}$ be a poset on $[n]$, and let $\mathcal{C}$ be an $[n,k]$ $\mathbb{P}$-code over $\mathbb{F}_q$. We have*

$$\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y) = \sum_{J\subseteq[n]} q^{k-\rho(\overline{[J]_{\mathbb{P}}})}\ (x-y)^{|\overline{J}|} y^{|J|},$$

where ρ is the rank function of $\mathcal{C}$.

Proof. By Corollary 2.4, Proposition 3.2 (iv) and Lemma 4.1 (ii), we have

$$\sum_{i=0}^{n} \mathcal{A}_{i,\mathbb{P}}(\mathcal{C})\ x^{n-i}y^{i} = \sum_{i=0}^{n}\left(\sum_{J\subseteq[n]}(-1)^{i-|J|}\binom{|\overline{J}|}{n-i}|\mathcal{C}_{[J]_{\mathbb{P}}}|\right)x^{n-i}y^{i}$$
$$= \sum_{J\subseteq[n]} q^{k-\rho(\overline{[J]_{\mathbb{P}}})}\ (x-y)^{|\overline{J}|}y^{|J|}.$$

□

4.2. r**-th higher** $\mathbb{P}$**-weight enumerator** $\mathcal{W}^{r}_{\mathcal{C},\mathbb{P}}(x,y)$**.** For $0\le r\le k$, we define r-th higher $\mathbb{P}$-weight enumerator $\mathcal{W}^{r}_{\mathcal{C},\mathbb{P}}(x,y)$ of $\mathcal{C}$ as

$$\mathcal{W}^{r}_{\mathcal{C},\mathbb{P}}(x,y) := \sum_{\substack{\mathcal{D}\subseteq\mathcal{C}\\ \dim(\mathcal{D})=r}} x^{n-w_{\mathbb{P}}(\mathcal{D})}y^{w_{\mathbb{P}}(\mathcal{D})} = \sum_{i=0}^{n}\mathcal{A}^{r}_{i,\mathbb{P}}(\mathcal{C})\ x^{n-i}y^{i},$$

where $w_{\mathbb{P}}(\mathcal{D}) = |\langle \mathrm{supp}(\mathcal{D})\rangle_{\mathbb{P}}|$, $\mathrm{supp}(\mathcal{D}) = \bigcup_{v\in\mathcal{D}}\mathrm{supp}(v)$ and $\mathcal{A}^{r}_{i,\mathbb{P}}(\mathcal{C}) = |\{\mathcal{D}\subseteq\mathcal{C} : w_{\mathbb{P}}(\mathcal{D}) = i, \dim(\mathcal{D}) = r\}|$. It is easy to prove that $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y) = x^{n}+(q-1)\mathcal{W}^{1}_{\mathcal{C},\mathbb{P}}(x,y)$.

Lemma 4.3. *For $0\le i\le n$ and $0\le r\le k$, we have*

(i) $\sum_{j=0}^{n}\binom{n-j}{n-i}\mathcal{A}^{r}_{j,\mathbb{P}}(\mathcal{C}) = \sum_{\substack{J\subseteq[n]\\|J|=i}}\begin{bmatrix}k-\rho(\overline{[J]_{\mathbb{P}}})\\ r\end{bmatrix}_q.$

(ii) $\mathcal{A}^{r}_{n-i,\mathbb{P}}(\mathcal{C}) = \sum_{j=i}^{n}(-1)^{j-i}\binom{j}{i}\sum_{\substack{J\subseteq[n]\\|J|=n-j}}\begin{bmatrix}k-\rho(\overline{[J]_{\mathbb{P}}})\\ r\end{bmatrix}_q = \sum_{J\subseteq[n]}(-1)^{|\overline{J}|-i}\binom{|\overline{J}|}{i}\begin{bmatrix}k-\rho(\overline{[J]_{\mathbb{P}}})\\ r\end{bmatrix}_q.$

Proof. (i) The proof is similar to that of Lemma 4.1 (i). Here we count $|\{(\mathcal{D},J) : \mathcal{D}\subseteq\mathcal{C},\ \dim(\mathcal{D}) = r, \langle\mathrm{supp}(\mathcal{D})\rangle_{\mathbb{P}}\subseteq J, |J| = i\}|$. (ii) The proof is similar to that of Lemma 4.1 (ii). □

