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A NEW CLASS OF DOUBLE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. The objective of this paper is to evaluate two double integrals involving generalized hypergeometric functions which are given in two unified forms and each of which contains 25 double integrals. The results are derived with the help of Edwards's double integral and the generalized Watson's summation theorem due to Lavoie *et al.*.

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1. Introduction and Preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integers, respectively, and

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}.$$

The natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric series ${}_pF_q$ ($p,\, q \in \mathbb{N}_0$) defined by (see [1], [6, p. 73] and [7, pp. 71-75]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p;\\ \beta_1,\,\ldots,\,\beta_q;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z), \end{aligned} \tag{1}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [7, p. 2 and p. 5]):

$$\begin{aligned} (\lambda)_n : &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ &= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \end{aligned} \tag{2}$$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here an empty product is interpreted as 1, and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1,\,\ldots,\,\alpha_p$, and the denominator parameters $\beta_1,\,\ldots,\,\beta_q$ take on complex values, provided that no zeros appear in the denominator of (1), that is,

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\; j = 1,\,\ldots,\,q). \tag{3}$$

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For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, among an extensive literature, one may refer to [1, 6, 7].

It is worthy of note that whenever the generalized hypergeometirc function ${}_pF_q$ (including ${}_2F_1$) with its specified argument (for example, unit argument) can be summed to be expressed in terms of the Gamma functions, the result may be very important from both theoretical and applicable points of view. Here, the classical summation theorems for the hypergeometric series ${}_2F_1$ such as those of Gauss and Gauss second, Kummer, and Bailey; Watson's, Dixon's, Whipple's and Saalschütz's summation theorems for the series ${}_3F_2$ and others play important roles in theory and application. During 1992-1996, in a series of works [3, 4, 5], Lavoie *et al.* have generalized the above mentioned classical summation theorems for ${}_3F_2$ of Watson, Dixon, and Whipple and presented a large number of special and limiting cases of their results. Those results have also been obtained and verified with the help of computer programs (for example, Mathematica).

In our present investigation, we recall the following classical Watson's summation theorem (see, *e.g.*, [1, 6]; see also [7, p. 351]):

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b, \quad c\,; \\ \frac{1}{2}(a+b+1),\ 2c\,; \end{matrix}\ 1\right] \\
&\quad = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)},
\end{aligned}
\tag{4}
$$

provided $\Re(2c-a-b) > -1$.

Lavoie *et al.* [3] established a generalization of (4), which contains twenty five identities closely related to (4), recorded in the following single form:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b, \quad c\,; \\ \frac{1}{2}(a+b+i+1),\ 2c+j\,; \end{matrix}\ 1\right] = \mathcal{A}_{j,i}\, 2^{a+b+i-2} \\
&\times \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(c+[j/2]+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}(a+b+|i+j|-j-1)\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\,\Gamma(b)} \\
&\times\Bigg\{\mathcal{B}_{j,i}\,\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1-(-1)^i\right)\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}+[j/2]-\frac{1}{4}(-1)^j\,\left(1-(-1)^i\right)\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}+[j/2]\right)} \\
&\quad +\mathcal{C}_{j,i}\,\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+[(j+1)/2]+\frac{1}{4}(-1)^j\,\left(1-(-1)^i\right)\right)\Gamma\left(c-\frac{1}{2}b+[(j+1)/2]\right)}\Bigg\} \\
&:= \Omega
\end{aligned}
\tag{5}
$$

for $i,\ j = 0,\ \pm1,\ \pm2$. Here, $[x]$ denotes the greatest integer less than or equal to x and $|x|$ is the absolute value of x. The coefficients $\mathcal{A}_{j,i}$, $\mathcal{B}_{j,i}$ and $\mathcal{C}_{j,i}$ are given in the tables below.

Here, in this paper, we aim to evaluate the following two classes of (presumably) new and (potentially) useful integrals associated with generalized hypergeometric functions:

$$\int_0^1 \int_0^1 y^c (1-x)^{c-1} (1-y)^{c+\ell} (1-xy)^{-2c-\ell} \times {}_3F_2\left[\begin{matrix} a,\, b,\, 2c+\ell+1; \\ \frac{1}{2}(a+b+i+1),\, 2c+j; \end{matrix} \frac{y(1-x)}{1-xy}\right] dx\, dy$$

and

$$\int_0^1 \int_0^1 y^{c+\ell+1} (1-x)^{c+\ell} (1-y)^{c-1} (1-xy)^{-2c-\ell} \times {}_3F_2\left[\begin{matrix} a,\, b,\, 2c+\ell+1; \\ \frac{1}{2}(a+b+i+1),\, 2c+j; \end{matrix} \frac{1-y}{1-xy}\right] dx\, dy$$

$$(\ell \in \mathbb{Z} \quad \text{and} \quad i,\, j = 0,\, \pm 1,\, \pm 2)$$

by mainly using the generalized Watson's summation theorem due to Lavoie *et al.* (5) and the following double integral due to Edwards [2]:

$$\int_0^1 \int_0^1 y^{\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta}\, dxdy = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{6}$$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. One hundred interesting general double integrals are also given as special cases of the main results.

2. General integral formulas

Here we present two classes of integral formulas involving the generalized hypergeometric functions ${}_3F_2$, which are asserted by the following theorem.

Theorem 2.1. *The following double integral formulas hold true:*

$$\begin{aligned} &\int_0^1 \int_0^1 y^c (1-x)^{c-1} (1-y)^{c+\ell} (1-xy)^{-2c-\ell} \\ &\quad \times {}_3F_2\left[\begin{matrix} a,\, b,\, 2c+\ell+1; \\ \frac{1}{2}(a+b+i+1),\, 2c+j; \end{matrix} \frac{y(1-x)}{1-xy}\right] dx\, dy \\ &\quad = \frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\Omega \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\int_0^1 \int_0^1 y^{c+\ell+1} (1-x)^{c+\ell} (1-y)^{c-1} (1-xy)^{-2c-\ell} \\ &\quad \times {}_3F_2\left[\begin{matrix} a,\, b,\, 2c+\ell+1; \\ \frac{1}{2}(a+b+i+1),\, 2c+j; \end{matrix} \frac{1-y}{1-xy}\right] dx\, dy \\ &\quad = \frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\Omega, \end{aligned} \tag{8}$$

where Ω is given in (5)*, $\ell \in \mathbb{Z}$, and $i,\, j = 0,\, \pm 1,\, \pm 2$, and provided $\Re(c+\ell) > 0$ $(\ell \in \mathbb{N}_0)$; $\Re(c) > -\ell$ $(\ell \in \mathbb{Z} \setminus \mathbb{N}_0)$ and $\Re(2c-a-b+i+2j+1) > 0$ $(i,\, j = 0,\, \pm 1,\, \pm 2)$.*

Proof. Let $\mathcal{L}$ be the left-hand side of (7). Expressing the ${}_3F_2$ in (7) as the corresponding summation in (1) and interchanging the order of integral and summation, which is guaranteed by the uniform convergence of the series on the interval, we obtain

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(2c+\ell+1)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(2c+j)_n\,n!} \times \int_0^1\int_0^1 y^{c+n}\,(1-x)^{c+n-1}\,(1-y)^{c+\ell}\,(1-xy)^{-2c-\ell-n}\,dxdy.$$

Evaluating the the double integral with the aid of (6), after a little simplification, we get

$$\mathcal{L} = \frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)} \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(c)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(2c+j)_n\,n!}.$$

Using (1), we have

$$\mathcal{L} = \frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\, {}_3F_2\left[\begin{matrix} a,\quad b,\quad c\,; \\ \frac{1}{2}(a+b+i+1),\ 2c+j\,; \end{matrix}\ 1\right],$$

which, upon evaluating with the aid of (5), is led to the right-hand side of (7).