Theorem 4.4. *Let $\mathbb{P}$ be a poset on $[n]$. Let $\mathcal{C}$ be an $[n,k]$ $\mathbb{P}$-code over $\mathbb{F}_q$. For $0\le r\le k$,*

$$\mathcal{W}^{r}_{\mathcal{C},\mathbb{P}}(x,y) = \sum_{J\subseteq[n]}\begin{bmatrix}k-\rho(\overline{[J]_{\mathbb{P}}})\\ r\end{bmatrix}_q (x-y)^{|\overline{J}|}\ y^{|J|},$$

where ρ is the rank function of $\mathcal{C}$.

Proof. It is proved by Proposition 3.2 (iv) and Lemma 4.3 (ii). The proof is similar to that of Theorem 4.2. □

4.3. **Extension** $\mathbb{P}$**-weight enumerator** $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,t)$**.** Given an $[n,k]$ $\mathbb{P}$-code $\mathcal{C}$ over $\mathbb{F}_q$ with a generator matrix $\mathcal{G}$ and a positive integer m, we consider the extension code over $\mathbb{F}_{q^m}$ having $\mathcal{G}$ as its generator matrix, and denote the extension code of $\mathcal{C}$ by $\mathcal{C}\otimes\mathbb{F}_{q^m}$. Note that the dimension of $\mathcal{C}\otimes\mathbb{F}_{q^m}$ over $\mathbb{F}_{q^m}$ is equal to that of $\mathcal{C}$ over $\mathbb{F}_q$, and for any $J\subseteq[n]$, $\rho(J)$ over $\mathbb{F}_q$ is identical to $\rho(J)$ over $\mathbb{F}_{q^m}$. Therefore, by

Lemma 4.1 (i), for $0 \le i \le n$,

$$\sum_{j=0}^{n} \binom{n-j}{n-i} \mathcal{A}_{j,\mathbb{P}}(\mathcal{C} \otimes \mathbb{F}_{q^m}) = \sum_{\substack{J \subseteq [n] \\ |J|=i}} (q^m)^{k-\rho(\overline{[J]_{\mathbb{P}}})} \tag{3}$$

and

$$\mathcal{W}_{\mathcal{C}\otimes\mathbb{F}_{q^m},\mathbb{P}}(x,y) = \sum_{J \subseteq [n]} (q^m)^{k-\rho(\overline{[J]_{\mathbb{P}}})} (x-y)^{|\overline{J}|} y^{|J|}. \tag{4}$$

Using t as a variable for q^m in (4), we define the extension $\mathbb{P}$-weight enumerator $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,t)$ of $\mathcal{C}$ by

$$\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,t) := \sum_{J \subseteq [n]} t^{k-\rho(\overline{[J]_{\mathbb{P}}})} (x-y)^{|\overline{J}|} y^{|J|}.$$

Note that $\mathcal{W}_{\mathcal{C}\otimes\mathbb{F}_{q^m},\mathbb{P}}(x,y) = \mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,q^m)$ and $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y) = \mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,q)$.

4.4. **$\mathbb{P}$-Tutte polynomial $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y)$.** Original Tutte polynomials play an important role in graph theory [7]. So we generalize the original Tutte polynomials with respect to poset codes to make the connection between poset codes and graph theory.

Let $\mathbb{P}$ be a poset on $[n]$. Let $\mathcal{C}$ be an $[n,k]$ $\mathbb{P}$-code over $\mathbb{F}_q$, and let $\mathcal{C}^\perp$ be the dual code of $\mathcal{C}$. Let ρ and $\rho^\perp$ be the rank and corank functions of $\mathcal{C}$, respectively. The $\mathbb{P}$-rank generating function $\mathcal{R}_{\mathcal{C},\mathbb{P}}(x,y)$ of $\mathcal{C}$ is defined by

$$\mathcal{R}_{\mathcal{C},\mathbb{P}}(x,y) := \sum_{J \subseteq [n]} x^{k-\rho(\overline{[J]_{\mathbb{P}}})} y^{|\overline{J}|-\rho(\overline{[J]_{\mathbb{P}}})}.$$

And we define $\mathbb{P}$-Tutte polynomial $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y)$ of $\mathcal{C}$ as follows:

$$\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) := R_{\mathcal{C},\mathbb{P}}(x-1,y-1) = \sum_{J \subseteq [n]} (x-1)^{k-\rho(\overline{[J]_{\mathbb{P}}})} (y-1)^{|\overline{J}|-\rho(\overline{[J]_{\mathbb{P}}})}.$$

Note that $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) = \mathcal{T}_{\mathcal{C}\otimes\mathbb{F}_{q^m},\mathbb{P}}(x,y)$ for any positive integer m by the definition of $\mathbb{P}$-Tutte polynomial. If $\mathbb{P}$ is an anti-chain on $[n]$, we have that $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) = \mathcal{T}_{\mathcal{C}^\perp,\mathbb{P}^\perp}(y,x)$ and the classical MacWilliams identity (5) can be obtained by using this duality [3].

$$\mathcal{W}_{\mathcal{C}^\perp}(x,y) = \frac{1}{|\mathcal{C}|} \mathcal{W}_{\mathcal{C}}(x+(q-1)y, x-y). \tag{5}$$

However, in general, we know $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) \neq \mathcal{T}_{\mathcal{C}^\perp,\mathbb{P}^\perp}(y,x)$ for an arbitrary poset $\mathbb{P}$.

Remark 4.5. *If the $\mathbb{P}$-rank generating function $\mathcal{R}_{\mathcal{C},\mathbb{P}}(x,y)$ is defined by*

$$\mathcal{R}_{\mathcal{C},\mathbb{P}}(x,y) = \sum_{J \subseteq [n]} x^{k-\rho(\overline{[J]_{\mathbb{P}}})} y^{|\overline{\langle J \rangle_{\mathbb{P}}}|-\rho(\overline{\langle J \rangle_{\mathbb{P}}})},$$

then the $\mathbb{P}$-Tutte polynomial $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y)$ is as follows:

$$\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) = \mathcal{R}_{\mathcal{C},\mathbb{P}}(x-1,y-1) = \sum_{J \subseteq [n]} (x-1)^{k-\rho(\overline{[J]_{\mathbb{P}}})} (y-1)^{|\overline{\langle J \rangle_{\mathbb{P}}}|-\rho(\overline{\langle J \rangle_{\mathbb{P}}})}.$$

Under this definition, we have $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) = \mathcal{T}_{\mathcal{C}^\perp,\mathbb{P}^\perp}(y,x)$, however, the relation (9) cannot be derived.

5. Relations among the weight enumerators

By (1), (3), Lemma 4.3 (*ii*) and Theorem 4.4, for a positive integer m, Lemma 5.1 is obtained.

Lemma 5.1. *For $0 \leq i \leq n$ and $m \geq 1$,*

(*i*) $\mathcal{A}_{i,\mathbb{P}}(\mathcal{C} \otimes \mathbb{F}_{q^m}) = \sum_{r=0}^{m} [m,r]_q \, \mathcal{A}^r_{i,\mathbb{P}}(\mathcal{C})$.

(*ii*) $\mathcal{W}_{\mathcal{C}\otimes\mathbb{F}_{q^m},\mathbb{P}}(x,y) = \sum_{r=0}^{m} [m,r]_q \mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y)$.

Proof. (*i*)

$$\begin{aligned}
\mathcal{A}_{i,\mathbb{P}}(\mathcal{C} \otimes \mathbb{F}_{q^m}) &= \sum_{j=n-i}^{n} (-1)^{j-n+i} \binom{j}{n-i} \sum_{\substack{J\subseteq[n] \\ |J|=n-j}} \left(\sum_{r=0}^{m} [m,r]_q \begin{bmatrix} k-\rho(\overline{[J]_{\mathbb{P}}}) \\ r \end{bmatrix}_q \right) \\
&= \sum_{r=0}^{m} [m,r]_q \left(\sum_{j=n-i}^{n} (-1)^{j-n+i} \binom{j}{n-i} \sum_{\substack{J\subseteq[n] \\ |J|=n-j}} \begin{bmatrix} k-\rho(\overline{[J]_{\mathbb{P}}}) \\ r \end{bmatrix}_q \right) \\
&= \sum_{r=0}^{m} [m,r]_q \, \mathcal{A}^r_{i,\mathbb{P}}(\mathcal{C}).
\end{aligned}$$