The formula (8) can be established in the same process as in the proof of (7). So details of its proof are omitted. This completes the proof. □

3. Special Cases

Here, as special cases of the main results (7) and (8), we present one hundred interesting integral formulas, which are given in the following four corollaries. In fact, in (7) and (8), for $n \in \mathbb{N}$, letting $b = -2n$ and replacing a by $a + 2n$, or, letting $b = -2n - 1$ and replacing a by $a + 2n + 1$, we find that, in each case, one of the two terms appearing on the right-hand sides of (7) and (8) will vanish. Then, under the given conditions, it is easy to get one hundred desired integral formulas.

Corollary 3.1. *The following integral formulas hold true:*

(9)
$$\int_0^1\int_0^1 y^{c}\,(1-x)^{c-1}\,(1-y)^{c+\ell}\,(1-xy)^{-2c-\ell} \times {}_3F_2\left[\begin{matrix} -2n,\ a+2n,\ 2c+\ell+1; \\ \frac{1}{2}(a+i+1),\ 2c+j; \end{matrix}\ \frac{y(1-x)}{1-xy}\right] dxdy$$
$$= \mathcal{D}_{i,j}\,\frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)_n}$$
$$:= \Omega_1 \quad \left(n \in \mathbb{N}_0;\ \ell \in \mathbb{Z};\ i,\ j = 0,\ \pm1,\ \pm2\right),$$

where $\mathcal{D}_{i,j}$ are given in the table below.

Corollary 3.2. *The following integral formulas hold true:*

(10)
$$\begin{aligned}
&\int_0^1\int_0^1 y^c\,(1-x)^{c-1}\,(1-y)^{c+\ell}\,(1-xy)^{-2c-\ell}\\
&\quad\times {}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ 2c+\ell+1;\\ \frac{1}{2}(a+i+1),\ 2c+j;\end{matrix}\ \frac{y(1-x)}{1-xy}\right]dxdy\\
&=\mathcal{E}_{i,j}\,\frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(3-(-1)^i\right)\right)_n}\\
&:=\Omega_2\quad \left(n\in\mathbb{N}_0;\ \ell\in\mathbb{Z};\ i,\ j=0,\ \pm1,\ \pm2\right),
\end{aligned}$$

where $\mathcal{E}_{i,j}$ are given in the table below.

Corollary 3.3. *The following integral formulas hold true:*

(11)
$$\begin{aligned}
&\int_0^1\int_0^1 y^{c+\ell+1}\,(1-x)^{c+\ell}\,(1-y)^{c-1}\,(1-xy)^{-2c-\ell}\\
&\quad\times {}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ 2c+\ell+1;\\ \frac{1}{2}(a+i+1),\ 2c+j;\end{matrix}\ \frac{1-y}{1-xy}\right]dxdy\\
&=\Omega_1\quad \left(n\in\mathbb{N}_0;\ \ell\in\mathbb{Z};\ i,\ j=0,\ \pm1,\ \pm2\right),
\end{aligned}$$

where Ω_1 is defined as in (9).

Corollary 3.4. *The following integral formulas hold true:*

(12)
$$\begin{aligned}
&\int_0^1\int_0^1 y^{c+\ell+1}\,(1-x)^{c+\ell}\,(1-y)^{c-1}\,(1-xy)^{-2c-\ell}\\
&\quad\times {}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ 2c+\ell+1;\\ \frac{1}{2}(a+i+1),\ 2c+j;\end{matrix}\ \frac{1-y}{1-xy}\right]dxdy\\
&=\Omega_2\quad \left(n\in\mathbb{N}_0;\ \ell\in\mathbb{Z};\ i,\ j=0,\ \pm1,\ \pm2\right),
\end{aligned}$$

where Ω_2 is defined as in (10).