(*ii*)

$$\begin{aligned}
\mathcal{W}_{\mathcal{C}\otimes\mathbb{F}_{q^m},\mathbb{P}}(x,y) &= \sum_{J\subseteq[n]} (q^m)^{k-\rho(\overline{[J]_{\mathbb{P}}})} (x-y)^{|\overline{J}|} y^{|J|} \\
&= \sum_{J\subseteq[n]} \left(\sum_{r=0}^{m} [m,r]_q \begin{bmatrix} k-\rho(\overline{[J]_{\mathbb{P}}}) \\ r \end{bmatrix}_q \right) (x-y)^{|\overline{J}|} y^{|J|} \\
&= \sum_{J\subseteq[n]} \left(\sum_{r=0}^{k} [m,r]_q \begin{bmatrix} k-\rho(\overline{[J]_{\mathbb{P}}}) \\ r \end{bmatrix}_q \right) (x-y)^{|\overline{J}|} y^{|J|} \\
&= \sum_{r=0}^{k} [m,r]_q \, \mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y).
\end{aligned}$$

□

By Lemma 5.1 (*ii*), we have

$$\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,t) = \sum_{r=0}^{k} \left(\prod_{j=0}^{r-1} (t-q^j) \right) \mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y). \tag{6}$$

Therefore, from (2) and Theorem 4.4, r-th higher $\mathbb{P}$-weight enumerator $\mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y)$ can be represented in terms of extension $\mathbb{P}$-weight enumerators $\{\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,q^j)\}_{j=0}^{r}$

as follows: for $0 \le r \le k$,

$$\begin{aligned}
\mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y) &= \sum_{J\subseteq[n]} \begin{bmatrix} k-\rho(\overline{[J]_{\mathbb{P}}}) \\ r \end{bmatrix}_q (x-y)^{|\overline{J}|} y^{|J|} \\
&= \frac{1}{[r,r]_q} \sum_{J\subseteq[n]} \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix}_q (-1)^{r-j} q^{\frac{(r-j-1)(r-j)}{2}} (q^j)^{k-\rho(\overline{[J]_{\mathbb{P}}})} (x-y)^{|J|} y^{|\overline{J}|} \\
&= \frac{1}{[r,r]_q} \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix}_q (-1)^{r-j} q^{\frac{(r-j-1)(r-j)}{2}} \sum_{J\subseteq[n]} (q^j)^{k-\rho(\overline{[J]_{\mathbb{P}}})} (x-y)^{|J|} y^{|\overline{J}|} \\
&= \frac{1}{[r,r]_q} \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix}_q (-1)^{r-j} q^{\frac{(r-j-1)(r-j)}{2}} \mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,q^j). \qquad (7)
\end{aligned}$$

From Theorem 4.2, $\mathbb{P}$-weight enumerator $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y)$ can be expressed by $\mathbb{P}$-Tutte polynomial $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y)$ as follows.

$$\begin{aligned}
\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y) &= \sum_{J\subseteq[n]} q^{k-\rho(\overline{[J]_{\mathbb{P}}})} \ (x-y)^{|\overline{J}|} y^{|J|} \\
&= (x-y)^k \ y^{n-k} \sum_{J\subseteq[n]} q^{k-\rho(\overline{[J]_{\mathbb{P}}})} \ (x-y)^{|\overline{J}|-k} \ y^{|J|-n+k} \\
&= (x-y)^k \ y^{n-k} \sum_{J\subseteq[n]} \left(\frac{qy}{x-y}\right)^{k-\rho(\overline{[J]_{\mathbb{P}}})} \left(\frac{x-y}{y}\right)^{|\overline{J}|-\rho(\overline{[J]_{\mathbb{P}}})} \\
&= (x-y)^k \ y^{n-k} \ \mathcal{R}_{\mathcal{C},\mathbb{P}} \left(\frac{qy}{x-y}, \frac{x-y}{y}\right) \\
&= (x-y)^k \ y^{n-k} \ \mathcal{T}_{\mathcal{C},\mathbb{P}} \left(\frac{x+(q-1)y}{x-y}, \frac{x}{y}\right). \qquad (8)
\end{aligned}$$