We conclude this paper by giving further special cases of (9) and (10). Setting $i=j=0$ in (9) and (10) yields the following integral formulas:

(13)
$$\begin{aligned}
&\int_0^1\int_0^1 y^c\,(1-x)^{c-1}\,(1-y)^{c+\ell}\,(1-xy)^{-2c-\ell}\\
&\quad\times {}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ 2c+\ell+1;\\ \frac{1}{2}(a+1),\ 2c;\end{matrix}\ \frac{y(1-x)}{1-xy}\right]dx\,dy\\
&=\frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n}\quad \left(n\in\mathbb{N}_0;\ \ell\in\mathbb{Z}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 y^c\,(1-x)^{c-1}\,(1-y)^{c+\ell}\,(1-xy)^{-2c-\ell}\\
&\quad\times {}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ 2c+\ell+1;\\ \tfrac{1}{2}(a+1),\ 2c;\end{matrix}\ \frac{y(1-x)}{1-xy}\right]dx\,dy\\
&=0\quad (n\in\mathbb{N}_0;\ \ell\in\mathbb{Z}).
\end{aligned}\tag{14}$$

Remark 1. The result (14) is interesting as it can be seen that for all $\ell\in\mathbb{Z}$, the value of the double integral is zero.

TABLE 1. Table for $\mathcal{A}_{j,i}$

$j \diagdown i$	2	1	0	-1	-2
-2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{2(c-1)}$	$\frac{1}{c-1}$	$\frac{1}{2(c-1)}$
-1	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{a-b}$	1	1	1
0	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{a-b}$	1	2	1
1	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b)}$	1	2	2
2	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$	$\frac{1}{2(c+1)(a-b)}$	$\frac{1}{2(c+1)}$	$\frac{2}{c+1}$	$\frac{2}{c+1}$

Here

$$\mathcal{B}_{-2,2} := c(a+b-1)-(a+1)(b+1)+2;$$

$$\mathcal{B}_{-2,0} := (c-a-1)(c-b-1)+(c-1)(c-2);$$

$$\mathcal{B}_{-2,-1} := 2(c-1)(c-2)-(a-b)(c-b-1);$$

$$\begin{aligned}\mathcal{B}_{-2,-2} :=&2(c-1)(c-2)\{(2c-1)(a+b-1)-a(a+1)-b(b+1)+2\}\\ &-(a-b-1)(a-b+1)\{(c-1)(2c-a-b-3)+ab\};\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{2,2} :=&2c(c+1)\{(2c+1)(a+b-1)-a(a-1)-b(b-1)\}\\ &-(a-b-1)(a-b+1)\{(c+1)(2c-a-b+1)+ab\};\end{aligned}$$

$$\mathcal{B}_{-1,-2} := 2(c-1)(a+b-1)-(a-b)^2+1;$$

$$\mathcal{B}_{0,2} := a(2c-a)+b(2c-b)-2c+1;$$

TABLE 2. Table for $\mathcal{B}_{j,i}$

$j \diagdown i$	2	1	0	-1	-2
-2	$\mathcal{B}_{-2,2}$	$c-b-1$	$\mathcal{B}_{-2,0}$	$\mathcal{B}_{-2,-1}$	$\mathcal{B}_{-2,-2}$
-1	$a-b+1$	1	1	$2c-a+b-2$	$\mathcal{B}_{-1,-2}$
0	$\mathcal{B}_{0,2}$	1	1	1	$\mathcal{B}_{0,-2}$
1	$\mathcal{B}_{1,2}$	$2c-a+b$	1	1	$a+b-1$
2	$\mathcal{B}_{2,2}$	$\mathcal{B}_{2,1}$	$\mathcal{B}_{2,0}$	$c-b+1$	$\mathcal{B}_{2,-2}$

$$\mathcal{B}_{0,-2} := a(2c-a) + b(2c-b) - 2c + 1;$$

$$\mathcal{B}_{1,2} := 2c(a+b-1) - (a-b)^2 + 1;$$

$$\mathcal{B}_{2,1} := 2c(c+1) - (a-b)(c-b+1);$$

$$\mathcal{B}_{2,0} := (c-a+1)(c-b+1) + c(c+1);$$

$$\mathcal{B}_{2,-2} := c(a+b-1) - (a-1)(b-1).$$

Here

$$\mathcal{C}_{-2,-1} := 2(c-1)(c-2) + (a-b)(c-a-1);$$
$$\mathcal{C}_{-2,-2} := 4(2c-a+b-3)(2c+a-b-3);$$
$$\mathcal{C}_{-1,-2} := 8c^2 - 2(c-1)(a+b+7) - (a-b)^2 - 7;$$
$$\mathcal{C}_{1,2} := -8c^2 + 2c(a+b-1) + (a-b)^2 - 1;$$
$$\mathcal{C}_{2,2} := -4(2c+a-b+1)(2c-a+b+1);$$
$$\mathcal{C}_{2,1} := -2c(c+1) - (a-b)(c-a+1).$$

Here

$$\mathcal{D}_{2,-2} := \frac{(a+1)\{(c-1)(a-1) + 2n(a+2n)\}}{(c-1)(a+4n-1)(a+4n+1)};$$

$$\mathcal{D}_{2,-1} := \frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)};$$

$$\mathcal{D}_{2,0} := \frac{(a+1)\{(a-1)(2c-a-1) - 4n(a+2n)\}}{(2c-a-1)(a+4n+1)(a+4n-1)};$$

TABLE 3. Table for $\mathcal{C}_{j,i}$

$j \setminus i$	2	1	0	-1	-2
-2	-4	$-(c-a-1)$	4	$\mathcal{C}_{-2,-1}$	$\mathcal{C}_{-2,-2}$
-1	$-(4c-a-b-3)$	-1	1	$2c+a-b-2$	$\mathcal{C}_{-1,-2}$
0	-8	-1	0	1	8
1	$\mathcal{C}_{1,2}$	$-(2c+a-b)$	-1	1	$4c-a-b+1$
2	$\mathcal{C}_{2,2}$	$\mathcal{C}_{2,1}$	-4	$c-a+1$	4

TABLE 4. Table for $\mathcal{D}_{i,j}$

$i \setminus j$	-2	-1	0	1	2
2	$\mathcal{D}_{2,-2}$	$\mathcal{D}_{2,-1}$	$\mathcal{D}_{2,0}$	$\mathcal{D}_{2,1}$	$\mathcal{D}_{2,2}$
1	$\frac{a(c+2n-1)}{(c-1)(a+4n)}$	$\frac{a}{a+4n}$	$\frac{a}{a+4n}$	$\frac{a(2c-a-4n)}{(2c-a)(a+4n)}$	$\mathcal{D}_{1,2}$
0	$\mathcal{D}_{0,-2}$	1	1	1	$\mathcal{D}_{0,2}$
-1	$\mathcal{D}_{-1,-2}$	$\frac{2c-a-4n-2}{2c-a-2}$	1	1	$\frac{c+2n+1}{c+1}$
-2	$\mathcal{D}_{-2,-2}$	$\mathcal{D}_{-2,-1}$	$\mathcal{D}_{-2,0}$	1	$\mathcal{D}_{-2,2}$

$$\mathcal{D}_{2,1} := \frac{(a+1)\{(a-1)(2c-a-1)-8n(a+2n)\}}{(2c-a-1)(a+4n+1)(a+4n-1)};$$

$$\mathcal{D}_{2,2} := \frac{N_{2,2}}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)},$$

where

$$\begin{aligned} N_{2,2} := (a+1)\Big\{&(a-1)(c+1)(2c-a+1)(2c-a-1)\\ &-2a\,n(6c+a+5)(2c-a+1)\\ &+4n^2\,(5a^2+4a-5-4c(3c-a+4)+64n^3(a+n)\Big\}; \end{aligned}$$

$$\mathcal{D}_{1,2} := \frac{a\{(c+1)(2c-a)-2n(2c+a+4n+2)\}}{(c+1)(2c-a)(a+4n)};$$