(8) shows the connection between $\mathbb{P}$-weight enumerator $\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y)$ and $\mathbb{P}$-Tutte polynomial $\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y)$. It is generalization of Greene theorem [2].

From (8), we get (9) easily.

$$\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,t) = (x-y)^k y^{n-k} \ \mathcal{T}_{\mathcal{C},\mathbb{P}} \left(\frac{x+(t-1)y}{x-y}, \frac{x}{y}\right). \qquad (9)$$

In (9), replacing x, y and t by $y, 1$ and $(x-1)(y-1)$ respectively, we have

$$\begin{aligned}
&\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) \\
&= (y-1)^{-k} \ \mathcal{W}_{\mathcal{C},\mathbb{P}}(y,1,(x-1)(y-1)) \qquad (10) \\
&= (y-1)^{-k} \sum_{r=0}^{k} \left(\prod_{j=0}^{r-1}((x-1)(y-1)-q^j)\right) \mathcal{W}^r_{\mathcal{C},\mathbb{P}}(y,1). \quad \text{(by (6))} \qquad (11)
\end{aligned}$$

By (7) and (9), we have, for $0 \le r \le k$,

$$\begin{aligned}
&\mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y) \\
&= \frac{(x-y)^k y^{n-k}}{[r,r]_q} \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix}_q (-1)^{r-j} q^{\frac{(r-j-1)(r-j)}{2}} \ \mathcal{T}_{\mathcal{C},\mathbb{P}} \left(\frac{x+(q^j-1)y}{x-y}, \frac{x}{y}\right). \qquad (12)
\end{aligned}$$

According to the above identities (6), (7), (9), (10), (11) and (12) for relations among weight polynomials, we have Theorem 5.2.

Theorem 5.2. *Let $\mathbb{P}$ be a poset on $[n]$ and let $\mathcal{C}$ be an $[n,k]$ $\mathbb{P}$-code over $\mathbb{F}_q$. Then we have the following identities.*

$$\begin{aligned}
\mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y) &= \frac{1}{[r,r]_q}\sum_{j=0}^{r}\begin{bmatrix} r \\ j\end{bmatrix}_q (-1)^{r-j} q^{\frac{(r-j-1)(r-j)}{2}} \mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,q^j) \\
&= \frac{(x-y)^k y^{n-k}}{[r,r]_q}\sum_{j=0}^{r}\begin{bmatrix} r \\ j\end{bmatrix}_q (-1)^{r-j} q^{\frac{(r-j-1)(r-j)}{2}} \ \mathcal{T}_{\mathcal{C},\mathbb{P}}\left(\frac{x+(q^j-1)y}{x-y}, \frac{x}{y}\right). \\
\mathcal{W}_{\mathcal{C},\mathbb{P}}(x,y,t) &= \sum_{r=0}^{k}\left(\prod_{j=0}^{r-1}(t-q^j)\right)\mathcal{W}^r_{\mathcal{C},\mathbb{P}}(x,y) \\
&= (x-y)^k y^{n-k} \ \mathcal{T}_{\mathcal{C},\mathbb{P}}\left(\frac{x+(t-1)y}{x-y}, \frac{x}{y}\right). \\
\mathcal{T}_{\mathcal{C},\mathbb{P}}(x,y) &= (y-1)^{-k} \ \mathcal{W}_{\mathcal{C},\mathbb{P}}(y,1,(x-1)(y-1)) \\
&= (y-1)^{-k}\sum_{r=0}^{k}\left(\prod_{j=0}^{r-1}((x-1)(y-1)-q^j)\right)\mathcal{W}^r_{\mathcal{C},\mathbb{P}}(y,1).
\end{aligned}$$

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