$$\mathcal{D}_{0,-2} := 1-\frac{2n(a+2n)}{(c-1)(2c-a-3)};$$

$$\mathcal{D}_{0,2} := 1-\frac{2n(a+2n)}{(c+1)(2c-a+1)};$$

$$\mathcal{D}_{-1,-2} := 1-\frac{2n(2c+a+4n-2)}{(c-1)(2c-a-4)};$$

$$\mathcal{D}_{-2,-2} := 1-\frac{N_{-2,-2}}{(a-1)(c-1)(2c-a-3)(2c-a-5)},$$

where

$$\begin{aligned} N_{-2,-2} :=&2an(6c+a-7)(2c-a-3)-4n^2\{5a^2-4a-21\\ &-4c(3c-a-8)\}-64n^3(a+n); \end{aligned}$$

$$\mathcal{D}_{-2,-1} := 1-\frac{8n(a+2n)}{(a-1)(2c-a-3)};$$

$$\mathcal{D}_{-2,0} := 1-\frac{4n(a+2n)}{(a-1)(2c-a-1)};$$

$$\mathcal{D}_{-2,2} := 1+\frac{2n(a+2n)}{(c+1)(a-1)}.$$

Here

$$\mathcal{E}_{2,-2} := \frac{(a+1)(2c-a-3)}{(c-1)(a+4n+1)(a+4n+3)};$$

$$\mathcal{E}_{2,-1} := \frac{(a+1)(4c-a-3)}{(2c-1)(a+4n+1)(a+4n+3)};$$

$$\mathcal{E}_{2,0} := \frac{2(a+1)}{(a+4n+1)(a+4n+3)};$$

$$\mathcal{E}_{2,1} := \frac{(a+1)\{(4c+a+3)(2c-a-1)-8n(a+2n+2)\}}{(a+4n+1)(a+4n+3)(2c+1)(2c-a-1)};$$

$$\mathcal{E}_{2,2} := \frac{(a+1)(2c+a+4n+3)(2c-a-4n-1)}{(a+4n+1)(a+4n+3)(c+1)(2c-a-1)};$$

$$\mathcal{E}_{1,2} := \frac{(c+a+2)(2c-a)-2n(3a-2c+4n+2)}{(c+1)(2c-a)(a+4n+2)};$$

$$\mathcal{E}_{-1,-2} := \frac{(c+a)(2c-a-4)-2n(3a-2c+4n+6)}{a(a-2c+4)(c-1)};$$

$$\mathcal{E}_{-2,-2} := \frac{(2c+a+4n-1)(2c-a-4n-5)}{(1-a)(c-1)(2c-a-5)};$$

TABLE 5. Table for $\mathcal{E}_{i,j}$

$i \diagdown j$	-2	-1	0	1	2
2	$\mathcal{E}_{2,-2}$	$\mathcal{E}_{2,-1}$	$\mathcal{E}_{2,0}$	$\mathcal{E}_{2,1}$	$\mathcal{E}_{2,2}$
1	$\frac{c-a-2n-2}{(c-1)(a+4n+2)}$	$\frac{2c-a-2}{(2c-1)(a+4n+2)}$	$\frac{1}{a+4n+2}$	$\frac{2c+a+4n+2}{(2c+1)(a+4n+2)}$	$\mathcal{E}_{1,2}$
0	$\frac{1}{1-c}$	$\frac{1}{1-2c}$	0	$\frac{1}{1+2c}$	$\frac{1}{1+c}$
-1	$\mathcal{E}_{-1,-2}$	$\frac{2c+a+4n}{a(1-2c)}$	$-\frac{1}{a}$	$\frac{a-2c}{a(2c+1)}$	$\frac{a-c+2n}{a(c+1)}$
-2	$\mathcal{E}_{-2,-2}$	$\mathcal{E}_{-2,-1}$	$\frac{2}{1-a}$	$\frac{4c-a+1}{(1-a)(2c+1)}$	$\frac{2c-a+1}{(1-a)(c+1)}$

$$\mathcal{E}_{-2,-1} := \frac{(4c+a-1)(2c-a-3)-8n(a+2n+2)}{(a-1)(a-2c+3)(2c-1)}.$$

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